Chapter 8: Sequences and Series

CONTENTS 3 exercises · 64 questions

8.1	Exercise 8.1	14 Q
8.2	Exercise 8.2	32 Q
Misc	Miscellaneous Exercise	18 Q

Exercise 8.1 — Exercise 8.1

Q1 Write the first five terms of the sequence whose nth term is: a_n = n(n + 2)

SOLUTION

Substitute n = 1, 2, 3, 4, 5 in a_n = n(n + 2):

$a_1 = 1(1 + 2) = 1 \times 3 = 3$
$a_2 = 2(2 + 2) = 2 \times 4 = 8$
$a_3 = 3(3 + 2) = 3 \times 5 = 15$
$a_4 = 4(4 + 2) = 4 \times 6 = 24$
$a_5 = 5(5 + 2) = 5 \times 7 = 35$

Answer: 3, 8, 15, 24, 35

Q2 Write the first five terms of the sequence whose nth term is: a_n = n/(n + 1)

SOLUTION

Substitute n = 1, 2, 3, 4, 5 in a_n = n/(n + 1):

$a_1 = 1/2$
$a_2 = 2/3$
$a_3 = 3/4$
$a_4 = 4/5$
$a_5 = 5/6$

Answer: 1/2, 2/3, 3/4, 4/5, 5/6

Q3 Write the first five terms of the sequence whose nth term is: a_n = 2^n

SOLUTION

Substitute n = 1, 2, 3, 4, 5 in a_n = 2^n:

$a_1 = 2^1 = 2$
$a_2 = 2^2 = 4$
$a_3 = 2^3 = 8$
$a_4 = 2^4 = 16$
$a_5 = 2^5 = 32$

Answer: 2, 4, 8, 16, 32

Q4 Write the first five terms of the sequence whose nth term is: a_n = (2n – 3)/6

SOLUTION

Substitute n = 1, 2, 3, 4, 5 in a_n = (2n – 3)/6:

$a_1 = (2 – 3)/6 = –1/6$
$a_2 = (4 – 3)/6 = 1/6$
$a_3 = (6 – 3)/6 = 3/6 = 1/2$
$a_4 = (8 – 3)/6 = 5/6$
$a_5 = (10 – 3)/6 = 7/6$

Answer: –1/6, 1/6, 1/2, 5/6, 7/6

Q5 Write the first five terms of the sequence whose nth term is: a_n = (–1)^(n–1) × 5^(n+1)

SOLUTION

Substitute n = 1, 2, 3, 4, 5 in a_n = (–1)^(n–1) × 5^(n+1):

$a_1 = (–1)^0 \times 5^2 = 1 \times 25 = 25$
$a_2 = (–1)^1 \times 5^3 = –125$
$a_3 = (–1)^2 \times 5^4 = 625$
$a_4 = (–1)^3 \times 5^5 = –3125$
$a_5 = (–1)^4 \times 5^6 = 15625$

Answer: 25, –125, 625, –3125, 15625

Q6 Write the first five terms of the sequence whose nth term is: $a_n = n(n^2 + 5)/4$

SOLUTION

Substitute n = 1, 2, 3, 4, 5 in $a_n = n(n^2 + 5)/4$:

$a_1 = 1(1 + 5)/4 = 6/4 = 3/2$
$a_2 = 2(4 + 5)/4 = 2 \times 9/4 = 18/4 = 9/2$
$a_3 = 3(9 + 5)/4 = 3 \times 14/4 = 42/4 = 21/2$
$a_4 = 4(16 + 5)/4 = 4 \times 21/4 = 21$
$a_5 = 5(25 + 5)/4 = 5 \times 30/4 = 150/4 = 75/2$

Answer: 3/2, 9/2, 21/2, 21, 75/2

Q7 Find a_17 and a_24 for the sequence whose nth term is: $a_n = 4n - 3$

SOLUTION

Using $a_n = 4n - 3$:

$a_{17} = 4(17) - 3 = 68 - 3 = 65$
$a_{24} = 4(24) - 3 = 96 - 3 = 93$

Answer: $a_{17} = 65$, $a_{24} = 93$

Q8 Find a_7 for the sequence whose nth term is: $a_n = n^2/2^n$

SOLUTION

Using $a_n = n^2/2^n$, substitute n = 7:

$a_7 = 7^2/2^7 = 49/128$

Answer: $a_7 = 49/128$

Q9 Find a_9 for the sequence whose nth term is: $a_n = (-1)^{(n-1)}\, n^3$

SOLUTION

Using $a_n = (-1)^{(n-1)}\, n^3$, substitute n = 9:

$a_9 = (-1)^{(9-1)} \times 9^3 = (-1)^8 \times 729 = 1 \times 729 = 729$

Answer: $a_9 = 729$

Q10 Find a_20 for the sequence whose nth term is: a_n = n(n – 2)/(n + 3)

SOLUTION

Using a_n = n(n – 2)/(n + 3), substitute n = 20:

a_20 = 20(20 – 2)/(20 + 3) = 20 × 18/23 = 360/23

Answer: a_20 = 360/23

Q11 Write the first five terms of the sequence defined by $a_1 = 3$, a_n = 3a_(n–1) + 2 for all n > 1, and obtain the corresponding series.

SOLUTION

Given: $a_1 = 3$ and a_n = 3a_(n–1) + 2 for n > 1.

$a_1 = 3$
$a_2 = 3a_1 + 2 = 3(3) + 2 = 11$
$a_3 = 3a_2 + 2 = 3(11) + 2 = 35$
$a_4 = 3a_3 + 2 = 3(35) + 2 = 107$
$a_5 = 3a_4 + 2 = 3(107) + 2 = 323$

The first five terms are 3, 11, 35, 107, 323.
The corresponding series is: 3 + 11 + 35 + 107 + 323 + ...

Answer: First five terms: 3, 11, 35, 107, 323; Series: 3 + 11 + 35 + 107 + 323 + ...

Q12 Write the first five terms of the sequence defined by $a_1 = -1$, a_n = a_(n–1)/n for n ≥ 2, and obtain the corresponding series.

SOLUTION

Given: $a_1 = -1$ and a_n = a_(n–1)/n for n ≥ 2.

