Chapter 9: Straight Lines

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Exercise 9.1 — Exercise 9.1

Q1 Draw a quadrilateral in the Cartesian plane, whose vertices are (–4, 5), (0, 7), (5, –5) and (–4, –2). Also, find its area.

SOLUTION

Let A = (–4, 5), B = (0, 7), C = (5, –5), D = (–4, –2).

Plot these points and join them in order to form quadrilateral ABCD.

To find the area, split the quadrilateral into two triangles: △ABC and △ACD.

Area of △ABC with A(–4,5), B(0,7), C(5,–5):

= (1/2)	x_A(y_B – y_C) + x_B(y_C – y_A) + x_C(y_A – y_B)	
= (1/2)	(–4)(7–(–5)) + 0(–5–5) + 5(5–7)	
= (1/2)	(–4)(12) + 0 + 5(–2)	
= (1/2)	–48 – 10	= (1/2)(58) = 29 square units

Area of △ACD with A(–4,5), C(5,–5), D(–4,–2):

= (1/2)	x_A(y_C – y_D) + x_C(y_D – y_A) + x_D(y_A – y_C)	
= (1/2)	(–4)(–5–(–2)) + 5(–2–5) + (–4)(5–(–5))	
= (1/2)	(–4)(–3) + 5(–7) + (–4)(10)	

= (1/2)	x_A(y_C – y_D) + x_C(y_D – y_A) + x_D(y_A – y_C)	
= (1/2)	12 – 35 – 40	= (1/2)(63) = 31.5 square units

Total area of quadrilateral ABCD = 29 + 31.5 = 60.5 square units.

Answer: Area of the quadrilateral = 60.5 square units.

Q2 The base of an equilateral triangle with side 2a lies along the y-axis such that the mid-point of the base is at the origin. Find vertices of the triangle.

SOLUTION

The base lies along the y-axis with midpoint at the origin.
So the two endpoints of the base are at (0, a) and (0, –a).

The third vertex lies on the x-axis (by symmetry about the x-axis).
Let the third vertex be (x, 0).

Since all sides are equal to 2a, the distance from (x, 0) to (0, a) must equal 2a:
$\sqrt{(x^2 + a^2)} = 2a$
$x^2 + a^2 = 4a^2$
$x^2 = 3a^2$
$x = \pm a\sqrt{3}$

So the third vertex is $(a\sqrt{3}, 0)$ or $(-a\sqrt{3}, 0)$.

Answer: The three vertices are (0, a), (0, –a), and $(a\sqrt{3}, 0)$. If the triangle is on the other side, the third vertex is $(-a\sqrt{3}, 0)$.

Q3 Find the distance between P (x1, y1) and Q (x2, y2) when: (i) PQ is parallel to the y-axis, (ii) PQ is parallel to the x-axis.

SOLUTION

(i) When PQ is parallel to the y-axis, the x-coordinates of P and Q are equal, so x1 = x2.
Distance PQ = $\sqrt{((x2-x1)^2 + (y2-y1)^2)} = \sqrt{(0 + (y2-y1)^2)} = |y2 - y1|$.

(ii) When PQ is parallel to the x-axis, the y-coordinates are equal, so y1 = y2.

Distance PQ = $\sqrt{((x2–x1)^2 + (y2–y1)^2)} = \sqrt{((x2–x1)^2 + 0)} = |x2 – x1|$.

Answer: (i) $|y2 – y1|$ (ii) $|x2 – x1|$

Q4 Find a point on the x-axis, which is equidistant from the points (7, 6) and (3, 4).

SOLUTION

Any point on the x-axis has the form A = (a, 0).

Given: distance from A to (7, 6) = distance from A to (3, 4).

$(a – 7)^2 + (0 – 6)^2 = (a – 3)^2 + (0 – 4)^2$
$a^2 – 14a + 49 + 36 = a^2 – 6a + 9 + 16$
$–14a + 85 = –6a + 25$
$–8a = –60$
$a = 15/2$

Answer: The required point on the x-axis is (15/2, 0).

Q5 Find the slope of a line, which passes through the origin, and the mid-point of the line segment joining the points P (0, –4) and B (8, 0).

SOLUTION

Mid-point of P(0, –4) and B(8, 0) is M = ((0+8)/2, (–4+0)/2) = (4, –2).

The line passes through the origin O(0, 0) and M(4, –2).

