Chapter 4: Complex Numbers and Quadratic Equations

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Exercise 4.1 — Exercise 4.1

Q1 Express the complex number (5i)(−3/5 i) in the form a + ib.

SOLUTION

We have (5i)(−3/5 i) = −3 × i² = −3 × (−1) = 3.
So the result is 3 + 0i.

Answer: 3 + 0i

Q2 Express $i^9 + i^{19}$ in the form a + ib.

SOLUTION

We use the pattern: for any integer k, i^(4k) = 1, i^(4k+1) = i, i^(4k+2) = −1, i^(4k+3) = −i.

$i^9 = i^{(4\times2 + 1)} = i$
$i^{19} = i^{(4\times4 + 3)} = -i$

Therefore $i^9 + i^{19} = i + (-i) = 0 = 0 + 0i$.

Answer: 0 + 0i

Q3 Express i^(−39) in the form a + ib.

SOLUTION

We write −39 = 4×(−10) + 1, so −39 ≡ 1 (mod 4).
Therefore $i^{(-39)} = i^1 = i = 0 + 1i$.

Alternatively: $i^{(-39)} = 1/i^{39} = 1/i^{(4\times9+3)} = 1/(-i) = 1/(-i) \times (i/i) = i/(-i^2) = i/1 = i$.

Answer: 0 + i

Q4 Express 3(7 + i7) + i(7 + i7) in the form a + ib.

SOLUTION

We factor out (7 + i7):
3(7 + i7) + i(7 + i7) = (3 + i)(7 + i7) = (3 + i) × 7(1 + i) = 7(3 + i)(1 + i)

Expanding: (3 + i)(1 + i) = 3 + 3i + i + i² = 3 + 4i − 1 = 2 + 4i

Therefore 7(2 + 4i) = 14 + 28i.

Alternatively: 3(7 + 7i) + i(7 + 7i) = 21 + 21i + 7i + 7i² = 21 + 28i − 7 = 14 + 28i.

Answer: 14 + 28i

Q5 Express (1 − i) − (−1 + i6) in the form a + ib.

SOLUTION

(1 − i) − (−1 + 6i) = 1 − i + 1 − 6i = 2 − 7i.

Answer: 2 − 7i

Q6 Express (1/5 + i 2/5)(−4 + i 5/2) in the form a + ib.

SOLUTION

Expand using (a + ib)(c + id) = (ac − bd) + i(ad + bc):
a = 1/5, b = 2/5, c = −4, d = 5/2

Real part: ac − bd = (1/5)(−4) − (2/5)(5/2) = −4/5 − 1 = −4/5 − 5/5 = −9/5
Imaginary part: ad + bc = (1/5)(5/2) + (2/5)(−4) = 1/2 − 8/5 = 5/10 − 16/10 = −11/10

Therefore the result is −9/5 − 11i/10.

Answer: −9/5 − 11i/10

Q7 Express {(1/3 + i√3) + (4 + i/3)} − (−4/3 + i) in the form a + ib.

SOLUTION

Subtracting (−4/3 + i) is the same as adding 4/3 and subtracting i.
Group the real and imaginary parts separately.

Real part: 1/3 + 4 + 4/3 = 1/3 + 4/3 + 4 = 5/3 + 12/3 = 17/3.
Imaginary part: √3 + 1/3 − 1 = √3 − 2/3.

Therefore the result is $17/3 + (\sqrt{3} - 2/3)i$.

Answer: $17/3 + (\sqrt{3} - 2/3)i$

Q8 Express $(1 - i)^4$ in the form $a + ib$.

SOLUTION

First compute $(1 - i)^2 = 1 - 2i + i^2 = 1 - 2i - 1 = -2i$.

Then $(1 - i)^4 = [(1 - i)^2]^2 = (-2i)^2 = 4i^2 = 4(-1) = -4$.

Therefore $(1 - i)^4 = -4 + 0i$.