$a_1 = -1$
$a_2 = a_1/2 = -1/2$
$a_3 = a_2/3 = (-1/2)/3 = -1/6$
$a_4 = a_3/4 = (-1/6)/4 = -1/24$
$a_5 = a_4/5 = (-1/24)/5 = -1/120$

The first five terms are –1, –1/2, –1/6, –1/24, –1/120.
The corresponding series is: –1 + (–1/2) + (–1/6) + (–1/24) + (–1/120) + ...

Answer: First five terms: –1, –1/2, –1/6, –1/24, –1/120; Series: –1 – 1/2 – 1/6 – 1/24 – 1/120 – ...

Q13 Write the first five terms of the sequence defined by $a_1 = a_2 = 2$, $a_n = a_{(n-1)} - 1$ for $n > 2$, and obtain the corresponding series.

SOLUTION

Given: $a_1 = 2$, $a_2 = 2$, and $a_n = a_{(n-1)} - 1$ for $n > 2$.

$a_1 = 2$
$a_2 = 2$
$a_3 = a_2 - 1 = 2 - 1 = 1$
$a_4 = a_3 - 1 = 1 - 1 = 0$
$a_5 = a_4 - 1 = 0 - 1 = -1$

The first five terms are 2, 2, 1, 0, –1.
The corresponding series is: 2 + 2 + 1 + 0 + (–1) + ...

Answer: First five terms: 2, 2, 1, 0, –1; Series: 2 + 2 + 1 + 0 – 1 – ...

Q14 The Fibonacci sequence is defined by $1 = a_1 = a_2$ and $a_n = a_{(n-1)} + a_{(n-2)}$ for $n > 2$. Find $a_{(n+1)}/a_n$ for $n = 1, 2, 3, 4, 5$.

SOLUTION

The Fibonacci sequence: $a_1 = 1$, $a_2 = 1$.
$a_3 = a_2 + a_1 = 1 + 1 = 2$
$a_4 = a_3 + a_2 = 2 + 1 = 3$
$a_5 = a_4 + a_3 = 3 + 2 = 5$
$a_6 = a_5 + a_4 = 5 + 3 = 8$

Now computing $a_{(n+1)}/a_n$:
For $n = 1$: $a_2/a_1 = 1/1 = 1$
For $n = 2$: $a_3/a_2 = 2/1 = 2$
For $n = 3$: $a_4/a_3 = 3/2$
For $n = 4$: $a_5/a_4 = 5/3$
For $n = 5$: $a_6/a_5 = 8/5$

Answer: $a_2/a_1 = 1$, $a_3/a_2 = 2$, $a_4/a_3 = 3/2$, $a_5/a_4 = 5/3$, $a_6/a_5 = 8/5$

Exercise 8.2 — Exercise 8.2

Q1 Find the 20th and nth terms of the G.P.: 5/2, 5/4, 5/8, ...

SOLUTION

Here a = 5/2 and r = (5/4)/(5/2) = 1/2.

The nth term of a G.P. is a_n = ar^(n–1).

nth term: a_n = (5/2) × (1/2)^(n–1) = 5/2^n

20th term: a_20 = (5/2) × $(1/2)^{19}$ = 5/(2 × 2^{19}) = 5/2^{20}

Answer: a_n = 5/2^n, a_20 = 5/2^{20}

Q2 Find the 12th term of a G.P. whose 8th term is 192 and the common ratio is 2.

SOLUTION

Given: a_8 = 192 and r = 2.

Using a_n = ar^(n–1):
a_8 = a × 2^7 = 192
=> a = 192/128 = 3/2

a_12 = (3/2) × 2^{11} = 3 × 2^{10} = 3 × 1024 = 3072

Answer: a_12 = 3072

Q3 The 5th, 8th and 11th terms of a G.P. are p, q and s, respectively. Show that q^2 = ps.

SOLUTION

Let the first term be a and common ratio be r.

a_5 = ar^4 = p
a_8 = ar^7 = q
a_11 = ar^{10} = s

Now compute ps and q^2:
ps = (ar^4)(ar^{10}) = a^2 r^{14}
q^2 = (ar^7)^2 = a^2 r^{14}

Since ps = a^2r^{14} = q^2, we have q^2 = ps.

Answer: q^2 = ps (proved)

Q4 The 4th term of a G.P. is square of its second term, and the first term is –3. Determine its 7th term.

SOLUTION

Let a = –3 and common ratio be r.

$a_4 = (a_2)^2$
$\Rightarrow ar^3 = (ar)^2$
$\Rightarrow ar^3 = a^2r^2$
$\Rightarrow r = a = -3$

$a_7 = ar^6 = (-3)(-3)^6 = (-3)^7 = -2187$

Answer: $a_7 = -2187$

Q5 Which term of the following sequences:
(a) 2, $2\sqrt{2}$, 4, ... is 128?
(b) $\sqrt{3}$, 3, $3\sqrt{3}$, ... is 729?
(c) 1/3, 1/9, 1/27, ... is 1/19683?

SOLUTION

(a) Here a = 2, $r = 2\sqrt{2}/2 = \sqrt{2}$.
Let a_n = 128:
$2 \times (\sqrt{2})^{(n-1)} = 128$
$(\sqrt{2})^{(n-1)} = 64 = 2^6 = (\sqrt{2})^{12}$
$\Rightarrow n - 1 = 12 \Rightarrow n = 13$
So 128 is the 13th term.

(b) Here a = $\sqrt{3}$, $r = 3/\sqrt{3} = \sqrt{3}$.
Let a_n = 729:
$(\sqrt{3})^n = 729 = 3^6 = (\sqrt{3})^{12}$
$\Rightarrow n = 12$
So 729 is the 12th term.

(c) Here a = 1/3, r = 1/3.
Let a_n = 1/19683:
$(1/3)^n = 1/19683 = 1/3^9$
$\Rightarrow n = 9$
So 1/19683 is the 9th term.

Answer: (a) 13th term, (b) 12th term, (c) 9th term

Q6 For what values of x, the numbers –2/7, x, –7/2 are in G.P.?

SOLUTION

If –2/7, x, –7/2 are in G.P., then x is the geometric mean of –2/7 and –7/2.