Slope = (–2 – 0)/(4 – 0) = –2/4 = –1/2.

Answer: Slope = –1/2.

Q6 Without using the Pythagoras theorem, show that the points (4, 4), (3, 5) and (–1, –1) are the vertices of a right angled triangle.

SOLUTION

Let A = (4, 4), B = (3, 5), C = (–1, –1).

Slope of AB: $m_{AB} = (5 – 4)/(3 – 4) = 1/(–1) = –1$.

Slope of BC: $m_{BC} = (–1 – 5)/(–1 – 3) = (–6)/(–4) = 3/2$.

Slope of AC: $m_{AC} = (–1 – 4)/(–1 – 4) = (–5)/(–5) = 1$.

Now, $m_{AB} \times m_{AC} = (–1)(1) = –1$.

Since the product of the slopes of AB and AC is –1, the lines AB and AC are perpendicular.

Therefore, the angle at vertex A is 90°, and the triangle is right-angled at A.

Answer: Since slope of AB × slope of AC = –1, sides AB ⊥ AC, so the triangle is right-angled at A(4, 4).

Q7 Find the slope of the line, which makes an angle of 30° with the positive direction of y-axis measured anticlockwise.

SOLUTION

The positive direction of the y-axis is at 90° from the positive x-axis.

An angle of 30° anticlockwise from the positive y-axis means the line makes an angle of 90° + 30° = 120° with the positive x-axis.

Slope = tan(120°) = tan(180° – 60°) = –tan(60°) = –√3.

Answer: Slope = –√3.

Q8 Without using distance formula, show that points (–2, –1), (4, 0), (3, 3) and (–3, 2) are the vertices of a parallelogram.

SOLUTION

Let A = (–2, –1), B = (4, 0), C = (3, 3), D = (–3, 2).

Slope of AB = (0 – (–1))/(4 – (–2)) = 1/6.
Slope of DC = (2 – 3)/(–3 – 3) = (–1)/(–6) = 1/6.

Since slope of AB = slope of DC, AB || DC.

Slope of BC = (3 – 0)/(3 – 4) = 3/(–1) = –3.
Slope of AD = (2 – (–1))/(–3 – (–2)) = 3/(–1) = –3.

Since slope of BC = slope of AD, BC || AD.

Both pairs of opposite sides are parallel, so ABCD is a parallelogram.

Answer: Since AB ∥ DC and BC ∥ AD (equal slopes for each pair), the quadrilateral is a parallelogram.

Q9 Find the angle between the x-axis and the line joining the points (3, –1) and (4, –2).

SOLUTION

Slope of the line through (3, –1) and (4, –2):
$m = (-2 - (-1))/(4 - 3) = (-1)/1 = -1$.

If θ is the inclination (angle with the positive x-axis), then:
$\tan\theta = m = -1$.

Since $\tan\theta = -1$ and $\theta \in [0°, 180°)$, we get $\theta = 135°$.

The angle between the x-axis and the line is 135°.

Answer: The angle between the x-axis and the line is 135°.

Q10 The slope of a line is double of the slope of another line. If tangent of the angle between them is 1/3, find the slopes of the lines.

SOLUTION

Let the slope of one line be m. Then the slope of the other line is 2m.

The tangent of the angle θ between them satisfies:
$\tan\theta = |(2m - m)/(1 + m\cdot 2m)| = |m/(1 + 2m^2)| = 1/3$.

Case 1: $m/(1 + 2m^2) = 1/3$
$3m = 1 + 2m^2$
$2m^2 - 3m + 1 = 0$
$(2m - 1)(m - 1) = 0$
$m = 1/2$ or $m = 1$.

Case 2: $m/(1 + 2m^2) = -1/3$
$-3m = 1 + 2m^2$
$2m^2 + 3m + 1 = 0$
$(2m + 1)(m + 1) = 0$
$m = -1/2$ or $m = -1$.

The pairs of slopes are:
- $m = 1/2$ and $2m = 1$
- $m = 1$ and $2m = 2$
- $m = -1/2$ and $2m = -1$
- $m = -1$ and $2m = -2$.

Answer: The slopes are 1 and 2, or 1/2 and 1, or –1 and –2, or –1/2 and –1.

Q11 A line passes through (x1, y1) and (h, k). If slope of the line is m, show that k – y1 = m (h – x1).