Answer: $-4 + 0i$

Q9 Express $(1/3 + 3i)^3$ in the form $a + ib$.

SOLUTION

Using $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ with $a = 1/3$ and $b = 3i$:

$a^3 = 1/27$
$3a^2b = 3 \times (1/9) \times 3i = i$
$3ab^2 = 3 \times (1/3) \times (3i)^2 = 3 \times (1/3) \times (-9) = -9$
$(3i)^3 = 27i^3 = 27(-i) = -27i$

Real part: $1/27 + (-9) = 1/27 - 243/27 = -242/27$
Imaginary part: $1 + (-27) = -26$

Therefore $(1/3 + 3i)^3 = -242/27 - 26i$.

Answer: $-242/27 - 26i$

Q10 Express $(-2 - i/3)^3$ in the form $a + ib$.

SOLUTION

Using $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ with $a = -2$ and $b = -i/3$:

$a^3 = (-2)^3 = -8$
$3a^2b = 3 \times 4 \times (-i/3) = -4i$
$(-i/3)^2 = (-1)^2(i^2)/9 = (1)(-1)/9 = -1/9$
$3ab^2 = 3 \times (-2) \times (-1/9) = 6/9 = 2/3$
$b^3 = (-i/3)^3 = (-1)^3 \times i^3 / 27 = (-1)(-i)/27 = i/27$

Real part: $-8 + 2/3 = -24/3 + 2/3 = -22/3$

Imaginary part: $-4 + 1/27 = -108/27 + 1/27 = -107/27$

Therefore $(-2 - i/3)^3 = -22/3 - 107i/27$.

Answer: $-22/3 - 107i/27$

Q11 Find the multiplicative inverse of $4 - 3i$.

SOLUTION

Let $z = 4 - 3i$.

Conjugate: $\bar{z} = 4 + 3i$.

$|z|^2 = 4^2 + (-3)^2 = 16 + 9 = 25$.

Multiplicative inverse: $z^{-1} = \bar{z}/|z|^2 = (4 + 3i)/25 = 4/25 + 3i/25$.

Answer: $4/25 + 3i/25$

Q12 Find the multiplicative inverse of $\sqrt{5} + 3i$.

SOLUTION

Let $z = \sqrt{5} + 3i$.

Conjugate: $\bar{z} = \sqrt{5} - 3i$.

$|z|^2 = (\sqrt{5})^2 + 3^2 = 5 + 9 = 14$.

Multiplicative inverse: $z^{-1} = \bar{z}/|z|^2 = (\sqrt{5} - 3i)/14 = \sqrt{5}/14 - 3i/14$.

Answer: $\sqrt{5}/14 - 3i/14$

Q13 Find the multiplicative inverse of $-i$.

SOLUTION

Let $z = -i = 0 - i$.

Conjugate: $\bar{z} = 0 + i = i$.

$|z|^2 = 0^2 + (-1)^2 = 1$.

Multiplicative inverse: $z^{-1} = \bar{z}/|z|^2 = i/1 = i$.

Verification: $(-i)(i) = -i^2 = -(-1) = 1$. ✓

Answer: i

Q14 Express the following expression in the form a + ib: $(3 + i\sqrt{5})(3 - i\sqrt{5}) / [(\sqrt{3} + \sqrt{2}\, i) - (\sqrt{3} - i\sqrt{2})]$

SOLUTION

Step 1: Compute the numerator.
$(3 + i\sqrt{5})(3 - i\sqrt{5}) = 3^2 - (i\sqrt{5})^2 = 9 - i^2 \times 5 = 9 - (-5) = 14.$

Step 2: Compute the denominator.
$(\sqrt{3} + \sqrt{2}\, i) - (\sqrt{3} - \sqrt{2}\, i) = \sqrt{3} + \sqrt{2}\, i - \sqrt{3} + \sqrt{2}\, i = 2\sqrt{2}\, i.$