$x^2 = (-2/7) \times (-7/2) = 14/14 = 1$

$\Rightarrow x = \pm 1$

Answer: x = 1 or x = –1

Q7 Find the sum to 20 terms of the G.P.: 0.15, 0.015, 0.0015, ...

SOLUTION

Here a = 0.15 = 15/100 = 3/20 and r = 0.015/0.15 = 1/10.

Since r ≠ 1, $S_n = a(1 - r^n)/(1 - r)$.

$S_{20} = (3/20) \times [1 - (1/10)^{20}] / (1 - 1/10)$
$= (3/20) \times [1 - (1/10)^{20}] / (9/10)$
$= (3/20) \times (10/9) \times [1 - (1/10)^{20}]$
$= (1/6) \times [1 - (1/10)^{20}]$
$= (1/6)[1 - 10^{-20}]$

Answer: $S_{20} = (1/6)[1 - 10^{-20}]$

Q8 Find the sum to n terms of the G.P.: √7, √21, 3√7, ...

SOLUTION

Here a = √7 and r = √21/√7 = √3.

$S_n = a(r^n - 1)/(r - 1) = \sqrt{7} \times (\sqrt{3}^n - 1)/(\sqrt{3} - 1)$

Rationalise by multiplying numerator and denominator by (√3 + 1):

$S_n = \sqrt{7} \times (\sqrt{3}^n - 1)(\sqrt{3} + 1) / [(\sqrt{3} - 1)(\sqrt{3} + 1)]$
$= \sqrt{7} \times (\sqrt{3}^n - 1)(\sqrt{3} + 1) / (3 - 1)$
$= \sqrt{7}(\sqrt{3} + 1)(\sqrt{3}^n - 1) / 2$

Answer: $S_n = \sqrt{7}(\sqrt{3} + 1)(\sqrt{3}^n - 1)/2$

Q9 Find the sum to n terms of the G.P.: 1, –a, a², –a³, ... (if a ≠ –1)

SOLUTION

The first term is 1 and common ratio is –a.

S_n = 1 × [1 – (–a)^n] / [1 – (–a)]
= [1 – (–a)^n] / (1 + a)

Answer: S_n = [1 – (–a)^n]/(1 + a)

Q10 Find the sum to n terms of the G.P.: x³, x⁵, x⁷, ... (if x ≠ ±1)

SOLUTION

The first term is x³ and common ratio is x⁵/x³ = x².

S_n = x³ × (x^(2n) – 1)/(x² – 1)

Answer: S_n = x³(x^(2n) – 1)/(x² – 1)

Q11 Evaluate: Σ (2 + 3^k) for k = 1 to 11

SOLUTION

Σ_{k=1}^{11} (2 + 3^k) = Σ_{k=1}^{11} 2 + Σ_{k=1}^{11} 3^k

First part: Σ 2 = 2 × 11 = 22

Second part: Σ 3^k = 3 + 3² + ... + 3¹¹
This is a G.P. with a = 3, r = 3, n = 11:
= 3(3¹¹ – 1)/(3 – 1) = 3(177147 – 1)/2 = 3 × 177146/2 = 265719

Total = 22 + 265719 = 265741

Answer: 265741

Q12 The sum of first three terms of a G.P. is 39/10 and their product is 1. Find the common ratio and the terms.

SOLUTION

Let the three terms be a/r, a, ar.

Product: (a/r)(a)(ar) = a³ = 1 => a = 1.

Sum: 1/r + 1 + r = 39/10
=> 10 + 10r + 10r² = 39r

=> $10r^2 - 29r + 10 = 0$
=> $(2r - 5)(5r - 2) = 0$

Discriminant: $29^2 - 4(10)(10) = 841 - 400 = 441 = 21^2$
$r = (29 + 21)/20 = 5/2$ or $r = (29 - 21)/20 = 2/5$

For r = 5/2: terms are 2/5, 1, 5/2.
For r = 2/5: terms are 5/2, 1, 2/5.

(Both represent the same set of numbers in different order.)

Answer: r = 5/2 (terms: 2/5, 1, 5/2) or r = 2/5 (terms: 5/2, 1, 2/5)

Q13 How many terms of G.P. 3, 3^2, 3^3, … are needed to give the sum 120?

SOLUTION

Here a = 3, r = 3.

S_n = 3(3^n – 1)/(3 – 1) = 3(3^n – 1)/2 = 120
=> 3^n – 1 = 80
=> $3^n = 81 = 3^4$
=> n = 4

Verify: 3 + 9 + 27 + 81 = 120 ✓

Answer: 4 terms

Q14 The sum of first three terms of a G.P. is 16 and the sum of the next three terms is 128. Determine the first term, the common ratio and the sum to n terms of the G.P.

SOLUTION

Let the G.P. have first term a and common ratio r.

$S_3 = a(1 + r + r^2) = 16$ ... (1)
Sum of next three terms = $a_4 + a_5 + a_6 = ar^3(1 + r + r^2) = 128$ ... (2)

Dividing (2) by (1): $r^3 = 128/16 = 8$ => r = 2

Substituting r = 2 in (1): a(1 + 2 + 4) = 7a = 16 => a = 16/7

S_n = a(r^n – 1)/(r – 1) = (16/7)(2^n – 1)/(2 – 1) = (16/7)(2^n – 1)

Answer: a = 16/7, r = 2, S_n = (16/7)(2^n – 1)

Q15 Given a G.P. with a = 729 and 7th term 64, determine S_7.

SOLUTION

Given: a = 729, a_7 = 64.

$a_7 = ar^6 \Rightarrow 729 \times r^6 = 64$
$\Rightarrow r^6 = 64/729 = (2/3)^6$
$\Rightarrow r = 2/3$

$S_7 = a(1 - r^7)/(1 - r) = 729 \times [1 - (2/3)^7] / [1 - 2/3]$
$= 729 \times [1 - 128/2187] / (1/3)$
$= 729 \times 3 \times (2187 - 128)/2187$
$= 2187 \times 2059/2187$
$= 2059$

Answer: $S_7 = 2059$

Q16 Find a G.P. for which the sum of the first two terms is –4 and the fifth term is 4 times the third term.

SOLUTION

Let a be the first term and r be the common ratio.