SOLUTION

The slope of a line through two points (x1, y1) and (h, k) is given by:
$m = (k – y1)/(h – x1)$.

Since the line is non-vertical (slope is defined), $h \neq x1$, so we can multiply both sides by (h – x1):
$m(h – x1) = k – y1$.

That is, $k – y1 = m(h – x1)$. Hence proved.

Answer: By the definition of slope: $m = (k – y1)/(h – x1)$, so $k – y1 = m(h – x1)$. Proved.

Exercise 9.2 — Exercise 9.2

Q1 Write the equations for the x-axis and y-axis.

SOLUTION

Every point on the x-axis has y-coordinate = 0, regardless of x.
So the equation of the x-axis is: y = 0.

Every point on the y-axis has x-coordinate = 0, regardless of y.
So the equation of the y-axis is: x = 0.

Answer: Equation of the x-axis: y = 0. Equation of the y-axis: x = 0.

Q2 Passing through the point (–4, 3) with slope 1/2.

SOLUTION

Using the point-slope form: $y - y_0 = m(x - x_0)$
with $(x_0, y_0) = (-4, 3)$ and $m = 1/2$:

$y - 3 = (1/2)(x - (-4))$
$y - 3 = (1/2)(x + 4)$
$2y - 6 = x + 4$
$x - 2y + 10 = 0.$

Answer: $x - 2y + 10 = 0.$

Q3 Passing through (0, 0) with slope m.

SOLUTION

Using the point-slope form with $(x_0, y_0) = (0, 0)$ and slope m:

$y - 0 = m(x - 0)$
$y = mx.$

Answer: $y = mx.$

Q4 Passing through $(2, 2\sqrt{3})$ and inclined with the x-axis at an angle of 75°.

SOLUTION

Slope = $\tan 75° = \tan(45° + 30°)$.

Using the addition formula:

$\tan 75° = (\tan 45° + \tan 30°)/(1 – \tan 45° \tan 30°) = (1 + 1/\sqrt{3})/(1 – 1/\sqrt{3})$
$= (\sqrt{3} + 1)/(\sqrt{3} – 1) = (\sqrt{3} + 1)^2/((\sqrt{3})^2 – 1^2) = (3 + 2\sqrt{3} + 1)/2 = (4 + 2\sqrt{3})/2 = 2 + \sqrt{3}.$

So $m = 2 + \sqrt{3}$.

Using point-slope form with $(x_0, y_0) = (2, 2\sqrt{3})$:
$y – 2\sqrt{3} = (2 + \sqrt{3})(x – 2)$
$y – 2\sqrt{3} = (2 + \sqrt{3})x – 2(2 + \sqrt{3})$
$y – 2\sqrt{3} = (2 + \sqrt{3})x – 4 – 2\sqrt{3}$
$y = (2 + \sqrt{3})x – 4.$

$(2 + \sqrt{3})x – y – 4 = 0.$

Answer: $(2 + \sqrt{3})x – y – 4 = 0.$

Q5 Intersecting the x-axis at a distance of 3 units to the left of origin with slope –2.

SOLUTION

A distance of 3 units to the left of origin on the x-axis gives the point (–3, 0).
This is the x-intercept, so $d = –3$.

Using the slope-intercept form $y = m(x – d)$:
$y = –2(x – (–3)) = –2(x + 3)$
$y = –2x – 6$
$2x + y + 6 = 0.$

Answer: $2x + y + 6 = 0.$

Q6 Intersecting the y-axis at a distance of 2 units above the origin and making an angle of 30° with positive direction of the x-axis.

SOLUTION

The line cuts the y-axis at 2 units above the origin, so y-intercept $c = 2$.

Slope $m = \tan 30° = 1/\sqrt{3}$.

Using the slope-intercept form $y = mx + c$:
$y = (1/\sqrt{3})x + 2$

Multiply through by $\sqrt{3}$:
$\sqrt{3}\, y = x + 2\sqrt{3}$
$x – \sqrt{3}\, y + 2\sqrt{3} = 0.$

Answer: $x - \sqrt{3}y + 2\sqrt{3} = 0$.

Q7 Passing through the points (–1, 1) and (2, –4).

SOLUTION

Slope $m = (-4 - 1)/(2 - (-1)) = -5/3$.

