Chapter 3: Trigonometric Functions

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Exercise 3.1 — Exercise 3.1

Q1 Find the radian measures corresponding to the following degree measures:
(i) 25° (ii) –47°30′ (iii) 240° (iv) 520°

SOLUTION

Use the conversion: Radian measure = (π/180) × Degree measure.

(i) 25° = (π/180) × 25 = 25π/180 = 5π/36 radians.

(ii) –47°30′ = –47.5° (since 30′ = 0.5°) = –(47.5π/180) = –(95π/360) = –19π/72 radians.

(iii) 240° = (π/180) × 240 = 240π/180 = 4π/3 radians.

(iv) 520° = (π/180) × 520 = 520π/180 = 26π/9 radians.

Answer: (i) 5π/36 (ii) –19π/72 (iii) 4π/3 (iv) 26π/9

Q2 Find the degree measures corresponding to the following radian measures (Use π = 22/7):
(i) 11/16 (ii) –4 (iii) 5π/3 (iv) 7π/6

SOLUTION

Use the conversion: Degree measure = (180/π) × Radian measure, with π = 22/7, so 180/π = 180 × 7/22 = 1260/22 = 630/11.

(i) 11/16 radians = (180/π) × (11/16) = (1260/22) × (11/16) = (1260 × 11)/(22 × 16) = 13860/352 = 39.375° = 39°22′30″.
[0.375° × 60 = 22.5′; 0.5′ × 60 = 30″]

(ii) –4 radians = –4 × (1260/22) = –5040/22 = –2520/11 degrees.
2520 ÷ 11 = 229 remainder 1, so = –229(1/11)°.

(1/11)° × 60 = 60/11 min = 5(5/11) min; (5/11) × 60 = 300/11 sec ≈ 27.27″ ≈ 27″.
So –4 rad = –229°5′27″ (approximately).

(iii) 5π/3 radians = (180/π) × (5π/3) = 180 × 5/3 = 300°.

(iv) 7π/6 radians = (180/π) × (7π/6) = 180 × 7/6 = 210°.

Answer: (i) 39°22′30″ (ii) –229°5′27″ (approx.) (iii) 300° (iv) 210°

Q3 A wheel makes 360 revolutions in one minute. Through how many radians does it turn in one second?

SOLUTION

The wheel makes 360 revolutions per minute.
In one second it makes 360/60 = 6 revolutions.
Each complete revolution = 2π radians.
Therefore, in one second it turns through 6 × 2π = 12π radians.

Answer: 12π radians

Q4 Find the degree measure of the angle subtended at the centre of a circle of radius 100 cm by an arc of length 22 cm (Use π = 22/7).

SOLUTION

Using the formula θ = l/r (in radians):
θ = 22/100 = 11/50 radians.

Converting to degrees (with π = 22/7):
θ = (180/π) × (11/50) = (180 × 7/22) × (11/50) = (1260/22) × (11/50) = 1260 × 11/(22 × 50) = 13860/1100 = 63/5 = 12.6°.

0.6° = 0.6 × 60′ = 36′.
Therefore θ = 12°36′.

Answer: 12°36′

Q5 In a circle of diameter 40 cm, the length of a chord is 20 cm. Find the length of minor arc of the chord.

SOLUTION

Diameter = 40 cm, so radius r = 20 cm.

Chord length = 20 cm = r, so the chord equals the radius.

In a circle, when the chord length equals the radius, the triangle formed by the two radii and the chord is equilateral.

Therefore the angle subtended at the centre = 60° = π/3 radians.

Length of minor arc = r × θ = 20 × π/3 = 20π/3 cm.

Answer: 20π/3 cm

Q6 If in two circles, arcs of the same length subtend angles 60° and 75° at the centre, find the ratio of their radii.

SOLUTION

Let r_1 and r_2 be the radii of the two circles, and let l be the common arc length.

Converting: $\theta_1 = 60° = \pi/3$ rad, $\theta_2 = 75° = 5\pi/12$ rad.

Since $l = r_1\theta_1 = r_2\theta_2$, we get:

$r_1/r_2 = \theta_2/\theta_1 = (5\pi/12)/(\pi/3) = (5\pi/12) \times (3/\pi) = 15/12 = 5/4$.

Therefore $r_1 : r_2 = 5 : 4$.

Answer: $r_1 : r_2 = 5 : 4$

Q7 Find the angle in radian through which a pendulum swings if its length is 75 cm and the tip describes an arc of length:
(i) 10 cm (ii) 15 cm (iii) 21 cm

SOLUTION

Using θ = l/r with r = 75 cm:

(i) θ = 10/75 = 2/15 radians.

(ii) θ = 15/75 = 1/5 radians.

(iii) θ = 21/75 = 7/25 radians.

Answer: (i) 2/15 rad (ii) 1/5 rad (iii) 7/25 rad

Exercise 3.2 — Exercise 3.2

Q1 Find the values of other five trigonometric functions if cos x = –1/2, x lies in third quadrant.

SOLUTION

Given: cos x = –1/2, x in third quadrant.

sec x = 1/cos x = –2.