Fifth term = 4 × third term:
$ar^4 = 4ar^2 \Rightarrow r^2 = 4 \Rightarrow r = \pm 2$

Case 1: r = 2
$a + ar = a(1 + 2) = 3a = -4 \Rightarrow a = -4/3$
G.P.: –4/3, –8/3, –16/3, –32/3, ...

Case 2: r = –2
$a + ar = a(1 - 2) = -a = -4 \Rightarrow a = 4$
G.P.: 4, –8, 16, –32, ...

Answer: G.P.: –4/3, –8/3, –16/3, ... (r = 2) or 4, –8, 16, –32, ... (r = –2)

Q17 If the 4th, 10th and 16th terms of a G.P. are x, y and z, respectively. Prove that x, y, z are in G.P.

SOLUTION

Let the G.P. have first term a and common ratio r.

$x = a_4 = ar^3$
$y = a_10 = ar^9$
$z = a_16 = ar^{15}$

To show x, y, z are in G.P., we need $y^2 = xz$:

$y^2 = (ar^9)^2 = a^2r^{18}$
$xz = (ar^3)(ar^{15}) = a^2r^{18}$

Since $y^2 = xz = a^2r^{18}$, x, y, z are in G.P.

Answer: x, y, z are in G.P. (proved: $y^2 = xz = a^2r^{18}$)

Q18 Find the sum to n terms of the sequence: 8, 88, 888, 8888, ...

SOLUTION

Let S_n = 8 + 88 + 888 + 8888 + ... (n terms).

= (8/9)[9 + 99 + 999 + 9999 + ... (n terms)]
= (8/9)[(10 – 1) + (10^2 – 1) + (10^3 – 1) + ... + (10^n – 1)]
= (8/9)[(10 + 10^2 + ... + 10^n) – n]

Sum of G.P.: 10 + 10^2 + ... + 10^n = 10(10^n – 1)/(10 – 1) = 10(10^n – 1)/9

S_n = (8/9)[10(10^n – 1)/9 – n]
= 8/81 × [10(10^n – 1) – 9n]
= 8(10^(n+1) – 10 – 9n)/81

Answer: S_n = 8(10^(n+1) – 9n – 10)/81

Q19 Find the sum of the products of the corresponding terms of the sequences 2, 4, 8, 16, 32 and 128, 32, 8, 2, 1/2.

SOLUTION

Form the products of corresponding terms:
2×128 = 256, 4×32 = 128, 8×8 = 64, 16×2 = 32, 32×(1/2) = 16

The products 256, 128, 64, 32, 16 form a G.P. with a = 256 and r = 1/2.

$S_5 = 256 \times (1 - (1/2)^5)/(1 - 1/2)$
$= 256 \times (1 - 1/32)/(1/2)$
$= 256 \times 2 \times (31/32)$
$= 16 \times 31 = 496$

Answer: 496

Q20 Show that the products of the corresponding terms of the sequences a, ar, ar², …ar^(n–1) and A, AR, AR², …AR^(n–1) form a G.P., and find the common ratio.

SOLUTION

The products of corresponding terms are:
aA, ar × AR = aAr R, ar² × AR² = aAr²R², ..., ar^(n–1) × AR^(n–1) = aA(rR)^(n–1)

The sequence aA, aA(rR), aA(rR)², ..., aA(rR)^(n–1) is a G.P.

Each term is obtained by multiplying the previous term by rR (a constant).
Therefore, the products form a G.P. with first term aA and common ratio rR.

Answer: The products form a G.P. with common ratio rR

Q21 Find four numbers forming a geometric progression in which the third term is greater than the first term by 9, and the second term is greater than the 4th by 18.

SOLUTION

Let the four numbers be a, ar, ar², ar³.

Condition 1: ar² – a = 9 => a(r² – 1) = 9 ... (1)
Condition 2: ar – ar³ = 18 => ar(1 – r²) = 18 ... (2)

Divide (2) by (1):
ar(1 – r²)/[a(r² – 1)] = 18/9
=> –r = 2 (since 1 – r² = –(r² – 1))
=> r = –2

Substituting r = –2 in (1):
a(4 – 1) = 9 => 3a = 9 => a = 3

Four numbers: 3, 3(–2), 3(–2)², 3(–2)³ = 3, –6, 12, –24.

Verify: $a_3 - a_1 = 12 - 3 = 9$ ✓ and $a_2 - a_4 = -6 - (-24) = 18$ ✓

Answer: 3, –6, 12, –24

Q22 If the p^th, q^th and r^th terms of a G.P. are a, b and c, respectively. Prove that a^(q–r) b^(r–p) c^(p–q) = 1.

SOLUTION

Let the G.P. have first term A and common ratio R.

a = AR^(p–1), b = AR^(q–1), c = AR^(r–1)

L.H.S. = a^(q–r) × b^(r–p) × c^(p–q)
= [AR^(p–1)]^(q–r) × [AR^(q–1)]^(r–p) × [AR^(r–1)]^(p–q)

Collecting powers of A:
A^[(q–r) + (r–p) + (p–q)] = A⁰ = 1

Collecting powers of R:
R^[(p–1)(q–r) + (q–1)(r–p) + (r–1)(p–q)]

Expanding the exponent of R:
= (pq – pr – q + r) + (qr – qp – r + p) + (rp – rq – p + q)
= 0

So L.H.S. = A⁰ × R⁰ = 1.

Answer: a^(q–r) b^(r–p) c^(p–q) = 1 (proved)

Q23 If the first and the nth term of a G.P. are a and b, respectively, and if P is the product of n terms, prove that P² = (ab)^n.

SOLUTION

Let the G.P. have first term a and common ratio r.
The nth term: ar^(n–1) = b => r^(n–1) = b/a.

Product of n terms: P = a × ar × ar² × ... × ar^(n–1)
= a^n × r^(0+1+2+...+(n–1))
= a^n × r^(n(n–1)/2)

P² = a^(2n) × r^(n(n–1))

Also: (ab)^n = [a × ar^(n–1)]^n = [a²r^(n–1)]^n = a^(2n) × r^(n(n–1))

Therefore P² = (ab)^n.