Using the two-point form with $(x_1, y_1) = (-1, 1)$:

$y - 1 = (-5/3)(x - (-1))$

$y - 1 = (-5/3)(x + 1)$

$3(y - 1) = -5(x + 1)$

$3y - 3 = -5x - 5$

$5x + 3y + 2 = 0$.

Answer: $5x + 3y + 2 = 0$.

Q8 The vertices of $\triangle PQR$ are P(2, 1), Q(–2, 3) and R(4, 5). Find equation of the median through the vertex R.

SOLUTION

The median from R goes to the midpoint of PQ.

Midpoint M of PQ = $((2 + (-2))/2, (1 + 3)/2) = (0, 2)$.

Slope of RM = $(5 - 2)/(4 - 0) = 3/4$.

Using point-slope form with R(4, 5):

$y - 5 = (3/4)(x - 4)$

$4(y - 5) = 3(x - 4)$

$4y - 20 = 3x - 12$

$3x - 4y + 8 = 0$.

Answer: $3x - 4y + 8 = 0$.

Q9 Find the equation of the line passing through (–3, 5) and perpendicular to the line through the points (2, 5) and (–3, 6).

SOLUTION

Slope of the line through (2, 5) and (–3, 6):

$m_1 = (6 - 5)/(-3 - 2) = 1/(-5) = -1/5$.

The required line is perpendicular to this, so its slope:

m = –1/m1 = –1/(–1/5) = 5.

Using point-slope form with (–3, 5):
y – 5 = 5(x – (–3))
y – 5 = 5x + 15
5x – y + 20 = 0.

Answer: 5x – y + 20 = 0.

Q10 A line perpendicular to the line segment joining the points (1, 0) and (2, 3) divides it in the ratio 1:n. Find the equation of the line.

SOLUTION

The point dividing the segment from (1, 0) to (2, 3) in the ratio 1:n is:
P = ((n·1 + 1·2)/(1 + n), (n·0 + 1·3)/(1 + n)) = ((n + 2)/(n + 1), 3/(n + 1)).

Slope of the segment from (1, 0) to (2, 3):
m1 = (3 – 0)/(2 – 1) = 3.

Slope of the perpendicular line: m = –1/3.

Using point-slope form with P = ((n+2)/(n+1), 3/(n+1)):
y – 3/(n+1) = (–1/3)(x – (n+2)/(n+1))

Multiply through by 3(n+1):
3(n+1)y – 9 = –(n+1)x + (n+2)
(n+1)x + 3(n+1)y = n + 2 + 9 = n + 11.

Thus: (1 + n)(x + 3y) = n + 11.

Answer: (1 + n)(x + 3y) = n + 11.

Q11 Find the equation of a line that cuts off equal intercepts on the coordinate axes and passes through the point (2, 3).

SOLUTION

Let the equal intercepts be a on both axes. Then the intercept form is:
x/a + y/a = 1, i.e., x + y = a.

Since the line passes through (2, 3):
2 + 3 = a => a = 5.

The equation of the line is x + y = 5, or x + y – 5 = 0.

Answer: x + y – 5 = 0.

Q12 Find equation of the line passing through the point (2, 2) and cutting off intercepts on the axes whose sum is 9.

SOLUTION

Let the x-intercept be a and y-intercept be b, with a + b = 9, so b = 9 – a.

Intercept form: x/a + y/b = 1.

Since (2, 2) lies on the line:
2/a + 2/(9–a) = 1
2(9–a) + 2a = a(9–a)
18 – 2a + 2a = 9a – a²
18 = 9a – a²
a² – 9a + 18 = 0
(a – 3)(a – 6) = 0
a = 3 or a = 6.

Case 1: a = 3, b = 6 => x/3 + y/6 = 1 => 2x + y – 6 = 0.
Case 2: a = 6, b = 3 => x/6 + y/3 = 1 => x + 2y – 6 = 0.

Answer: 2x + y – 6 = 0 or x + 2y – 6 = 0.

Q13 Find equation of the line through the point (0, 2) making an angle 2π/3 with the positive x-axis. Also, find the equation of line parallel to it and crossing the y-axis at a distance of 2 units below the origin.

SOLUTION

Slope = tan(2π/3) = tan(120°) = –tan(60°) = –√3.

Line through (0, 2) with slope –√3 and y-intercept 2:
y = –√3 x + 2
√3 x + y – 2 = 0.

