Chapter 7: Binomial Theorem

CONTENTS — 2 exercises · 20 questions

7.1	Exercise 7.1	14 Q
Misc	Miscellaneous Exercise	6 Q

Exercise 7.1 — Exercise 7.1

Q1 Expand the expression: $(1 - 2x)^5$

SOLUTION

Using the Binomial Theorem: $(a + b)^n = \Sigma\, C(n,r)\, a^{(n-r)}\, b^r$

Here $a = 1$, $b = -2x$, $n = 5$.

$(1 - 2x)^5 = C(5,0)(1)^5 + C(5,1)(1)^4(-2x) + C(5,2)(1)^3(-2x)^2 + C(5,3)(1)^2(-2x)^3 + C(5,4)(1)(-2x)^4 + C(5,5)(-2x)^5$

$= 1 + 5(-2x) + 10(4x^2) + 10(-8x^3) + 5(16x^4) + (-32x^5)$

$= 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$

Answer: $(1 - 2x)^5 = 1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$

Q2 Expand the expression: $(2/x - x/2)^5$

SOLUTION

Here $a = 2/x$, $b = -x/2$, $n = 5$.

$(2/x - x/2)^5 = C(5,0)(2/x)^5 + C(5,1)(2/x)^4(-x/2) + C(5,2)(2/x)^3(-x/2)^2 + C(5,3)(2/x)^2(-x/2)^3 + C(5,4)(2/x)(-x/2)^4 + C(5,5)(-x/2)^5$

Term 1: $(2/x)^5 = 32/x^5$

Term 2: $5 \cdot (16/x^4) \cdot (-x/2) = 5 \cdot (-8/x^3) = -40/x^3$

Term 3: $10 \cdot (8/x^3) \cdot (x^2/4) = 10 \cdot (2/x) = 20/x$

Term 4: $10 \cdot (4/x^2) \cdot (-x^3/8) = 10 \cdot (-x/2) = -5x$

Term 5: $5 \cdot (2/x) \cdot (x^4/16) = 5 \cdot (x^3/8) = 5x^3/8$

Term 6: $(-x/2)^5 = -x^5/32$

$= 32/x^5 - 40/x^3 + 20/x - 5x + (5x^3)/8 - x^5/32$

Answer: $(2/x - x/2)^5 = 32/x^5 - 40/x^3 + 20/x - 5x + 5x^3/8 - x^5/32$

Q3 Expand the expression: $(2x - 3)^6$

SOLUTION

Here a = 2x, b = –3, n = 6.

$(2x - 3)^6 = C(6,0)(2x)^6 + C(6,1)(2x)^5(-3) + C(6,2)(2x)^4(-3)^2 + C(6,3)(2x)^3(-3)^3 + C(6,4)(2x)^2(-3)^4 + C(6,5)(2x)(-3)^5 + C(6,6)(-3)^6$

$= 64x^6 + 6(32x^5)(-3) + 15(16x^4)(9) + 20(8x^3)(-27) + 15(4x^2)(81) + 6(2x)(-243) + 729$

$= 64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729$

Answer: $(2x - 3)^6 = 64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729$

Q4 Expand the expression: $(x/3 + 1/x)^5$

SOLUTION

Here a = x/3, b = 1/x, n = 5.

$(x/3 + 1/x)^5 = C(5,0)(x/3)^5 + C(5,1)(x/3)^4(1/x) + C(5,2)(x/3)^3(1/x)^2 + C(5,3)(x/3)^2(1/x)^3 + C(5,4)(x/3)(1/x)^4 + C(5,5)(1/x)^5$

Term 1: $x^5/243$
Term 2: $5 \cdot (x^4/81) \cdot (1/x) = 5x^3/81$
Term 3: $10 \cdot (x^3/27) \cdot (1/x^2) = 10x/27$
Term 4: $10 \cdot (x^2/9) \cdot (1/x^3) = 10/(9x)$
Term 5: $5 \cdot (x/3) \cdot (1/x^4) = 5/(3x^3)$
Term 6: $1/x^5$

$= x^5/243 + 5x^3/81 + 10x/27 + 10/(9x) + 5/(3x^3) + 1/x^5$

Answer: $(x/3 + 1/x)^5 = x^5/243 + 5x^3/81 + 10x/27 + 10/(9x) + 5/(3x^3) + 1/x^5$

Q5 Expand the expression: $(x + 1/x)^6$

SOLUTION

Here a = x, b = 1/x, n = 6.