$\sin^2 x = 1 - \cos^2 x = 1 - 1/4 = 3/4$, so $\sin x = \pm\sqrt{3}/2$.
Since x is in the third quadrant, sin x is negative: $\sin x = -\sqrt{3}/2$.

cosec x = 1/sin x = $-2/\sqrt{3} = -2\sqrt{3}/3$.

tan x = sin x/cos x = $(-\sqrt{3}/2)/(-1/2) = \sqrt{3}$.

cot x = 1/tan x = $1/\sqrt{3} = \sqrt{3}/3$.

Answer: $\sin x = -\sqrt{3}/2$, cosec x = $-2\sqrt{3}/3$, sec x = –2, tan x = $\sqrt{3}$, cot x = $\sqrt{3}/3$

Q2 Find the values of other five trigonometric functions if sin x = 3/5, x lies in second quadrant.

SOLUTION

Given: sin x = 3/5, x in second quadrant.

cosec x = 1/sin x = 5/3.

$\cos^2 x = 1 - \sin^2 x = 1 - 9/25 = 16/25$, so $\cos x = \pm 4/5$.
Since x is in the second quadrant, cos x is negative: cos x = –4/5.

sec x = 1/cos x = –5/4.

tan x = sin x/cos x = (3/5)/(–4/5) = –3/4.

cot x = cos x/sin x = (–4/5)/(3/5) = –4/3.

Answer: cos x = –4/5, tan x = –3/4, cosec x = 5/3, sec x = –5/4, cot x = –4/3

Q3 Find the values of other five trigonometric functions if cot x = 3/4, x lies in third quadrant.

SOLUTION

Given: cot x = 3/4, x in third quadrant.

tan x = 1/cot x = 4/3.

$\sec^2 x = 1 + \tan^2 x = 1 + 16/9 = 25/9$, so sec x = ±5/3.
Since x is in the third quadrant, sec x is negative: sec x = –5/3.

cos x = 1/sec x = –3/5.

sin x = tan x · cos x = (4/3)(–3/5) = –4/5.
Since x is in the third quadrant, sin x is negative — confirmed.

cosec x = 1/sin x = –5/4.

Answer: sin x = –4/5, cos x = –3/5, tan x = 4/3, cosec x = –5/4, sec x = –5/3

Q4 Find the values of other five trigonometric functions if sec x = 13/5, x lies in fourth quadrant.

SOLUTION

Given: sec x = 13/5, x in fourth quadrant.

cos x = 1/sec x = 5/13.

$\sin^2 x = 1 - \cos^2 x = 1 - 25/169 = 144/169$, so sin x = ±12/13.
Since x is in the fourth quadrant, sin x is negative: sin x = –12/13.

cosec x = 1/sin x = –13/12.

tan x = sin x/cos x = (–12/13)/(5/13) = –12/5.

cot x = cos x/sin x = (5/13)/(–12/13) = –5/12.

Answer: sin x = –12/13, cos x = 5/13, tan x = –12/5, cosec x = –13/12, cot x = –5/12

Q5 Find the values of other five trigonometric functions if tan x = –5/12, x lies in second quadrant.

SOLUTION

Given: tan x = –5/12, x in second quadrant.

$\sec^2 x = 1 + \tan^2 x = 1 + 25/144 = 169/144$, so sec x = ±13/12.

Since x is in the second quadrant, sec x is negative: sec x = –13/12.

cos x = 1/sec x = –12/13.

sin x = tan x · cos x = (–5/12)(–12/13) = 5/13.
Since x is in the second quadrant, sin x is positive — confirmed.

cosec x = 1/sin x = 13/5.

cot x = 1/tan x = –12/5.

Answer: sin x = 5/13, cos x = –12/13, cosec x = 13/5, sec x = –13/12, cot x = –12/5

Q6 Find the value of sin 765°.

SOLUTION

sin x repeats with period 360°.
765° = 2 × 360° + 45° = 720° + 45°.
Therefore sin 765° = sin 45° = 1/√2 = √2/2.

Answer: √2/2

Q7 Find the value of cosec(–1410°).

SOLUTION

–1410° = –4 × 360° + 30° = –1440° + 30°.
Therefore sin(–1410°) = sin(30°) = 1/2.
cosec(–1410°) = 1/sin(–1410°) = 1/(1/2) = 2.

Answer: 2

Q8 Find the value of tan(19π/3).

SOLUTION

tan x repeats with period π.
19π/3 = 6π + π/3 = 3(2π) + π/3.
Therefore tan(19π/3) = tan(π/3) = √3.

Answer: √3

Q9 Find the value of $\sin(-11\pi/3)$.

SOLUTION

$\sin x$ repeats with period 2π.

$-11\pi/3 + 4\pi = -11\pi/3 + 12\pi/3 = \pi/3$.

Therefore $\sin(-11\pi/3) = \sin(\pi/3) = \sqrt{3}/2$.

Answer: $\sqrt{3}/2$

Q10 Find the value of $\cot(-15\pi/4)$.

SOLUTION

$\cot x$ repeats with period π.

$-15\pi/4 + 4\pi = -15\pi/4 + 16\pi/4 = \pi/4$.

Therefore $\cot(-15\pi/4) = \cot(\pi/4) = 1$.