Step 3: Divide.
$14 / (2\sqrt{2}\, i) = 7/(\sqrt{2}\, i).$

Rationalize by multiplying numerator and denominator by $(-i)$:
$7/(\sqrt{2}\, i) \times (-i)/(-i) = -7i/(\sqrt{2} \times (-i^2)) = -7i/(\sqrt{2} \times 1) = -7i/\sqrt{2}.$

Multiply numerator and denominator by $\sqrt{2}$:
$-7\sqrt{2}\, i / 2.$

Therefore the result is $0 - 7\sqrt{2}\, i/2.$

Answer: $0 - 7\sqrt{2}\, i/2$

Misc — Miscellaneous Exercise

Q1 Evaluate: $[i^{18} + (1/i)^{25}]^3$.

SOLUTION

First, simplify i^{18} and $(1/i)^{25}$.

$i^{18} = i^{(4\times4 + 2)} = i^2 = -1$.

$(1/i)^{25} = i^{(-25)}$. Since $-25 = 4\times(-7) + 3$, we get $i^{(-25)} = i^3 = -i$.
Alternatively: $(1/i)^{25} = (1/i)^{25}$. Note $1/i = -i$, so $(-i)^{25} = (-1)^{25} \times i^{25} = (-1)(i) = -i$.

So $i^{18} + (1/i)^{25} = -1 + (-i) = -1 - i$.

Now compute $(-1 - i)^3$ using $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ with $a = -1$, $b = -i$:
$a^3 = -1$
$3a^2b = 3(1)(-i) = -3i$
$3ab^2 = 3(-1)(i^2) = 3(-1)(-1) = 3$
$b^3 = (-i)^3 = -(i^3) = -(-i) = i$

Sum: $(-1 + 3) + (-3i + i) = 2 - 2i$.

Therefore $[i^{18} + (1/i)^{25}]^3 = 2 - 2i$.

Answer: $2 - 2i$

Q2 For any two complex numbers z_1 and z_2, prove that $\mathrm{Re}(z_1 z_2) = \mathrm{Re}\, z_1\, \mathrm{Re}\, z_2 - \mathrm{Im}\, z_1\, \mathrm{Im}\, z_2$.

SOLUTION

Let $z_1 = a + ib$ and $z_2 = c + id$, where a, b, c, d are real numbers.

Then $z_1 z_2 = (a + ib)(c + id) = (ac - bd) + i(ad + bc)$.

Therefore $\mathrm{Re}(z_1 z_2) = ac - bd$.

Also, $\mathrm{Re}\, z_1 = a$, $\mathrm{Re}\, z_2 = c$, $\mathrm{Im}\, z_1 = b$, $\mathrm{Im}\, z_2 = d$.

So $\mathrm{Re}\, z_1\, \mathrm{Re}\, z_2 - \mathrm{Im}\, z_1\, \mathrm{Im}\, z_2 = ac - bd$.

Hence $\mathrm{Re}(z_1 z_2) = \mathrm{Re}\, z_1\, \mathrm{Re}\, z_2 - \mathrm{Im}\, z_1\, \mathrm{Im}\, z_2$. (Proved)

Answer: Proved: $\mathrm{Re}(z_1 z_2) = ac - bd = \mathrm{Re}\, z_1\, \mathrm{Re}\, z_2 - \mathrm{Im}\, z_1\, \mathrm{Im}\, z_2$.

Q3 Reduce $[1/(1 - 4i) - 2/(1 + i)] \times [(3 - 4i)/(5 + i)]$ to the standard form.

SOLUTION

Step 1: Compute $1/(1 - 4i)$.
$1/(1 - 4i) = (1 + 4i)/[(1 - 4i)(1 + 4i)] = (1 + 4i)/(1 + 16) = (1 + 4i)/17$.

Step 2: Compute $2/(1 + i)$.
$2/(1 + i) = 2(1 - i)/[(1 + i)(1 - i)] = 2(1 - i)/2 = 1 - i$.