A line parallel to this has the same slope –√3.
Crossing the y-axis at 2 units below the origin means y-intercept = –2.

Second line: y = –√3 x – 2
√3 x + y + 2 = 0.

Answer: First line: √3 x + y – 2 = 0. Parallel line: √3 x + y + 2 = 0.

Q14 The perpendicular from the origin to a line meets it at the point (–2, 9), find the equation of the line.

SOLUTION

The perpendicular from the origin O(0, 0) meets the line at (–2, 9).

Slope of the perpendicular segment from O to (–2, 9):
$m_{perp} = (9 – 0)/(–2 – 0) = –9/2$.

Since the required line is perpendicular to this segment:
Slope of the required line = $–1/(–9/2) = 2/9$.

The line passes through (–2, 9) with slope 2/9:
$y – 9 = (2/9)(x – (–2))$
$9(y – 9) = 2(x + 2)$
$9y – 81 = 2x + 4$
$2x – 9y + 85 = 0$.

Answer: $2x – 9y + 85 = 0$.

Q15 The length L (in centimetre) of a copper rod is a linear function of its Celsius temperature C. In an experiment, if L = 124.942 when C = 20 and L = 125.134 when C = 110, express L in terms of C.

SOLUTION

Since L is a linear function of C, we have $L = mC + b$ for constants m and b.

Using the two data points:
$124.942 = 20m + b$ ... (1)
$125.134 = 110m + b$ ... (2)

Subtracting (1) from (2):
$0.192 = 90m$
$m = 0.192/90 = 4/1875$.

From (1): $b = 124.942 – 20 \times (4/1875) = 124.942 – 80/1875$.
$80/1875 \approx 0.04267$, so $b \approx 124.899$.

Exact: $b = 124.942 – 80/1875 = (124.942 \times 1875 – 80)/1875 = (234267.25)/1875$.
Alternatively, using the standard formula:
$L – 124.942 = (0.192/90)(C – 20)$
$L = 124.942 + (0.192/90)(C – 20)$.

Or equivalently, L = (4/1875)C + b where b = 124.942 – (4/1875)×20.

Answer: L = 124.942 + (0.192/90)(C – 20), i.e. L = (4/1875)C + (124.942 – 80/1875).

Q16 The owner of a milk store finds that, he can sell 980 litres of milk each week at Rs 14/litre and 1220 litres of milk each week at Rs 16/litre. Assuming a linear relationship between selling price and demand, how many litres could he sell weekly at Rs 17/litre?

SOLUTION

Let x = selling price (Rs/litre) and y = litres sold per week.

Given points: (14, 980) and (16, 1220).

Slope: m = (1220 – 980)/(16 – 14) = 240/2 = 120.

Using point-slope form with (14, 980):
y – 980 = 120(x – 14)
y = 980 + 120(x – 14).

At x = 17:
y = 980 + 120(17 – 14) = 980 + 120 × 3 = 980 + 360 = 1340.

Answer: He could sell 1340 litres per week at Rs 17/litre.

Q17 P (a, b) is the mid-point of a line segment between axes. Show that equation of the line is x/a + y/b = 2.

SOLUTION

Let the line cut the x-axis at A(α, 0) and the y-axis at B(0, β).

P(a, b) is the midpoint of AB, so:
(α + 0)/2 = a => α = 2a.
(0 + β)/2 = b => β = 2b.

The intercept form of the line (x-intercept 2a, y-intercept 2b):
x/(2a) + y/(2b) = 1.

Multiply both sides by 2:
x/a + y/b = 2. Hence proved.

Answer: Since intercepts are 2a and 2b, intercept form gives x/(2a) + y/(2b) = 1, i.e. x/a + y/b = 2. Proved.

Q18 Point R (h, k) divides a line segment between the axes in the ratio 1 : 2. Find equation of the line.

SOLUTION

Let the line cut the x-axis at A(a, 0) and the y-axis at B(0, b).

R(h, k) divides AB in the ratio 1:2 (from A), so:
h = (1·0 + 2·a)/(1 + 2) = 2a/3 => a = 3h/2.
k = (1·b + 2·0)/(1 + 2) = b/3 => b = 3k.

The intercept form with x-intercept a = 3h/2 and y-intercept b = 3k:
x/(3h/2) + y/(3k) = 1
2x/(3h) + y/(3k) = 1.