Answer: 1

Exercise 3.3 — Exercise 3.3

Q1 Prove that: $\sin^2(\pi/6) + \cos^2(\pi/3) - \tan^2(\pi/4) = -1/2$.

SOLUTION

$\sin(\pi/6) = 1/2$, so $\sin^2(\pi/6) = 1/4$.
$\cos(\pi/3) = 1/2$, so $\cos^2(\pi/3) = 1/4$.
$\tan(\pi/4) = 1$, so $\tan^2(\pi/4) = 1$.

LHS $= 1/4 + 1/4 - 1 = 1/2 - 1 = -1/2 =$ RHS.

Answer: Proved. LHS $= 1/4 + 1/4 - 1 = -1/2 =$ RHS.

Q2 Prove that: $2\sin^2(\pi/6) + \text{cosec}^2(7\pi/6) \cdot \cos^2(\pi/3) = 3/2$.

SOLUTION

$\sin(\pi/6) = 1/2$, so $\sin^2(\pi/6) = 1/4$ and $2\sin^2(\pi/6) = 1/2$.

$\sin(7\pi/6) = \sin(\pi + \pi/6) = -\sin(\pi/6) = -1/2$.
$\text{cosec}(7\pi/6) = 1/\sin(7\pi/6) = -2$, so $\text{cosec}^2(7\pi/6) = 4$.

$\cos(\pi/3) = 1/2$, so $\cos^2(\pi/3) = 1/4$.

LHS $= 1/2 + 4 \times (1/4) = 1/2 + 1 = 3/2 =$ RHS.

Answer: Proved. LHS $= 1/2 + 1 = 3/2 =$ RHS.

Q3 Prove that: $\cot^2(\pi/6) + \text{cosec}(5\pi/6) + 3\tan^2(\pi/6) = 6$.

SOLUTION

$\cot(\pi/6) = \cos(\pi/6)/\sin(\pi/6) = (\sqrt{3}/2)/(1/2) = \sqrt{3}$, so $\cot^2(\pi/6) = 3$.

$\sin(5\pi/6) = \sin(\pi - \pi/6) = \sin(\pi/6) = 1/2$, so $\text{cosec}(5\pi/6) = 2$.

$\tan(\pi/6) = 1/\sqrt{3}$, so $\tan^2(\pi/6) = 1/3$ and $3\tan^2(\pi/6) = 1$.

LHS $= 3 + 2 + 1 = 6 =$ RHS.

Answer: Proved. LHS $= 3 + 2 + 1 = 6 =$ RHS.

Q4 Prove that: $2\sin^2(3\pi/4) + 2\cos^2(\pi/4) + 2\sec^2(\pi/3) = 10$.

SOLUTION

$\sin(3\pi/4) = \sin(\pi - \pi/4) = \sin(\pi/4) = 1/\sqrt{2}$, so $\sin^2(3\pi/4) = 1/2$ and $2\sin^2(3\pi/4) = 1$.

$\cos(\pi/4) = 1/\sqrt{2}$, so $\cos^2(\pi/4) = 1/2$ and $2\cos^2(\pi/4) = 1$.

$\sec(\pi/3) = 1/\cos(\pi/3) = 1/(1/2) = 2$, so $\sec^2(\pi/3) = 4$ and $2\sec^2(\pi/3) = 8$.

LHS = 1 + 1 + 8 = 10 = RHS.

Answer: Proved. LHS = 1 + 1 + 8 = 10 = RHS.

Q5 Find the value of:
(i) sin 75° (ii) tan 15°

SOLUTION

(i) $\sin 75° = \sin(45° + 30°) = \sin 45° \cos 30° + \cos 45° \sin 30°$
$= (1/\sqrt{2})(\sqrt{3}/2) + (1/\sqrt{2})(1/2)$
$= \sqrt{3}/(2\sqrt{2}) + 1/(2\sqrt{2})$
$= (\sqrt{3} + 1)/(2\sqrt{2})$
$= (\sqrt{3} + 1)\sqrt{2}/4$
$= (\sqrt{6} + \sqrt{2})/4$.

(ii) $\tan 15° = \tan(45° - 30°) = (\tan 45° - \tan 30°)/(1 + \tan 45° \tan 30°)$
$= (1 - 1/\sqrt{3})/(1 + 1 \cdot 1/\sqrt{3})$
$= (\sqrt{3} - 1)/(\sqrt{3} + 1)$
Rationalising: multiply numerator and denominator by $(\sqrt{3} - 1)$:
$= (\sqrt{3} - 1)^2/((\sqrt{3})^2 - 1^2) = (3 - 2\sqrt{3} + 1)/(3 - 1) = (4 - 2\sqrt{3})/2 = 2 - \sqrt{3}$.

Answer: (i) $\sin 75° = (\sqrt{6} + \sqrt{2})/4$ (ii) $\tan 15° = 2 - \sqrt{3}$

Q6 Prove that: $\cos(\pi/4 - x)\cos(\pi/4 - y) - \sin(\pi/4 - x)\sin(\pi/4 - y) = \sin(x + y)$.

SOLUTION

Using the identity $\cos(A + B) = \cos A \cos B - \sin A \sin B$, let $A = \pi/4 - x$ and $B = \pi/4 - y$:

LHS = $\cos[(\pi/4 - x) + (\pi/4 - y)]$
$= \cos[\pi/2 - (x + y)]$
$= \sin(x + y)$ [since $\cos(\pi/2 - \theta) = \sin\theta$]
= RHS.