$(x + 1/x)^6 = C(6,0)x^6 + C(6,1)x^5(1/x) + C(6,2)x^4(1/x^2) + C(6,3)x^3(1/x^3) + C(6,4)x^2(1/x^4) + C(6,5)x(1/x^5) + C(6,6)(1/x^6)$

$= x^6 + 6x^4 + 15x^2 + 20 + 15/x^2 + 6/x^4 + 1/x^6$

Answer: $(x + 1/x)^6 = x^6 + 6x^4 + 15x^2 + 20 + 15/x^2 + 6/x^4 + 1/x^6$

Q6 Using binomial theorem, evaluate: $(96)^3$

SOLUTION

Write $96 = 100 - 4$.

$(96)^3 = (100 - 4)^3$
$= C(3,0)(100)^3 - C(3,1)(100)^2(4) + C(3,2)(100)(4)^2 - C(3,3)(4)^3$
$= 1000000 - 3(10000)(4) + 3(100)(16) - 64$
$= 1000000 - 120000 + 4800 - 64$
$= 884736$

Answer: $(96)^3 = 884736$

Q7 Using binomial theorem, evaluate: $(102)^5$

SOLUTION

Write $102 = 100 + 2$.

$(102)^5 = (100 + 2)^5$
$= C(5,0)(100)^5 + C(5,1)(100)^4(2) + C(5,2)(100)^3(2^2) + C(5,3)(100)^2(2^3) + C(5,4)(100)(2^4) + C(5,5)(2^5)$
$= 10000000000 + 5(100000000)(2) + 10(1000000)(4) + 10(10000)(8) + 5(100)(16) + 32$
$= 10000000000 + 1000000000 + 40000000 + 800000 + 8000 + 32$
$= 11040808032$

Answer: $(102)^5 = 11040808032$

Q8 Using binomial theorem, evaluate: $(101)^4$

SOLUTION

Write $101 = 100 + 1$.

$(101)^4 = (100 + 1)^4$
$= C(4,0)(100)^4 + C(4,1)(100)^3(1) + C(4,2)(100)^2(1)^2 + C(4,3)(100)(1)^3 + C(4,4)(1)^4$
$= 100000000 + 4(1000000) + 6(10000) + 4(100) + 1$
$= 100000000 + 4000000 + 60000 + 400 + 1$
$= 104060401$

Answer: $(101)^4 = 104060401$

Q9 Using binomial theorem, evaluate: $(99)^5$

SOLUTION

Write $99 = 100 - 1$.

$(99)^5 = (100 - 1)^5$
$= C(5,0)(100)^5 - C(5,1)(100)^4 + C(5,2)(100)^3 - C(5,3)(100)^2 + C(5,4)(100) - C(5,5)$
$= 10000000000 - 5(100000000) + 10(1000000) - 10(10000) + 5(100) - 1$
$= 10000000000 - 500000000 + 10000000 - 100000 + 500 - 1$
$= 9509900499$

Answer: $(99)^5 = 9509900499$

Q10 Using Binomial Theorem, indicate which number is larger: $(1.1)^{10000}$ or 1000.

SOLUTION

Write $1.1 = 1 + 0.1$.

$(1.1)^{10000} = (1 + 0.1)^{10000}$
$= C(10000,0) + C(10000,1)(0.1) + C(10000,2)(0.1)^2 + \ldots$ (all terms positive)

Taking only the first two terms:
$(1.1)^{10000} > 1 + 10000 \times 0.1 = 1 + 1000 = 1001 > 1000$

Since all remaining terms are positive, $(1.1)^{10000} > 1000$.