Step 3: Subtract.
$1/(1 - 4i) - 2/(1 + i) = (1 + 4i)/17 - (1 - i)$
$= (1 + 4i)/17 - 17(1 - i)/17$
$= [(1 + 4i) - (17 - 17i)]/17$
$= (1 + 4i - 17 + 17i)/17$
$= (-16 + 21i)/17$.

Step 4: Compute $(3 - 4i)/(5 + i)$.
$(3 - 4i)/(5 + i) = (3 - 4i)(5 - i)/[(5 + i)(5 - i)] = (15 - 3i - 20i + 4i^2)/26$
$= (15 - 23i - 4)/26 = (11 - 23i)/26$.

Step 5: Multiply the two results.
$(-16 + 21i)/17 \times (11 - 23i)/26$
$= (-16 + 21i)(11 - 23i) / 442$.

Expand the numerator:
$(-16)(11) = -176$
$(-16)(-23i) = 368i$
$(21i)(11) = 231i$
$(21i)(-23i) = -483i^2 = 483$

Numerator $= (-176 + 483) + (368 + 231)i = 307 + 599i$.

Therefore the standard form is $307/442 + 599i/442$.

Answer: $307/442 + 599i/442$

Q4 If $x - iy = \sqrt{[(a - ib)/(c - id)]}$, prove that $(x^2 + y^2)^2 = (a^2 + b^2)/(c^2 + d^2)$.

SOLUTION

Given: $x - iy = \sqrt{[(a - ib)/(c - id)]}$.

Step 1: Square both sides.
$(x - iy)^2 = (a - ib)/(c - id)$.

Step 2: Take modulus of both sides.
$|(x - iy)^2| = |(a - ib)/(c - id)|$

Using	z^2	=	z	² on the left and	z_1/z_2	=	z_1	/	z_2	on the right:
	x − iy	² =	a − ib	/	c − id					

$(x^2 + y^2) = \sqrt{(a^2 + b^2)}/\sqrt{(c^2 + d^2)}$.

Step 3: Square both sides again.
$(x^2 + y^2)^2 = (a^2 + b^2)/(c^2 + d^2)$. (Proved)

Answer: Proved: $(x^2 + y^2)^2 = (a^2 + b^2)/(c^2 + d^2)$.

Q5 If $z_1 = 2 - i$, $z_2 = 1 + i$, find $|(z_1 + z_2 + 1)/(z_1 - z_2 + 1)|$.

SOLUTION

Compute $z_1 + z_2 + 1$:
$(2 - i) + (1 + i) + 1 = 4 + 0i = 4$.

Compute $z_1 - z_2 + 1$:
$(2 - i) - (1 + i) + 1 = (2 - i - 1 - i + 1) = 2 - 2i$.

	$z_1 + z_2 + 1$	=	4	= 4.
	$z_1 - z_2 + 1$	=	2 − 2i	$= \sqrt{(4 + 4)} = \sqrt{8} = 2\sqrt{2}$.

Therefore $|(z_1 + z_2 + 1)/(z_1 - z_2 + 1)| = 4/(2\sqrt{2}) = 2/\sqrt{2} = \sqrt{2}$.

Answer: $\sqrt{2}$

Q6 If $a + ib = (x + i)^2/(2x^2 + 1)$, prove that $a^2 + b^2 = (x^2 + 1)^2/(2x^2 + 1)^2$.

SOLUTION

We have $a + ib = (x + i)^2/(2x^2 + 1)$.

Expand $(x + i)^2$: $x^2 + 2xi + i^2 = x^2 - 1 + 2xi$.

So $a + ib = (x^2 - 1)/(2x^2 + 1) + i \times 2x/(2x^2 + 1)$.

Hence $a = (x^2 - 1)/(2x^2 + 1)$ and $b = 2x/(2x^2 + 1)$.