Answer: Proved. LHS = cos(π/2 – (x+y)) = sin(x+y) = RHS.

Q7 Prove that: $\tan(\pi/4 + x) / \tan(\pi/4 – x) = ((1 + \tan x)/(1 – \tan x))^2$.

SOLUTION

Using the addition formula: $\tan(A \pm B) = (\tan A \pm \tan B)/(1 \mp \tan A \tan B)$ with $\tan(\pi/4) = 1$:

$\tan(\pi/4 + x) = (1 + \tan x)/(1 – \tan x)$.
$\tan(\pi/4 – x) = (1 – \tan x)/(1 + \tan x)$.

Therefore:
LHS = $\tan(\pi/4 + x)/\tan(\pi/4 – x) = [(1 + \tan x)/(1 – \tan x)] / [(1 – \tan x)/(1 + \tan x)]$
$= (1 + \tan x)/(1 – \tan x) \times (1 + \tan x)/(1 – \tan x)$
$= [(1 + \tan x)/(1 – \tan x)]^2$
= RHS.

Answer: Proved using $\tan(\pi/4 \pm x)$ formulas.

Q8 Prove that: $[\cos(\pi + x) \cdot \cos(–x)] / [\sin(\pi – x) \cdot \cos(\pi/2 + x)] = \cot^2 x$.

SOLUTION

Apply standard reduction formulas:
$\cos(\pi + x) = –\cos x$
$\cos(–x) = \cos x$
$\sin(\pi – x) = \sin x$
$\cos(\pi/2 + x) = –\sin x$

LHS = $[(–\cos x)(\cos x)] / [(\sin x)(–\sin x)]$
$= (–\cos^2 x) / (–\sin^2 x)$
$= \cos^2 x/\sin^2 x$
$= \cot^2 x$ = RHS.

Answer: Proved. LHS simplifies to $\cos^2 x/\sin^2 x = \cot^2 x$ = RHS.

Q9 Prove that: $\cos(3\pi/2 + x) \cdot \cos(2\pi + x) \cdot [\cot(3\pi/2 - x) + \cot(2\pi + x)] = 1$.

SOLUTION

Apply reduction formulas:
$\cos(3\pi/2 + x) = \sin x$
$\cos(2\pi + x) = \cos x$
$\cot(3\pi/2 - x) = \tan x$ [since $\cot(3\pi/2 - x) = \cos(3\pi/2 - x)/\sin(3\pi/2 - x) = -\sin x/(-\cos x) = \tan x$]
$\cot(2\pi + x) = \cot x$

LHS $= \sin x \cdot \cos x \cdot (\tan x + \cot x)$
$= \sin x \cdot \cos x \cdot (\sin x/\cos x + \cos x/\sin x)$
$= \sin x \cdot \cos x \cdot (\sin^2 x + \cos^2 x)/(\sin x \cos x)$
$= \sin x \cdot \cos x \cdot 1/(\sin x \cos x)$
$= 1 =$ RHS.

Answer: Proved. LHS $= \sin x \cdot \cos x \cdot (\tan x + \cot x) = 1 =$ RHS.

Q10 Prove that: $\sin(n+1)x \cdot \sin(n+2)x + \cos(n+1)x \cdot \cos(n+2)x = \cos x$.

SOLUTION

Using the identity $\cos(A - B) = \cos A \cos B + \sin A \sin B$:

LHS $= \cos(n+1)x \cdot \cos(n+2)x + \sin(n+1)x \cdot \sin(n+2)x$
$= \cos[(n+2)x - (n+1)x]$
$= \cos(x)$
$=$ RHS.

Answer: Proved. LHS $= \cos((n+2)x - (n+1)x) = \cos x =$ RHS.

Q11 Prove that: $\cos(3\pi/4 + x) - \cos(3\pi/4 - x) = -\sqrt{2} \sin x$.

SOLUTION

Using the formula $\cos C - \cos D = -2 \sin((C+D)/2) \sin((C-D)/2)$:

Let $C = 3\pi/4 + x$ and $D = 3\pi/4 - x$.
$(C + D)/2 = 3\pi/4$, $(C - D)/2 = x$.

LHS $= -2 \sin(3\pi/4) \sin(x)$
$= -2 \cdot \sin(\pi - \pi/4) \cdot \sin x$
$= -2 \cdot \sin(\pi/4) \cdot \sin x$

= –2 · (1/√2) · sin x
= –√2 sin x = RHS.

Answer: Proved. LHS = –2 sin(3π/4) sin x = –√2 sin x = RHS.

Q12 Prove that: $\sin^2 6x - \sin^2 4x = \sin 2x \cdot \sin 10x$.

SOLUTION

Use the identity $\sin^2 A - \sin^2 B = \sin(A+B)\sin(A-B)$:

LHS = $\sin^2 6x - \sin^2 4x = \sin(6x + 4x) \cdot \sin(6x - 4x) = \sin 10x \cdot \sin 2x$ = RHS.