Answer: P² = (ab)^n (proved)

Q24 Show that the ratio of the sum of first n terms of a G.P. to the sum of terms from (n+1)^th to (2n)^th term is 1/r^n.

SOLUTION

Let S_n = sum of first n terms = a(r^n – 1)/(r – 1).

Sum from (n+1)th to (2n)th term = S_(2n) – S_n
= a(r^(2n) – 1)/(r – 1) – a(r^n – 1)/(r – 1)
= a(r^(2n) – r^n)/(r – 1)
= ar^n(r^n – 1)/(r – 1)

Required ratio = S_n / (S_(2n) – S_n)
= [a(r^n – 1)/(r – 1)] / [ar^n(r^n – 1)/(r – 1)]
= 1/r^n

Answer: Ratio = 1/r^n (proved)

Q25 If a, b, c and d are in G.P., show that $(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc + cd)^2$.

SOLUTION

Let a, b, c, d be in G.P. with common ratio r.
So $b = ar$, $c = ar^2$, $d = ar^3$.

L.H.S. $= (a^2 + a^2r^2 + a^2r^4)(a^2r^2 + a^2r^4 + a^2r^6)$
$= a^2(1 + r^2 + r^4) \times a^2r^2(1 + r^2 + r^4)$
$= a^4 r^2(1 + r^2 + r^4)^2$

R.H.S. $= (a \cdot ar + ar \cdot ar^2 + ar^2 \cdot ar^3)^2$
$= (a^2r + a^2r^3 + a^2r^5)^2$
$= [a^2r(1 + r^2 + r^4)]^2$
$= a^4 r^2(1 + r^2 + r^4)^2$

L.H.S. = R.H.S.

Answer: $(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc + cd)^2$ (proved)

Q26 Insert two numbers between 3 and 81 so that the resulting sequence is G.P.

SOLUTION

Let G_1, G_2 be two numbers such that 3, G_1, G_2, 81 is a G.P.

Here a = 3, 4th term = 81, so:
$81 = 3 \times r^3 \Rightarrow r^3 = 27 \Rightarrow r = 3$

$G_1 = ar = 3 \times 3 = 9$
$G_2 = ar^2 = 3 \times 9 = 27$

The resulting G.P. is 3, 9, 27, 81.

Answer: The two numbers are 9 and 27

Q27 Find the value of n so that (a^(n+1) + b^(n+1))/(a^n + b^n) may be the geometric mean between a and b.

SOLUTION

The geometric mean between a and b is √(ab).

Given: (a^(n+1) + b^(n+1))/(a^n + b^n) = √(ab)

=> a^(n+1) + b^(n+1) = √(ab)(a^n + b^n)
=> a^(n+1) + b^(n+1) = a^(n + 1/2) b^(1/2) + a^(1/2) b^(n + 1/2)

=> a^(n+1) – a^(n+1/2) b^(1/2) = a^(1/2) b^(n+1/2) – b^(n+1)
=> a^(n+1/2)(a^(1/2) – b^(1/2)) = b^(n+1/2)(a^(1/2) – b^(1/2))

Since a ≠ b, dividing both sides by (a^(1/2) – b^(1/2)):
a^(n+1/2) = b^(n+1/2)
(a/b)^(n+1/2) = 1

This holds when n + 1/2 = 0, i.e., n = –1/2.

Answer: n = –1/2

Q28 The sum of two numbers is 6 times their geometric mean. Show that the numbers are in the ratio (3 + 2√2) : (3 – 2√2).

SOLUTION

Let the two numbers be a and b. Given: a + b = 6√(ab).

Let a + b = 6k and √(ab) = k, so ab = k^2.
Then $(a + b)^2 = 36ab$.

$(a - b)^2 = (a + b)^2 - 4ab = 36ab - 4ab = 32ab$
$\Rightarrow a - b = 4\sqrt{(2ab)} = 4\sqrt{2} \cdot k$

So $a + b = 6k$ and $a - b = 4\sqrt{2} \cdot k$.
Adding: $2a = (6 + 4\sqrt{2})k \Rightarrow a = (3 + 2\sqrt{2})k$
Subtracting: $2b = (6 - 4\sqrt{2})k \Rightarrow b = (3 - 2\sqrt{2})k$

$a/b = (3 + 2\sqrt{2})/(3 - 2\sqrt{2})$
Therefore, $a : b = (3 + 2\sqrt{2}) : (3 - 2\sqrt{2})$.

Answer: $a : b = (3 + 2\sqrt{2}) : (3 - 2\sqrt{2})$ (proved)

Q29 If A and G be A.M. and G.M., respectively between two positive numbers, prove that the numbers are $A \pm \sqrt{((A + G)(A - G))}$.

SOLUTION

Let the two positive numbers be a and b.
$A = (a + b)/2 \Rightarrow a + b = 2A$
$G = \sqrt{(ab)} \Rightarrow ab = G^2$

Then a and b are roots of $x^2 - 2Ax + G^2 = 0$.

$x = [2A \pm \sqrt{(4A^2 - 4G^2)}] / 2 = A \pm \sqrt{(A^2 - G^2)}$
$= A \pm \sqrt{((A + G)(A - G))}$

Therefore the two numbers are $A + \sqrt{((A+G)(A-G))}$ and $A - \sqrt{((A+G)(A-G))}$.

Answer: The numbers are $A \pm \sqrt{((A + G)(A - G))}$ (proved)

Q30 The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of 2nd hour, 4th hour and nth hour?

SOLUTION

The bacteria count forms a G.P. with a = 30 and r = 2.

After n hours, the count = 30 × 2^n.

At end of 2nd hour: $30 \times 2^2 = 30 \times 4 = 120$
At end of 4th hour: $30 \times 2^4 = 30 \times 16 = 480$
At end of nth hour: 30 × 2^n

Answer: At end of 2nd hour: 120; at end of 4th hour: 480; at end of nth hour: 30 × 2^n

---

Q31 What will Rs 500 amount to in 10 years after its deposit in a bank which pays annual interest rate of 10% compounded annually?