Answer: $(1.1)^{10000}$ is larger than 1000.

Q11 Find $(a + b)^4 - (a - b)^4$. Hence, evaluate $(\sqrt{3} + \sqrt{2})^4 - (\sqrt{3} - \sqrt{2})^4$.

SOLUTION

Using the Binomial Theorem:
$(a + b)^4 = C(4,0)a^4 + C(4,1)a^3b + C(4,2)a^2b^2 + C(4,3)ab^3 + C(4,4)b^4$
$= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$(a - b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$

Subtracting:
$(a + b)^4 - (a - b)^4 = (a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4) - (a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4)$
$= 8a^3b + 8ab^3$

$= 8ab(a^2 + b^2)$

Now substitute $a = \sqrt{3}$, $b = \sqrt{2}$:

$(\sqrt{3} + \sqrt{2})^4 - (\sqrt{3} - \sqrt{2})^4 = 8(\sqrt{3})(\sqrt{2})((\sqrt{3})^2 + (\sqrt{2})^2)$

$= 8\sqrt{6}\,(3 + 2)$

$= 8\sqrt{6} \times 5$

$= 40\sqrt{6}$

Answer: $(a + b)^4 - (a - b)^4 = 8ab(a^2 + b^2)$; for $a = \sqrt{3}$, $b = \sqrt{2}$, the value is $40\sqrt{6}$.

Q12 Find $(x + 1)^6 + (x - 1)^6$. Hence or otherwise evaluate $(\sqrt{2} + 1)^6 + (\sqrt{2} - 1)^6$.

SOLUTION

Using the Binomial Theorem:

$(x + 1)^6 = x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$

$(x - 1)^6 = x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$

Adding (odd-power terms cancel):

$(x + 1)^6 + (x - 1)^6 = 2x^6 + 30x^4 + 30x^2 + 2$

$= 2(x^6 + 15x^4 + 15x^2 + 1)$

Substitute $x = \sqrt{2}$:

$(\sqrt{2} + 1)^6 + (\sqrt{2} - 1)^6 = 2((\sqrt{2})^6 + 15(\sqrt{2})^4 + 15(\sqrt{2})^2 + 1)$

$= 2(8 + 15 \times 4 + 15 \times 2 + 1)$

$= 2(8 + 60 + 30 + 1)$

$= 2 \times 99$

$= 198$

Answer: $(x + 1)^6 + (x - 1)^6 = 2(x^6 + 15x^4 + 15x^2 + 1)$; for $x = \sqrt{2}$, the value is 198.

Q13 Show that 9^(n+1) – 8n – 9 is divisible by 64, whenever n is a positive integer.

SOLUTION

Write $9 = 1 + 8$, so $9^{(n+1)} = (1 + 8)^{(n+1)}$.

By the Binomial Theorem:

$(1 + 8)^{(n+1)} = C(n+1,0) + C(n+1,1)(8) + C(n+1,2)(8^2) + C(n+1,3)(8^3) + \ldots + C(n+1,n+1)(8^{(n+1)})$

$= 1 + (n+1)\cdot 8 + C(n+1,2)\cdot 64 + C(n+1,3)\cdot 8^3 + \ldots + 8^{(n+1)}$

Therefore:

9^(n+1) = 1 + 8(n+1) + 64[C(n+1,2) + C(n+1,3)·8 + … + 8^(n-1)]
= 1 + 8n + 8 + 64·k, where k = C(n+1,2) + C(n+1,3)·8 + … + 8^(n-1) is a positive integer.

So: 9^(n+1) – 8n – 9 = (1 + 8n + 8 + 64k) – 8n – 9 = 64k

Hence 9^(n+1) – 8n – 9 is divisible by 64 for every positive integer n.

Answer: 9^(n+1) – 8n – 9 = 64k for some positive integer k, so it is divisible by 64.

Q14 Prove that Σ (r=0 to n) 3^r · C(n,r) = 4^n.

SOLUTION

We use the Binomial Theorem with a = 1, b = 3:

(1 + 3)^n = Σ (r=0 to n) C(n,r) · 1^(n-r) · 3^r
= Σ (r=0 to n) C(n,r) · 3^r

But (1 + 3)^n = 4^n.