Now compute $a^2 + b^2$:

$a^2 + b^2 = [(x^2 - 1)^2 + 4x^2]/(2x^2 + 1)^2$

$= [x^4 - 2x^2 + 1 + 4x^2]/(2x^2 + 1)^2$

$= [x^4 + 2x^2 + 1]/(2x^2 + 1)^2$

$= (x^2 + 1)^2/(2x^2 + 1)^2$. (Proved)

Answer: Proved: $a^2 + b^2 = (x^2 + 1)^2/(2x^2 + 1)^2$.

Q7 Let $z_1 = 2 - i$, $z_2 = -2 + i$. Find (i) $\text{Re}(z_1 z_2 / \bar{z}_1)$, (ii) $\text{Im}(1/(z_1 \bar{z}_1))$.

SOLUTION

(i) Compute $z_1 z_2$:

$z_1 z_2 = (2 - i)(-2 + i) = -4 + 2i + 2i - i^2 = -4 + 4i - (-1) = -3 + 4i$.

Compute $\bar{z}_1$ (conjugate of z_1):

$\bar{z}_1 = 2 + i$.

Compute $(-3 + 4i)/(2 + i)$: multiply numerator and denominator by $(2 - i)$:

$(-3 + 4i)(2 - i)/[(2 + i)(2 - i)] = (-6 + 3i + 8i - 4i^2)/5$

$= (-6 + 11i + 4)/5 = (-2 + 11i)/5$.

Therefore $\text{Re}(z_1 z_2 / \bar{z}_1) = -2/5$.

(ii) Compute $z_1 \bar{z}_1$:

$z_1 \bar{z}_1 = |z_1|^2 = 2^2 + (-1)^2 = 4 + 1 = 5$ (a positive real number).

Therefore $1/(z_1 \bar{z}_1) = 1/5$, which is real.

$\text{Im}(1/(z_1 \bar{z}_1)) = 0$.

Answer: (i) $\text{Re}(z_1 z_2/\bar{z}_1) = -2/5$; (ii) $\text{Im}(1/(z_1 \bar{z}_1)) = 0$.

Q8 Find the real numbers x and y if $(x - iy)(3 + 5i)$ is the conjugate of $-6 - 24i$.

SOLUTION

The conjugate of $-6 - 24i$ is $-6 + 24i$.

So $(x - iy)(3 + 5i) = -6 + 24i$.

Expand the left side:

$(x - iy)(3 + 5i) = 3x + 5xi - 3yi - 5yi^2 = 3x + 5y + (5x - 3y)i$.

Equating real and imaginary parts:

$3x + 5y = -6$ ... (1)

$5x - 3y = 24$ ... (2)

Multiply (1) by 3 and (2) by 5:
$9x + 15y = -18$ ... (3)
$25x - 15y = 120$ ... (4)

Adding (3) and (4): $34x = 102 \Rightarrow x = 3$.
Substituting in (1): $9 + 5y = -6 \Rightarrow 5y = -15 \Rightarrow y = -3$.

Therefore $x = 3$, $y = -3$.

Answer: $x = 3, y = -3$

Q9 Find the modulus of $(1 + i)/(1 - i) - (1 - i)/(1 + i)$.

SOLUTION

First simplify $(1 + i)/(1 - i)$:
$(1 + i)/(1 - i) = (1 + i)^2/[(1 - i)(1 + i)] = (1 + 2i + i^2)/2 = (1 + 2i - 1)/2 = 2i/2 = i$.

Next simplify $(1 - i)/(1 + i)$:
$(1 - i)/(1 + i) = (1 - i)^2/[(1 + i)(1 - i)] = (1 - 2i + i^2)/2 = (1 - 2i - 1)/2 = -2i/2 = -i$.

Therefore $(1 + i)/(1 - i) - (1 - i)/(1 + i) = i - (-i) = 2i$.

Modulus $= |2i| = 2$.