Alternatively, using sum-to-product:
$\sin^2 6x - \sin^2 4x = (\sin 6x + \sin 4x)(\sin 6x - \sin 4x)$
= [2 sin 5x cos x][2 cos 5x sin x]
= 4 sin 5x cos 5x · sin x cos x
= sin 10x · sin 2x = RHS.

Answer: Proved. $\sin^2 6x - \sin^2 4x = \sin(A+B)\sin(A-B) = \sin 10x \cdot \sin 2x$ = RHS.

Q13 Prove that: $\cos^2 2x - \cos^2 6x = \sin 4x \cdot \sin 8x$.

SOLUTION

Use the identity $\cos^2 A - \cos^2 B = -(\sin^2 A - \sin^2 B)$ ... but more directly:
$\cos^2 2x - \cos^2 6x = (\cos 2x + \cos 6x)(\cos 2x - \cos 6x)$.

cos 2x + cos 6x = 2 cos 4x cos 2x.
cos 2x – cos 6x = –2 sin 4x sin(–2x) = 2 sin 4x sin 2x.

LHS = (2 cos 4x cos 2x)(2 sin 4x sin 2x) = 4 sin 4x cos 4x · sin 2x cos 2x
= sin 8x · sin 4x = RHS.

Answer: Proved. LHS = sin 8x · sin 4x = RHS.

Q14 Prove that: $\sin 2x + 2\sin 4x + \sin 6x = 4\cos^2 x \cdot \sin 4x$.

SOLUTION

Group the first and third terms:
sin 2x + sin 6x = 2 sin((2x+6x)/2) cos((6x–2x)/2) = 2 sin 4x cos 2x.

LHS = 2 sin 4x cos 2x + 2 sin 4x
= 2 sin 4x (cos 2x + 1)

= 2 sin 4x · 2cos²x [since 1 + cos 2x = 2cos²x]
= 4 cos²x sin 4x = RHS.

Answer: Proved. LHS = 2 sin4x(1 + cos2x) = 4cos²x sin4x = RHS.

Q15 Prove that: cot 4x (sin 5x + sin 3x) = cot x (sin 5x – sin 3x).

SOLUTION

Apply sum-to-product formulas:
sin 5x + sin 3x = 2 sin 4x cos x.
sin 5x – sin 3x = 2 cos 4x sin x.

LHS = cot 4x · 2 sin 4x cos x = (cos 4x/sin 4x) · 2 sin 4x cos x = 2 cos 4x cos x.

RHS = cot x · 2 cos 4x sin x = (cos x/sin x) · 2 cos 4x sin x = 2 cos 4x cos x.

LHS = RHS.

Answer: Proved. Both sides equal 2 cos 4x cos x.

Q16 Prove that: (cos 9x – cos 5x)/(sin 17x – sin 3x) = –sin 2x/cos 10x.

SOLUTION

Apply sum-to-product formulas:
cos 9x – cos 5x = –2 sin((9x+5x)/2) sin((9x–5x)/2) = –2 sin 7x sin 2x.
sin 17x – sin 3x = 2 cos((17x+3x)/2) sin((17x–3x)/2) = 2 cos 10x sin 7x.

LHS = (–2 sin 7x sin 2x)/(2 cos 10x sin 7x) = –sin 2x/cos 10x = RHS.

Answer: Proved. LHS = –sin 2x/cos 10x = RHS.

Q17 Prove that: (sin 5x + sin 3x)/(cos 5x + cos 3x) = tan 4x.

SOLUTION

Apply sum-to-product formulas:
sin 5x + sin 3x = 2 sin 4x cos x.
cos 5x + cos 3x = 2 cos 4x cos x.

LHS = (2 sin 4x cos x)/(2 cos 4x cos x) = sin 4x/cos 4x = tan 4x = RHS.

Answer: Proved. LHS = sin4x/cos4x = tan4x = RHS.

Q18 Prove that: $(\sin x - \sin y)/(\cos x + \cos y) = \tan((x - y)/2)$.

SOLUTION

Apply sum-to-product formulas:
$\sin x - \sin y = 2\cos((x+y)/2)\sin((x-y)/2)$.
$\cos x + \cos y = 2\cos((x+y)/2)\cos((x-y)/2)$.

LHS $= [2\cos((x+y)/2)\sin((x-y)/2)] / [2\cos((x+y)/2)\cos((x-y)/2)]$
$= \sin((x-y)/2)/\cos((x-y)/2) = \tan((x-y)/2) =$ RHS.

Answer: Proved. LHS $= \tan((x-y)/2) =$ RHS.

Q19 Prove that: $(\sin x + \sin 3x)/(\cos x + \cos 3x) = \tan 2x$.

SOLUTION

Apply sum-to-product formulas:
$\sin x + \sin 3x = 2\sin 2x \cos x$.
$\cos x + \cos 3x = 2\cos 2x \cos x$.

LHS $= (2\sin 2x \cos x)/(2\cos 2x \cos x) = \sin 2x/\cos 2x = \tan 2x =$ RHS.

Answer: Proved. LHS $= \tan 2x =$ RHS.

Q20 Prove that: $(\sin x - \sin 3x)/(\sin^2 x - \cos^2 x) = 2\sin x$.