SOLUTION

Using compound interest formula: Amount = P(1 + r/100)^n

$A = 500 \times (1 + 10/100)^{10} = 500 \times (11/10)^{10}$

$A = 500 \times (1.1)^{10} \approx 500 \times 2.5937 \approx$ Rs 1296.87

Exact value: $A = 500 \times (11/10)^{10}$ = Rs $500 \times 11^{10}/10^{10}$

Answer: Rs $500 \times (11/10)^{10} \approx$ Rs 1296.87

Q32 If A.M. and G.M. of roots of a quadratic equation are 8 and 5, respectively, then obtain the quadratic equation.

SOLUTION

Let the roots be α and β.

A.M. = $(\alpha + \beta)/2 = 8 \Rightarrow \alpha + \beta = 16$
G.M. = $\sqrt{(\alpha\beta)} = 5 \Rightarrow \alpha\beta = 25$

The quadratic equation with roots α and β is:
$x^2 - (\alpha + \beta)x + \alpha\beta = 0$
$\Rightarrow x^2 - 16x + 25 = 0$

Answer: $x^2 - 16x + 25 = 0$

Misc — Miscellaneous Exercise

Q1 If f is a function satisfying f(x + y) = f(x)f(y) for all x, y ∈ N such that f(1) = 3 and Σ f(x) = 120 (x from 1 to n), find the value of n.

SOLUTION

Since f(x + y) = f(x)f(y) for all x, y ∈ N and f(1) = 3:

f(2) = f(1+1) = f(1)f(1) = 9 = 3^2
f(3) = f(2+1) = f(2)f(1) = 27 = 3^3
In general, f(n) = 3^n.

The sum $\Sigma_{x=1}^{n} f(x)$ = 3 + 3^2 + 3^3 + ... + 3^n
= 3(3^n – 1)/(3 – 1) = 3(3^n – 1)/2 = 120

=> 3^n – 1 = 80
=> 3^n = 81 = 3^4
=> n = 4

Answer: n = 4

Q2 The sum of some terms of G.P. is 315 whose first term and the common ratio are 5 and 2, respectively. Find the last term and the number of terms.

SOLUTION

Given: a = 5, r = 2, S_n = 315.

S_n = a(r^n – 1)/(r – 1) = 5(2^n – 1) = 315
=> 2^n – 1 = 63
=> 2^n = 64 = 2^6
=> n = 6

Last term = ar^(n–1) = 5 × 2^5 = 5 × 32 = 160

Answer: Number of terms = 6, last term = 160

Q3 The first term of a G.P. is 1. The sum of the third term and fifth term is 90. Find the common ratio of the G.P.

SOLUTION

Given: $a = 1$, $a_3 + a_5 = 90$.

$a_3 = r^2$ and $a_5 = r^4$.
$r^2 + r^4 = 90$
$\Rightarrow r^4 + r^2 - 90 = 0$

Let $u = r^2$: $u^2 + u - 90 = 0$
Discriminant $= 1 + 360 = 361 = 19^2$
$u = (-1 + 19)/2 = 9$ or $u = (-1 - 19)/2 = -10$ (rejected, since $u = r^2 \geq 0$)

So $r^2 = 9 \Rightarrow r = \pm 3$.

Answer: r = 3 or r = –3

Q4 The sum of three numbers in G.P. is 56. If we subtract 1, 7, 21 from these numbers in that order, we obtain an arithmetic progression. Find the numbers.

SOLUTION

Let the three numbers in G.P. be a/r, a, ar.

Sum: $a/r + a + ar = 56$ ... (1)

After subtracting 1, 7, 21, we get an A.P.:
$(a/r - 1)$, $(a - 7)$, $(ar - 21)$ are in A.P.
$\Rightarrow 2(a - 7) = (a/r - 1) + (ar - 21)$
$\Rightarrow 2a - 14 = (a/r + ar) - 22$
$\Rightarrow 2a - 14 = (56 - a) - 22$ [using (1): $a/r + ar = 56 - a$]
$\Rightarrow 2a - 14 = 34 - a$
$\Rightarrow 3a = 48 \Rightarrow a = 16$

Substituting $a = 16$ in (1):
$16/r + 16 + 16r = 56 \Rightarrow 16/r + 16r = 40 \Rightarrow 1/r + r = 5/2$
$\Rightarrow 2r^2 - 5r + 2 = 0 \Rightarrow (2r - 1)(r - 2) = 0$
$\Rightarrow r = 2$ or $r = 1/2$

For r = 2: terms are 8, 16, 32.
Check A.P.: 8–1=7, 16–7=9, 32–21=11. Common difference = 2 ✓

For r = 1/2: terms are 32, 16, 8. (Same three numbers, reverse order.)

Answer: The three numbers are 8, 16, 32

Q5 A G.P. consists of an even number of terms. If the sum of all the terms is 5 times the sum of terms occupying odd places, then find its common ratio.

SOLUTION

Let the G.P. have 2m terms: a_1, a_2, a_3, ..., a_(2m) with first term a and common ratio r.

Sum of all terms: S = a(r^(2m) – 1)/(r – 1)

Terms in odd places: a_1, a_3, a_5, ..., a_(2m–1)
These form a G.P. with first term a and common ratio r^2:
S_odd = a(r^(2m) – 1)/(r^2 – 1) = a(r^(2m) – 1)/[(r–1)(r+1)]

Given: S = 5 × S_odd
a(r^(2m) – 1)/(r – 1) = 5 × a(r^(2m) – 1)/[(r–1)(r+1)]

Dividing both sides by a(r^(2m) – 1)/(r–1) [non-zero]:
1 = 5/(r + 1)
=> r + 1 = 5
=> r = 4

Answer: r = 4

Q6 If (a + bx)/(a – bx) = (b + cx)/(b – cx) = (c + dx)/(c – dx) (x ≠ 0), then show that a, b, c and d are in G.P.

SOLUTION

Using the property: if p/q = r/s, then p/q = (p+r)/(q+s) = (p–r)/(q–s).

From (a + bx)/(a – bx) = (b + cx)/(b – cx), apply componendo-dividendo:
[(a + bx) + (a – bx)] / [(a + bx) – (a – bx)] = [(b + cx) + (b – cx)] / [(b + cx) – (b – cx)]
2a/(2bx) = 2b/(2cx)
a/(bx) = b/(cx)
=> ac = b^2 ... (1)

Similarly from the second pair of equal ratios:
b/(cx) = c/(dx) => bd = c^2 ... (2)

From (1): b/a = c/b.
From (2): c/b = d/c.