Answer: 2

Q10 If $(x + iy)^3 = u + iv$, then show that $u/x + v/y = 4(x^2 - y^2)$.

SOLUTION

Expand $(x + iy)^3$ using the binomial theorem:
$(x + iy)^3 = x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3$
$= x^3 + 3x^2yi + 3x(i^2y^2) + i^3 y^3$
$= x^3 + 3x^2yi - 3xy^2 - iy^3$
$= (x^3 - 3xy^2) + i(3x^2y - y^3)$.

So $u = x^3 - 3xy^2$ and $v = 3x^2y - y^3$.

Therefore:
$u/x = (x^3 - 3xy^2)/x = x^2 - 3y^2$
$v/y = (3x^2y - y^3)/y = 3x^2 - y^2$

$u/x + v/y = (x^2 - 3y^2) + (3x^2 - y^2) = 4x^2 - 4y^2 = 4(x^2 - y^2)$. (Proved)

Answer: Proved: u/x + v/y = 4(x² − y²).

Q11 If α and β are different complex numbers with |β| = 1, then find |(β − α)/(1 − ᾱβ)|.

SOLUTION

Let w = (β − α)/(1 − ᾱβ). We compute |w|.

|w|² = |β − α|²/|1 − ᾱβ|².

Numerator:	β − α	² = (β − α)(β − ᾱ) =	β	² − βᾱ − ᾱβ +	α	²
= 1 − βᾱ − ᾱβ +	α	² (since	β	= 1).		

Denominator:	1 − ᾱβ	² = (1 − ᾱβ)(1 − αβ)		
= 1 − αβ − ᾱβ +	α	²	β	²
= 1 − αβ − ᾱβ +	α	² (since	β	² = 1).

Note that βᾱ = conjugate of (ᾱβ) = αβ, so the expressions in the numerator and denominator are identical.

Therefore |w|² = 1, which gives |w| = 1.

Hence |(β − α)/(1 − ᾱβ)| = 1.

Answer: 1

Q12 Find the number of non-zero integral solutions of the equation |1 − i|^x = 2^x.

SOLUTION

Compute	1 − i	:
	1 − i	= √(1² + (−1)²) = √2.

So the equation becomes: (√2)^x = 2^x.

2^(x/2) = 2^x.

Since the bases are equal: x/2 = x.
=> x − x/2 = 0 => x/2 = 0 => x = 0.

The only solution is x = 0, which is not a non-zero integer.

Therefore there are no non-zero integral solutions.

Number of non-zero integral solutions = 0.

Answer: 0

Q13 If (a + ib)(c + id)(e + if)(g + ih) = A + iB, then show that (a² + b²)(c² + d²)(e² + f²)(g² + h²) = A² + B².

SOLUTION

Take the modulus of both sides of (a + ib)(c + id)(e + if)(g + ih) = A + iB.

Using the property	$z_1 z_2$	=	z_1		z_2	, applied repeatedly:				
	a + ib	×	c + id	×	e + if	×	g + ih	=	A + iB	.

Now square both sides. Since	z	² = (Re z)² + (Im z)²:								
	a + ib	² ×	c + id	² ×	e + if	² ×	g + ih	² =	A + iB	².

(a² + b²)(c² + d²)(e² + f²)(g² + h²) = A² + B². (Proved)

Answer: Proved: (a² + b²)(c² + d²)(e² + f²)(g² + h²) = A² + B².

Q14 If [(1 + i)/(1 − i)]^m = 1, then find the least positive integral value of m.

SOLUTION

First simplify (1 + i)/(1 − i):

(1 + i)/(1 − i) = (1 + i)²/[(1 − i)(1 + i)] = (1 + 2i + i²)/2 = (1 + 2i − 1)/2 = 2i/2 = i.

So the equation becomes i^m = 1.

We know $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$.

In general i^m = 1 when m is a multiple of 4.

The least positive integral value of m is 4.

Answer: m = 4