Multiply through by 3:
2x/h + y/k = 3
2kx + hy = 3hk
2kx + hy – 3hk = 0.

Answer: 2kx + hy – 3hk = 0.

Q19 By using the concept of equation of a line, prove that the three points (3, 0), (–2, –2) and (8, 2) are collinear.

SOLUTION

Find the equation of the line through (3, 0) and (–2, –2).

Slope m = (–2 – 0)/(–2 – 3) = (–2)/(–5) = 2/5.

Using point-slope form with (3, 0):
y – 0 = (2/5)(x – 3)
5y = 2x – 6
2x – 5y – 6 = 0.

Now check whether (8, 2) satisfies this equation:
2(8) – 5(2) – 6 = 16 – 10 – 6 = 0. ✓

Since (8, 2) lies on the line through (3, 0) and (–2, –2), all three points are collinear.

Answer: The equation of the line through (3,0) and (–2,–2) is 2x – 5y – 6 = 0. Since (8,2) satisfies it (16 – 10 – 6 = 0), all three points are collinear. Proved.

Exercise 9.3 — Exercise 9.3

Q1 Reduce the following equations into slope-intercept form and find their slopes and the y-intercepts.
(i) $x + 7y = 0$, (ii) $6x + 3y – 5 = 0$, (iii) $y = 0$.

SOLUTION

(i) $x + 7y = 0$
$7y = –x$
$y = (–1/7)x + 0$.
Slope $m = –1/7$, y-intercept $c = 0$.

(ii) $6x + 3y – 5 = 0$
$3y = –6x + 5$
$y = –2x + 5/3$.
Slope $m = –2$, y-intercept $c = 5/3$.

(iii) $y = 0$
This is already in the form $y = mx + c$ with $m = 0$ and $c = 0$.
Slope $m = 0$, y-intercept $c = 0$.

Answer: (i) Slope = –1/7, y-intercept = 0. (ii) Slope = –2, y-intercept = 5/3. (iii) Slope = 0, y-intercept = 0.

Q2 Reduce the following equations into intercept form and find their intercepts on the axes.
(i) $3x + 2y – 12 = 0$, (ii) $4x – 3y = 6$, (iii) $3y + 2 = 0$.

SOLUTION

(i) $3x + 2y – 12 = 0 \Rightarrow 3x + 2y = 12$.
Divide by 12: $x/4 + y/6 = 1$.
x-intercept $a = 4$, y-intercept $b = 6$.

(ii) $4x – 3y = 6$.
Divide by 6: $(4x)/6 – (3y)/6 = 1 \Rightarrow x/(3/2) + y/(–2) = 1$.
x-intercept $a = 3/2$, y-intercept $b = –2$.

(iii) $3y + 2 = 0 \Rightarrow y = –2/3$.
This is a horizontal line parallel to the x-axis.
It does not intersect the x-axis (no x-intercept).
y-intercept $b = –2/3$.

Intercept form: y/(–2/3) = 1, or equivalently, the line has no finite x-intercept.

Answer: (i) x/4 + y/6 = 1; x-intercept = 4, y-intercept = 6. (ii) x/(3/2) + y/(–2) = 1; x-intercept = 3/2, y-intercept = –2. (iii) Parallel to x-axis; no x-intercept, y-intercept = –2/3.

Q3 Find the distance of the point (–1, 1) from the line 12(x + 6) = 5(y – 2).

SOLUTION

First, expand and simplify the line equation:
12(x + 6) = 5(y – 2)
12x + 72 = 5y – 10
12x – 5y + 82 = 0.

Distance from (x1, y1) = (–1, 1):

d =	Ax1 + By1 + C	/ √(A² + B²)
=	12(–1) + (–5)(1) + 82	/ √(144 + 25)
=	–12 – 5 + 82	/ √169
=	65	/ 13

= 5 units.

Answer: Distance = 5 units.

Q4 Find the points on the x-axis, whose distances from the line x/3 + y/4 = 1 are 4 units.

SOLUTION

Rewrite x/3 + y/4 = 1 in standard form: multiply by 12:
4x + 3y – 12 = 0.

Let the point on the x-axis be (a, 0). Then:
d = |4a + 3(0) – 12| / √(16 + 9) = |4a – 12| / 5 = 4.

|4a – 12| = 20.