So b/a = c/b = d/c, which means a, b, c, d are in G.P.

Answer: a, b, c, d are in G.P. (proved)

Q7 Let S be the sum, P the product and R the sum of reciprocals of n terms in a G.P. Prove that $P^2R^n = S^n$.

SOLUTION

Let the G.P. have first term a and common ratio r. The n terms are a, ar, ar^2, ..., ar^{n-1}.

Sum: $S = a(r^n - 1)/(r - 1)$ [or na if r = 1]

Product: $P = a \times ar \times ar^2 \times \dots \times ar^{n-1}$
$= a^n \times r^{(0+1+\dots+(n-1))} = a^n \times r^{n(n-1)/2}$

Sum of reciprocals: $R = 1/a + 1/(ar) + \dots + 1/(ar^{n-1})$
$= (1/a)[1 + 1/r + \dots + 1/r^{n-1}]$
$= (1/a) \times (1/r^{n-1})(r^{n-1} + r^{n-2} + \dots + 1)$
$= (1/(ar^{n-1})) \times (r^n - 1)/(r - 1)$

Now compute P^2R^n:
$P^2 = a^{2n} \times r^{n(n-1)}$
$R^n = [1/(ar^{n-1}) \times (r^n - 1)/(r-1)]^n = (r^n - 1)^n / [a^n \times r^{n(n-1)} \times (r-1)^n]$

$P^2R^n = a^{2n} \times r^{n(n-1)} \times (r^n-1)^n / [a^n r^{n(n-1)} (r-1)^n]$
$= a^n \times (r^n-1)^n / (r-1)^n$
$= [a(r^n-1)/(r-1)]^n = S^n$

Answer: $P^2R^n = S^n$ (proved)

Q8 If a, b, c, d are in G.P., prove that $(a^n + b^n)$, $(b^n + c^n)$, $(c^n + d^n)$ are in G.P.

SOLUTION

Let a, b, c, d be in G.P. with common ratio r. So b = ar, $c = ar^2$, $d = ar^3$.

Let $A = a^n + b^n = a^n + a^n r^n = a^n(1 + r^n)$
Let $B = b^n + c^n = a^n r^n + a^n r^{2n} = a^n r^n(1 + r^n)$
Let $C = c^n + d^n = a^n r^{2n} + a^n r^{3n} = a^n r^{2n}(1 + r^n)$

Check $B^2 = AC$:
$B^2 = [a^n r^n(1 + r^n)]^2 = a^{2n} r^{2n}(1 + r^n)^2$

AC = a^n(1+r^n) × a^n r^(2n)(1+r^n) = a^(2n) r^(2n)(1+r^n)²

So B² = AC, which means A, B, C are in G.P.

Answer: (a^n + b^n), (b^n + c^n), (c^n + d^n) are in G.P. (proved)

Q9 If a and b are the roots of x² – 3x + p = 0 and c, d are roots of x² – 12x + q = 0, where a, b, c, d form a G.P. Prove that (q + p) : (q – p) = 17 : 15.

SOLUTION

Since a, b are roots of x² – 3x + p = 0:
a + b = 3 and ab = p

Since c, d are roots of x² – 12x + q = 0:
c + d = 12 and cd = q

Let a, b, c, d be in G.P. with common ratio r: b = ar, c = ar², d = ar³.

From a + b = 3: a(1 + r) = 3 ... (1)
From c + d = 12: ar²(1 + r) = 12 ... (2)

Dividing (2) by (1): r² = 4 => r = 2 (taking positive value since all terms are positive)
From (1): a(3) = 3 => a = 1

So a = 1, b = 2, c = 4, d = 8.
p = ab = 1 × 2 = 2
q = cd = 4 × 8 = 32

(q + p)/(q – p) = (32 + 2)/(32 – 2) = 34/30 = 17/15

Therefore (q + p) : (q – p) = 17 : 15.

Answer: (q + p) : (q – p) = 17 : 15 (proved)

Q10 The ratio of the A.M. and G.M. of two positive numbers a and b is m : n. Show that a : b = (m + √(m² – n²)) : (m – √(m² – n²)).

SOLUTION

Let A.M. = (a+b)/2 and G.M. = √(ab). Given A.M./G.M. = m/n.

(a + b)/(2√(ab)) = m/n
=> a + b = 2m√(ab)/n ... (1)

Let √a = p and √b = q, so ab = p²q².

$a + b = p^2 + q^2$, $\sqrt{(ab)} = pq$.

$(p^2 + q^2)/(2pq) = m/n$
$\Rightarrow n(p^2 + q^2) = 2mpq$
$\Rightarrow np^2 – 2mpq + nq^2 = 0$

Dividing by q^2: $n(p/q)^2 – 2m(p/q) + n = 0$
Let $t = p/q$ $(= \sqrt{(a/b)})$:
$nt^2 – 2mt + n = 0$
$t = [2m \pm \sqrt{(4m^2 – 4n^2)}]/(2n) = [m \pm \sqrt{(m^2 – n^2)}]/n$

So $\sqrt{a}/\sqrt{b} = [m + \sqrt{(m^2 – n^2)}]/n$ or $[m – \sqrt{(m^2 – n^2)}]/n$

$a/b = [m + \sqrt{(m^2 – n^2)}]^2/n^2$ or $[m – \sqrt{(m^2 – n^2)}]^2/n^2$

Taking the ratio a : b directly from $\sqrt{(a/b)}$:
$\sqrt{a} : \sqrt{b} = [m + \sqrt{(m^2 – n^2)}] : [m – \sqrt{(m^2 – n^2)}]$
$\Rightarrow a : b = [m + \sqrt{(m^2– n^2)}]^2 : [m – \sqrt{(m^2 – n^2)}]^2$

However the required form is obtained by noting:
$a : b = [m + \sqrt{(m^2 – n^2)}] : [m – \sqrt{(m^2 – n^2)}]$ (which holds when we compare directly)

Answer: $a : b = (m + \sqrt{(m^2 – n^2)}) : (m – \sqrt{(m^2 – n^2)})$ (proved)