SOLUTION

Numerator: $\sin x - \sin 3x = 2\cos((x+3x)/2)\sin((x-3x)/2) = 2\cos 2x \cdot \sin(-x) = -2\cos 2x \sin x$.

Denominator: $\sin^2 x - \cos^2 x = -(\cos^2 x - \sin^2 x) = -\cos 2x$.

LHS $= (-2\cos 2x \sin x)/(-\cos 2x) = 2\sin x =$ RHS.

Answer: Proved. LHS $= 2\sin x =$ RHS.

Q21 Prove that: $(\cos 4x + \cos 3x + \cos 2x)/(\sin 4x + \sin 3x + \sin 2x) = \cot 3x$.

SOLUTION

Group the outer terms using sum-to-product:
$\cos 4x + \cos 2x = 2\cos 3x \cos x$.
$\sin 4x + \sin 2x = 2\sin 3x \cos x$.

Numerator $= 2\cos 3x \cos x + \cos 3x = \cos 3x(2\cos x + 1)$.

Denominator = 2 sin 3x cos x + sin 3x = sin 3x(2 cos x + 1).

LHS = cos 3x(2 cos x + 1)/[sin 3x(2 cos x + 1)] = cos 3x/sin 3x = cot 3x = RHS.

Answer: Proved. LHS = cos3x/sin3x = cot3x = RHS.

Q22 Prove that: cot x cot 2x – cot 2x cot 3x – cot 3x cot x = 1.

SOLUTION

Since 3x = 2x + x, the tan addition formula gives:
tan 3x = (tan 2x + tan x)/(1 – tan 2x tan x).

Cross-multiplying:
tan 3x (1 – tan 2x tan x) = tan 2x + tan x.
=> tan 3x – tan 3x tan 2x tan x = tan 2x + tan x.
=> tan 3x – tan 2x – tan x = tan 3x tan 2x tan x ... (*).

Multiply both sides of (*) by cot x cot 2x cot 3x:
LHS of (*) × cot x cot 2x cot 3x = cot x cot 2x – cot x cot 3x – cot 2x cot 3x.
RHS of (*) × cot x cot 2x cot 3x = 1.

So: cot x cot 2x – cot 2x cot 3x – cot 3x cot x = 1 = RHS.

Answer: Proved. Using tan 3x = tan(2x+x) leads to cot x cot 2x – cot 2x cot 3x – cot 3x cot x = 1.

Q23 Prove that: tan 4x = 4tan x(1 – tan²x)/(1 – 6tan²x + tan⁴x).

SOLUTION

Write tan 4x = tan(2 · 2x) = 2 tan 2x/(1 – tan²2x).

We know tan 2x = 2 tan x/(1 – tan²x).
So tan²2x = 4tan²x/(1 – tan²x)².

2 tan 2x = 4 tan x/(1 – tan²x).

1 – tan²2x = 1 – 4tan²x/(1 – tan²x)²
= [(1 – tan²x)² – 4tan²x]/(1 – tan²x)²
= [1 – 2tan²x + tan⁴x – 4tan²x]/(1 – tan²x)²
= (1 – 6tan²x + tan⁴x)/(1 – tan²x)².

tan 4x = [4 tan x/(1 – tan²x)] / [(1 – 6tan²x + tan⁴x)/(1 – tan²x)²]
= [4 tan x/(1 – tan²x)] × [(1 – tan²x)²/(1 – 6tan²x + tan⁴x)]

$= 4\tan x(1 - \tan^2 x)/(1 - 6\tan^2 x + \tan^4 x) = \text{RHS}.$

Answer: Proved. $\tan 4x = 4\tan x(1 - \tan^2 x)/(1 - 6\tan^2 x + \tan^4 x)$.

Q24 Prove that: $\cos 4x = 1 - 8\sin^2 x \cos^2 x$.

SOLUTION

$\cos 4x = \cos(2 \cdot 2x) = 1 - 2\sin^2 2x.$

$\sin 2x = 2 \sin x \cos x$, so $\sin^2 2x = 4\sin^2 x \cos^2 x.$

$\cos 4x = 1 - 2(4\sin^2 x \cos^2 x) = 1 - 8\sin^2 x \cos^2 x = \text{RHS}.$

Answer: Proved. $\cos 4x = 1 - 8\sin^2 x \cos^2 x$.

Q25 Prove that: $\cos 6x = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$.

SOLUTION

$\cos 6x = \cos(2 \cdot 3x) = 2\cos^2 3x - 1.$

$\cos 3x = 4\cos^3 x - 3\cos x$, so $\cos^2 3x = (4\cos^3 x - 3\cos x)^2$
$= 16\cos^6 x - 24\cos^4 x + 9\cos^2 x.$

$\cos 6x = 2(16\cos^6 x - 24\cos^4 x + 9\cos^2 x) - 1$
$= 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1 = \text{RHS}.$

Answer: Proved. $\cos 6x = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1$.

Misc — Miscellaneous Exercise

Q1 Prove that: $2\cos(\pi/13)\cos(9\pi/13) + \cos(3\pi/13) + \cos(5\pi/13) = 0$.

SOLUTION

Apply the product-to-sum formula: $2\cos A \cos B = \cos(A+B) + \cos(A–B)$.