Case 1: 4a – 12 = 20 => 4a = 32 => a = 8.
Case 2: 4a – 12 = –20 => 4a = –8 => a = –2.

Answer: The points are (8, 0) and (–2, 0).

Q5 Find the distance between parallel lines
(i) 15x + 8y – 34 = 0 and 15x + 8y + 31 = 0
(ii) l(x + y) + p = 0 and l(x + y) – r = 0.

SOLUTION

(i) Lines: 15x + 8y – 34 = 0 and 15x + 8y + 31 = 0.
Here A = 15, B = 8, C1 = –34, C2 = 31.
d = |C1 – C2| / √(A² + B²) = |–34 – 31| / √(225 + 64) = 65 / √289 = 65/17.

(ii) Lines: lx + ly + p = 0 and lx + ly – r = 0.
Here A = l, B = l, C1 = p, C2 = –r.
d = |C1 – C2| / √(A² + B²) = |p – (–r)| / √(l² + l²) = |p + r| / (l√2) = (p + r)/(l√2).
(Assuming p + r > 0 and l > 0.)

Answer: (i) d = 65/17. (ii) d = (p + r)/(l√2).

Q6 Find equation of the line parallel to the line 3x – 4y + 2 = 0 and passing through the point (–2, 3).

SOLUTION

A line parallel to 3x – 4y + 2 = 0 has the same coefficients of x and y:
3x – 4y + k = 0 for some constant k.

Since the line passes through (–2, 3):
3(–2) – 4(3) + k = 0
–6 – 12 + k = 0
k = 18.

The required equation is 3x – 4y + 18 = 0.

Answer: 3x – 4y + 18 = 0.

Q7 Find equation of the line perpendicular to the line x – 7y + 5 = 0 and having x-intercept 3.

SOLUTION

Slope of x – 7y + 5 = 0: 7y = x + 5 => y = (1/7)x + 5/7, so slope = 1/7.

A line perpendicular to this has slope m = –7.

The x-intercept is 3, so the line passes through (3, 0).

Using point-slope form:

$y - 0 = -7(x - 3)$

$y = -7x + 21$

$7x + y - 21 = 0.$

Answer: $7x + y - 21 = 0.$

Q8 Find angles between the lines $\sqrt{3}\,x + y = 1$ and $x + \sqrt{3}\,y = 1$.

SOLUTION

From $\sqrt{3}\,x + y = 1$: slope $m1 = -\sqrt{3}$.

From $x + \sqrt{3}\,y = 1$: slope $m2 = -1/\sqrt{3}$.

The acute angle θ between the lines satisfies:

$\tan\theta =$	$(m2 - m1)/(1 + m1\,m2)$	
$=$	$(-1/\sqrt{3} - (-\sqrt{3}))/(1 + (-\sqrt{3})(-1/\sqrt{3}))$	
$=$	$(-1/\sqrt{3} + \sqrt{3})/(1 + 1)$	
$=$	$(2/\sqrt{3})/2$	

$= 1/\sqrt{3}.$

So $\theta = 30°$.

The obtuse angle $= 180° - 30° = 150°$.

Answer: The angles between the lines are 30° and 150°.

Q9 The line through the points (h, 3) and (4, 1) intersects the line $7x - 9y - 19 = 0$ at right angle. Find the value of h.

SOLUTION

Slope of the line through (h, 3) and (4, 1):

$m1 = (1 - 3)/(4 - h) = -2/(4 - h).$

Slope of $7x - 9y - 19 = 0$: $-9y = -7x + 19 \Rightarrow y = (7/9)x - 19/9$.

Slope $m2 = 7/9$.

For perpendicularity: $m1 \times m2 = -1$:

$(-2/(4 - h)) \times (7/9) = -1$

$-14/(9(4 - h)) = -1$

$14 = 9(4 - h) = 36 - 9h$

9h = 22
h = 22/9.

Answer: h = 22/9.

Q10 Prove that the line through the point (x1, y1) and parallel to the line Ax + By + C = 0 is A(x – x1) + B(y – y1) = 0.

SOLUTION

The line Ax + By + C = 0 has slope m = –A/B (assuming B ≠ 0).

A line parallel to it also has slope –A/B and passes through (x1, y1).
By point-slope form:
y – y1 = (–A/B)(x – x1)
B(y – y1) = –A(x – x1)
A(x – x1) + B(y – y1) = 0.