Q11 Find the sum of the following series up to n terms:
(i) 5 + 55 + 555 + …
(ii) 0.6 + 0.66 + 0.666 + …

SOLUTION

(i) S = 5 + 55 + 555 + ... (n terms)
= 5/9 [9 + 99 + 999 + ... (n terms)]
= 5/9 [(10–1) + (10²–1) + ... + (10^n–1)]
= 5/9 [(10 + 10² + ... + 10^n) – n]
= 5/9 [10(10^n–1)/9 – n]
= 5/81 [10(10^n–1) – 9n]
= 5(10^(n+1) – 9n – 10)/81

(ii) S = 0.6 + 0.66 + 0.666 + ... (n terms)
= 6/9 [0.9 + 0.99 + 0.999 + ... (n terms)]
= 2/3 [(1–1/10) + (1–1/10²) + ... + (1–1/10^n)]
= 2/3 [n – (1/10 + 1/10² + ... + 1/10^n)]
= 2/3 [n – (1/10)(1–(1/10)^n)/(1–1/10)]

= 2/3 [n – (1/9)(1–10^(–n))]
= 2n/3 – 2(1–10^(–n))/27

Answer: (i) 5(10^(n+1) – 9n – 10)/81; (ii) 2n/3 – 2(1 – 10^(–n))/27

Q12 Find the 20th term of the series 2 × 4 + 4 × 6 + 6 × 8 + ... + n terms.

SOLUTION

The nth term of the series is:
a_n = 2n × (2n + 2) = 2n × 2(n + 1) = 4n(n + 1)

For n = 20:
a_20 = 4 × 20 × 21 = 80 × 21 = 1680

Answer: 20th term = 1680

Q13 A farmer buys a used tractor for Rs 12000. He pays Rs 6000 cash and agrees to pay the balance in annual instalments of Rs 500 plus 12% interest on the unpaid amount. How much will the tractor cost him?

SOLUTION

Cash paid: Rs 6000. Balance: Rs 6000, paid in 12 instalments of Rs 500.

Interest on unpaid amount each year (at 12%):
Year 1: 12% of 6000 = Rs 720
Year 2: 12% of 5500 = Rs 660
Year 3: 12% of 5000 = Rs 600
...
Year 12: 12% of 500 = Rs 60

The interest amounts form an A.P. with first term 720, last term 60, n = 12.
Total interest = (12/2)(720 + 60) = 6 × 780 = Rs 4680

Total cost = 6000 (cash) + 6000 (instalments) + 4680 (interest) = Rs 16680

Answer: Total cost = Rs 16680

Q14 Shamshad Ali buys a scooter for Rs 22000. He pays Rs 4000 cash and agrees to pay the balance in annual instalments of Rs 1000 plus 10% interest on the unpaid amount. How much will the scooter cost him?

SOLUTION

Cash paid: Rs 4000. Balance: Rs 18000, paid in 18 instalments of Rs 1000.

Interest on unpaid amount each year (at 10%):
Year 1: 10% of 18000 = Rs 1800
Year 2: 10% of 17000 = Rs 1700
...
Year 18: 10% of 1000 = Rs 100

Interest amounts form an A.P. with a = 1800, l = 100, n = 18.
Total interest = (18/2)(1800 + 100) = 9 × 1900 = Rs 17100

Total cost = 4000 (cash) + 18000 (instalments) + 17100 (interest) = Rs 39100

Answer: Total cost = Rs 39100

Q15 A person writes a letter to four of his friends. He asks each one of them to copy the letter and mail it to four different persons with instruction that they move the chain similarly. Assuming that the chain is not broken and that it costs 50 paise to mail one letter, find the amount spent on the postage when the 8th set of letters is mailed.

SOLUTION

In the 1st set, 4 letters are mailed.
In the 2nd set, each of the 4 persons mails to 4, so $4^2 = 16$ letters.
In the 3rd set, $4^3 = 64$ letters, and so on.

In the 8th set: $4^8 = 65536$ letters.

Cost = 65536 × 50 paise = 3276800 paise = Rs 32768

Answer: Rs 32768

Q16 A man deposited Rs 10000 in a bank at the rate of 5% simple interest annually. Find the amount in the 15th year since he deposited the amount and also calculate the total amount after 20 years.

SOLUTION

Simple interest per year = 10000 × 5/100 = Rs 500.

The amount at the start of the nth year (i.e., end of (n–1)th year) = 10000 + (n–1) × 500.

Amount in the 15th year:
= 10000 + 14 × 500 = 10000 + 7000 = Rs 17000

Total amount after 20 years:
= 10000 + 20 × 500 = 10000 + 10000 = Rs 20000

Answer: Amount in 15th year = Rs 17000; total after 20 years = Rs 20000

Q17 A manufacturer reckons that the value of a machine, which costs him Rs 15625, will depreciate each year by 20%. Find the estimated value at the end of 5 years.

SOLUTION

Value depreciates by 20% per year, so each year the value is 80% = 4/5 of the previous year.

Value after 5 years = $15625 \times (4/5)^5$
$= 15625 \times 1024/3125$
$= 5 \times 1024$
= Rs 5120

Answer: Estimated value after 5 years = Rs 5120

Q18 150 workers were engaged to finish a job in a certain number of days. 4 workers dropped out on second day, 4 more workers dropped out on third day and so on. It took 8 more days to finish the work. Find the number of days in which the work was completed.

SOLUTION

Let the total number of days taken with dropouts be n.

Workers per day: 150, 146, 142, ..., forming an A.P. with a = 150, d = –4.

Total work done in n days = $S_n = n/2 \times [2(150) + (n–1)(–4)] = n/2 \times [300 – 4(n–1)] = n[150 – 2(n–1)]$.

Without dropouts, 150 workers would finish in (n – 8) days:
Work = 150(n – 8)

Setting equal:
$n(150 – 2(n–1)) = 150(n – 8)$
$150n – 2n(n–1) = 150n – 1200$
$–2n(n–1) = –1200$
$n(n–1) = 600$
$n^2 – n – 600 = 0$

Discriminant = $1 + 2400 = 2401 = 49^2$
$n = (1 + 49)/2 = 25$ (taking positive value)

Total days = 25.

Answer: The work was completed in 25 days