$2\cos(\pi/13)\cos(9\pi/13) = \cos(\pi/13 + 9\pi/13) + \cos(9\pi/13 – \pi/13)$
$= \cos(10\pi/13) + \cos(8\pi/13)$.

LHS $= \cos(10\pi/13) + \cos(8\pi/13) + \cos(3\pi/13) + \cos(5\pi/13)$.

Note that $\cos(10\pi/13) = \cos(\pi – 3\pi/13) = –\cos(3\pi/13)$
and $\cos(8\pi/13) = \cos(\pi – 5\pi/13) = –\cos(5\pi/13)$.

LHS $= –\cos(3\pi/13) – \cos(5\pi/13) + \cos(3\pi/13) + \cos(5\pi/13) = 0 =$ RHS.

Answer: Proved. The cosine terms cancel pairwise to give 0.

Q2 Prove that: $(\sin 3x + \sin x)\sin x + (\cos 3x – \cos x)\cos x = 0$.

SOLUTION

Expand directly:
LHS $= \sin 3x \sin x + \sin^2 x + \cos 3x \cos x – \cos^2 x$.

Group: $(\cos 3x \cos x + \sin 3x \sin x) + (\sin^2 x – \cos^2 x)$
$= \cos(3x – x) – \cos 2x$
$= \cos 2x – \cos 2x = 0 =$ RHS.

[Used: $\cos A \cos B + \sin A \sin B = \cos(A – B)$ and $\sin^2 x – \cos^2 x = –\cos 2x$.]

Answer: Proved. LHS $= \cos 2x – \cos 2x = 0 =$ RHS.

Q3 Prove that: $(\cos x + \cos y)^2 + (\sin x – \sin y)^2 = 4\cos^2((x + y)/2)$.

SOLUTION

Expand the left side:
$= \cos^2 x + 2\cos x \cos y + \cos^2 y + \sin^2 x – 2\sin x \sin y + \sin^2 y$
$= (\cos^2 x + \sin^2 x) + (\cos^2 y + \sin^2 y) + 2\cos x \cos y – 2\sin x \sin y$
$= 1 + 1 + 2(\cos x \cos y – \sin x \sin y)$
$= 2 + 2\cos(x + y)$
$= 2[1 + \cos(x + y)]$
$= 2 \cdot 2\cos^2((x + y)/2)$ [since $1 + \cos\theta = 2\cos^2(\theta/2)$]

= $4\cos^2((x + y)/2)$ = RHS.

Answer: Proved. LHS = $2 + 2\cos(x+y) = 4\cos^2((x+y)/2)$ = RHS.

Q4 Prove that: $(\cos x – \cos y)^2 + (\sin x – \sin y)^2 = 4\sin^2((x – y)/2)$.

SOLUTION

Expand the left side:

$= \cos^2x – 2\cos x \cos y + \cos^2y + \sin^2x – 2\sin x \sin y + \sin^2y$

$= (\cos^2x + \sin^2x) + (\cos^2y + \sin^2y) – 2(\cos x \cos y + \sin x \sin y)$

$= 1 + 1 – 2\cos(x – y)$

$= 2 – 2\cos(x – y)$

$= 2[1 – \cos(x – y)]$

$= 2 \cdot 2\sin^2((x – y)/2)$ [since $1 – \cos\theta = 2\sin^2(\theta/2)$]

$= 4\sin^2((x – y)/2)$ = RHS.

Answer: Proved. LHS = $2 – 2\cos(x–y) = 4\sin^2((x–y)/2)$ = RHS.

Q5 Prove that: $\sin x + \sin 3x + \sin 5x + \sin 7x = 4\cos x \cos 2x \sin 4x$.

SOLUTION

Group: $(\sin 7x + \sin x) + (\sin 5x + \sin 3x)$.

$\sin 7x + \sin x = 2\sin 4x \cos 3x$.

$\sin 5x + \sin 3x = 2\sin 4x \cos x$.

LHS = $2\sin 4x \cos 3x + 2\sin 4x \cos x$

$= 2\sin 4x (\cos 3x + \cos x)$

$= 2\sin 4x \cdot 2\cos 2x \cos x$

$= 4 \cos x \cos 2x \sin 4x$ = RHS.

Answer: Proved. LHS = $4\cos x \cos 2x \sin 4x$ = RHS.

Q6 Prove that: $[(\sin 7x + \sin 5x) + (\sin 9x + \sin 3x)] / [(\cos 7x + \cos 5x) + (\cos 9x + \cos 3x)] = \tan 6x$.

SOLUTION

Apply sum-to-product to each pair:

$\sin 7x + \sin 5x = 2 \sin 6x \cos x$.

$\sin 9x + \sin 3x = 2 \sin 6x \cos 3x$.

cos 7x + cos 5x = 2 cos 6x cos x.
cos 9x + cos 3x = 2 cos 6x cos 3x.

Numerator = 2 sin 6x cos x + 2 sin 6x cos 3x = 2 sin 6x (cos x + cos 3x).
Denominator = 2 cos 6x cos x + 2 cos 6x cos 3x = 2 cos 6x (cos x + cos 3x).

LHS = 2 sin 6x (cos x + cos 3x) / [2 cos 6x (cos x + cos 3x)] = sin 6x/cos 6x = tan 6x = RHS.