Therefore: Σ (r=0 to n) 3^r · C(n,r) = 4^n. (Proved)

Answer: Σ (r=0 to n) C(n,r)·3^r = (1 + 3)^n = 4^n.

Misc — Miscellaneous Exercise

Q1 If a and b are distinct integers, prove that a – b is a factor of a^n – b^n, whenever n is a positive integer. [Hint: write a^n = (a – b + b)^n and expand]

SOLUTION

Write a = (a – b) + b, so a^n = ((a – b) + b)^n.

By the Binomial Theorem:
a^n = Σ (r=0 to n) C(n,r) (a – b)^r · b^(n-r)
= C(n,0)·b^n + C(n,1)(a–b)·b^(n-1) + C(n,2)(a–b)²·b^(n-2) + … + (a–b)^n
= b^n + (a–b)[C(n,1)·b^(n-1) + C(n,2)(a–b)·b^(n-2) + … + (a–b)^(n-1)]

Therefore:
a^n – b^n = (a – b)[C(n,1)·b^(n-1) + C(n,2)(a–b)·b^(n-2) + … + (a–b)^(n-1)]

The expression in brackets is an integer (since a, b are integers and n is a positive integer).
Hence (a – b) is a factor of a^n – b^n. (Proved)

Answer: a^n – b^n = (a – b) × [integer expression], so (a – b) divides a^n – b^n.

Q2 Evaluate: $(\sqrt{3} + \sqrt{2})^6 - (\sqrt{3} - \sqrt{2})^6$.

SOLUTION

Let a = √3, b = √2. Using the Binomial Theorem:
$(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$
$(a - b)^6 = a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$

Subtracting (even-power terms cancel):
$(a + b)^6 - (a - b)^6 = 2[6a^5b + 20a^3b^3 + 6ab^5]$
$= 2ab[6a^4 + 20a^2b^2 + 6b^4]$
$= 4ab[3a^4 + 10a^2b^2 + 3b^4]$

Substitute a = √3, b = √2 (so $a^2 = 3$, $b^2 = 2$, ab = √6):
$= 4\sqrt{6}\,[3(9) + 10(3)(2) + 3(4)]$
$= 4\sqrt{6}\,[27 + 60 + 12]$
$= 4\sqrt{6} \times 99$
$= 396\sqrt{6}$

Answer: $(\sqrt{3} + \sqrt{2})^6 - (\sqrt{3} - \sqrt{2})^6 = 396\sqrt{6}$

Q3 Find the value of $(a^2 + \sqrt{(a^2-1)})^4 + (a^2 - \sqrt{(a^2-1)})^4$.

SOLUTION

Let $x = a^2$, $y = \sqrt{(a^2-1)}$, so $y^2 = a^2-1$.

Using $(x+y)^4 + (x-y)^4$: odd-power terms cancel, giving:
$(x+y)^4 + (x-y)^4 = 2[C(4,0)x^4 + C(4,2)x^2y^2 + C(4,4)y^4]$
$= 2[x^4 + 6x^2y^2 + y^4]$

Substitute $x = a^2$, $y^2 = a^2-1$:
$= 2[(a^2)^4 + 6(a^2)^2(a^2-1) + (a^2-1)^2]$
$= 2[a^8 + 6a^4(a^2-1) + (a^4 - 2a^2 + 1)]$
$= 2[a^8 + 6a^6 - 6a^4 + a^4 - 2a^2 + 1]$
$= 2[a^8 + 6a^6 - 5a^4 - 2a^2 + 1]$
$= 2a^8 + 12a^6 - 10a^4 - 4a^2 + 2$

Answer: $(a^2 + \sqrt{(a^2-1)})^4 + (a^2 - \sqrt{(a^2-1)})^4 = 2a^8 + 12a^6 - 10a^4 - 4a^2 + 2$

Q4 Find an approximation of $(0.99)^5$ using the first three terms of its expansion.