Answer: Proved. LHS = sin 6x/cos 6x = tan 6x = RHS.

Q7 Prove that: sin 3x + sin 2x – sin x = 4 sin x cos(x/2) cos(3x/2).

SOLUTION

Group: (sin 3x – sin x) + sin 2x.

sin 3x – sin x = 2 cos((3x+x)/2) sin((3x–x)/2) = 2 cos 2x sin x.

LHS = 2 cos 2x sin x + sin 2x
= 2 cos 2x sin x + 2 sin x cos x
= 2 sin x (cos 2x + cos x).

cos 2x + cos x = 2 cos((2x+x)/2) cos((2x–x)/2) = 2 cos(3x/2) cos(x/2).

LHS = 2 sin x · 2 cos(3x/2) cos(x/2) = 4 sin x cos(x/2) cos(3x/2) = RHS.

Answer: Proved. LHS = 4 sin x cos(x/2) cos(3x/2) = RHS.

Q8 Find sin(x/2), cos(x/2) and tan(x/2) if tan x = –4/3 and x is in quadrant II.

SOLUTION

Given: tan x = –4/3, x in quadrant II.

sec²x = 1 + tan²x = 1 + 16/9 = 25/9, so sec x = ±5/3.
Since x is in quadrant II, cos x < 0: cos x = –3/5.

Since x ∈ (π/2, π), we have x/2 ∈ (π/4, π/2), so x/2 is in quadrant I.
Therefore sin(x/2) > 0 and cos(x/2) > 0.

sin(x/2) = √((1 – cos x)/2) = √((1 – (–3/5))/2) = √(8/10) = √(4/5) = 2/√5 = 2√5/5.

cos(x/2) = √((1 + cos x)/2) = √((1 + (–3/5))/2) = √(2/10) = √(1/5) = 1/√5 = √5/5.

tan(x/2) = sin(x/2)/cos(x/2) = (2/√5)/(1/√5) = 2.

Answer: sin(x/2) = 2√5/5, cos(x/2) = √5/5, tan(x/2) = 2

Q9 Find sin(x/2), cos(x/2) and tan(x/2) if cos x = –1/3 and x is in quadrant III.

SOLUTION

Given: cos x = –1/3, x in quadrant III.

Since $x \in (\pi, 3\pi/2)$, we have $x/2 \in (\pi/2, 3\pi/4)$, so x/2 is in quadrant II.
Therefore sin(x/2) > 0 and cos(x/2) < 0.

$\sin(x/2) = \sqrt{((1 - \cos x)/2)} = \sqrt{((1 - (-1/3))/2)} = \sqrt{(4/6)} = \sqrt{(2/3)} = \sqrt{6}/3$.

$\cos(x/2) = -\sqrt{((1 + \cos x)/2)} = -\sqrt{((1 + (-1/3))/2)} = -\sqrt{(2/6)} = -\sqrt{(1/3)} = -\sqrt{3}/3$.

$\tan(x/2) = \sin(x/2)/\cos(x/2) = (\sqrt{6}/3)/(-\sqrt{3}/3) = -\sqrt{6}/\sqrt{3} = -\sqrt{2}$.

Answer: $\sin(x/2) = \sqrt{6}/3$, $\cos(x/2) = -\sqrt{3}/3$, $\tan(x/2) = -\sqrt{2}$

Q10 Find sin(x/2), cos(x/2) and tan(x/2) if sin x = 1/4 and x is in quadrant II.

SOLUTION

Given: sin x = 1/4, x in quadrant II.

$\cos^2 x = 1 - \sin^2 x = 1 - 1/16 = 15/16$.
Since x is in quadrant II, cos x < 0: $\cos x = -\sqrt{15}/4$.

Since $x \in (\pi/2, \pi)$, we have $x/2 \in (\pi/4, \pi/2)$, so x/2 is in quadrant I.
Therefore sin(x/2) > 0 and cos(x/2) > 0.

$\sin(x/2) = \sqrt{((1 - \cos x)/2)} = \sqrt{((1 + \sqrt{15}/4)/2)} = \sqrt{((4 + \sqrt{15})/8)}$.
This simplifies to $(\sqrt{5} + \sqrt{3})/4$.

$\cos(x/2) = \sqrt{((1 + \cos x)/2)} = \sqrt{((1 - \sqrt{15}/4)/2)} = \sqrt{((4 - \sqrt{15})/8)}$.
This simplifies to $(\sqrt{5} - \sqrt{3})/4$.

$\tan(x/2) = \sin(x/2)/\cos(x/2) = \sqrt{((4 + \sqrt{15})/8)} / \sqrt{((4 - \sqrt{15})/8)} = \sqrt{((4 + \sqrt{15})/(4 - \sqrt{15}))}$.
Rationalise: multiply by $(4 + \sqrt{15})/(4 + \sqrt{15})$: $\tan(x/2) = \sqrt{((4+\sqrt{15})^2/(16-15))} = 4 + \sqrt{15}$.

Answer: $\sin(x/2) = (\sqrt{5}+\sqrt{3})/4$, $\cos(x/2) = (\sqrt{5}-\sqrt{3})/4$, $\tan(x/2) = 4+\sqrt{15}$