SOLUTION

Write $0.99 = 1 - 0.01$.

$(0.99)^5 = (1 - 0.01)^5$
$\approx C(5,0) - C(5,1)(0.01) + C(5,2)(0.01)^2$ (first three terms)
$= 1 - 5(0.01) + 10(0.0001)$
$= 1 - 0.05 + 0.001$
$= 0.951$

Answer: $(0.99)^5 \approx 0.951$

Q5 Expand using Binomial Theorem: $(1 + x/2 - 2/x)^4$, $x \neq 0$.

SOLUTION

Group the expression as A + B where $A = 1$, $B = x/2 - 2/x$.

$(1 + (x/2 - 2/x))^4 = C(4,0) + C(4,1)B + C(4,2)B^2 + C(4,3)B^3 + C(4,4)B^4$
$= 1 + 4B + 6B^2 + 4B^3 + B^4$

Compute each power of $B = x/2 - 2/x$:
$B = x/2 - 2/x$
$B^2 = x^2/4 - 2 + 4/x^2$

$B^3 = x^3/8 - 3x/2 + 6/x - 8/x^3$

$B^4 = x^4/16 - x^2 + 6 - 16/x^2 + 16/x^4$

Now multiply and collect:

$4B = 2x - 8/x$

$6B^2 = 3x^2/2 - 12 + 24/x^2$

$4B^3 = x^3/2 - 6x + 24/x - 32/x^3$

$B^4 = x^4/16 - x^2 + 6 - 16/x^2 + 16/x^4$

Summing all terms:

Constant: $1 - 12 + 6 = -5$

$x^4/16$ term: $x^4/16$

$x^3/2$ term: $x^3/2$

x^2 terms: $3x^2/2 - x^2 = x^2/2$

x terms: $2x - 6x = -4x$

$1/x$ terms: $-8/x + 24/x = 16/x$

$1/x^2$ terms: $24/x^2 - 16/x^2 = 8/x^2$

$1/x^3$ term: $-32/x^3$

$1/x^4$ term: $16/x^4$

$= x^4/16 + x^3/2 + x^2/2 - 4x - 5 + 16/x + 8/x^2 - 32/x^3 + 16/x^4$

Answer: $(1 + x/2 - 2/x)^4 = x^4/16 + x^3/2 + x^2/2 - 4x - 5 + 16/x + 8/x^2 - 32/x^3 + 16/x^4$

Q6 Find the expansion of $(3x^2 - 2ax + 3a^2)^3$ using binomial theorem.

SOLUTION

Group the expression as $A + B$ where $A = 3x^2 + 3a^2$, $B = -2ax$.

$(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$

Compute each part:

$A = 3x^2 + 3a^2$

$A^3 = (3x^2 + 3a^2)^3 = 27x^6 + 81x^4a^2 + 81x^2a^4 + 27a^6$

$3A^2B = 3(3x^2 + 3a^2)^2(-2ax)$

$A^2 = 9x^4 + 18x^2a^2 + 9a^4$

$3A^2B = 3(9x^4 + 18x^2a^2 + 9a^4)(-2ax) = -6ax(9x^4 + 18x^2a^2 + 9a^4)$

$= -54ax^5 - 108a^3x^3 - 54a^5x$

$3AB^2 = 3(3x^2 + 3a^2)(4a^2x^2) = 12a^2x^2(3x^2 + 3a^2) = 36a^2x^4 + 36a^4x^2$

$B^3 = (-2ax)^3 = -8a^3x^3$

Combining all terms:

x^6: $27x^6$

x^5: $-54ax^5$

x^4: $81a^2x^4 + 36a^2x^4 = 117a^2x^4$

x^3: $-108a^3x^3 - 8a^3x^3 = -116a^3x^3$

x^2: $81a^4x^2 + 36a^4x^2 = 117a^4x^2$

x: $-54a^5x$

constant: $27a^6$

$= 27x^6 - 54ax^5 + 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6$

Answer: $(3x^2 - 2ax + 3a^2)^3 = 27x^6 - 54ax^5 + 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6$