This holds even when B = 0 (vertical line): then A(x – x1) = 0 gives x = x1, which is parallel to x = –C/A. Hence proved.

Answer: Using the slope from Ax+By+C=0 and the given point, the parallel line is A(x – x1) + B(y – y1) = 0. Proved.

Q11 Two lines passing through the point (2, 3) intersect each other at an angle of 60°. If slope of one line is 2, find equation of the other line.

SOLUTION

Let the required slope be m. The given line has slope 2.

Using the angle formula with θ = 60°:

tan 60° =	(m – 2)/(1 + 2m)	
√3 =	(m – 2)/(1 + 2m)	.

Case 1: (m – 2)/(1 + 2m) = √3
m – 2 = √3(1 + 2m) = √3 + 2√3 m
m(1 – 2√3) = 2 + √3
m = (2 + √3)/(1 – 2√3).

Rationalise: m = (2 + √3)(1 + 2√3)/((1 – 2√3)(1 + 2√3)) = (2 + 4√3 + √3 + 6)/(1 – 12) = (8 + 5√3)/(–11).

Case 2: $(m - 2)/(1 + 2m) = -\sqrt{3}$
$m - 2 = -\sqrt{3}(1 + 2m) = -\sqrt{3} - 2\sqrt{3}\, m$
$m(1 + 2\sqrt{3}) = 2 - \sqrt{3}$
$m = (2 - \sqrt{3})/(1 + 2\sqrt{3}) = (2 - \sqrt{3})(1 - 2\sqrt{3})/((1 + 2\sqrt{3})(1 - 2\sqrt{3})) = (2 - 4\sqrt{3} - \sqrt{3} + 6)/(-11) = (8 - 5\sqrt{3})/(-11)$.

Line 1 ($m = -(8 + 5\sqrt{3})/11$) through (2, 3):
$11(y - 3) = -(8 + 5\sqrt{3})(x - 2)$
$(8 + 5\sqrt{3})x + 11y - (49 + 10\sqrt{3}) = 0$.

Line 2 ($m = -(8 - 5\sqrt{3})/11 = (5\sqrt{3} - 8)/11$) through (2, 3):
$11(y - 3) = (5\sqrt{3} - 8)(x - 2)$
$(8 - 5\sqrt{3})x + 11y - (49 - 10\sqrt{3}) = 0$.

Answer: $(8 + 5\sqrt{3})x + 11y = 49 + 10\sqrt{3}$, or $(8 - 5\sqrt{3})x + 11y = 49 - 10\sqrt{3}$.

Q12 Find the equation of the right bisector of the line segment joining the points (3, 4) and (–1, 2).

SOLUTION

The right bisector (perpendicular bisector) passes through the midpoint of the segment and is perpendicular to it.

Midpoint M = $((3 + (-1))/2, (4 + 2)/2) = (1, 3)$.

Slope of the segment from (–1, 2) to (3, 4):
$m_{seg} = (4 - 2)/(3 - (-1)) = 2/4 = 1/2$.

Slope of the perpendicular bisector: $m = -1/(1/2) = -2$.

Using point-slope form with M(1, 3):
$y - 3 = -2(x - 1)$
$y - 3 = -2x + 2$
$2x + y - 5 = 0$.

Answer: $2x + y - 5 = 0$.

Q13 Find the coordinates of the foot of perpendicular from the point (–1, 3) to the line 3x – 4y – 16 = 0.

SOLUTION

The foot of perpendicular from P(–1, 3) to line 3x – 4y – 16 = 0.

Slope of the line 3x – 4y – 16 = 0 is 3/4.
Slope of perpendicular from P: m = –4/3.

Equation of the perpendicular through P(–1, 3):
y – 3 = (–4/3)(x – (–1))
3(y – 3) = –4(x + 1)
3y – 9 = –4x – 4
4x + 3y – 5 = 0. ... (perpendicular line)

Solve simultaneously:
3x – 4y = 16 ... (given line)
4x + 3y = 5 ... (perpendicular)

Multiply the first by 3 and the second by 4:
9x – 12y = 48
16x + 12y = 20
Adding: 25x = 68 => x = 68/25.

Substitute back: 3(68/25) – 4y = 16 => 204/25 – 4y = 400/25 => 4y = –196/25 => y = –49/25.

Answer: Foot of perpendicular = (68/25, –49/25).

Q14 The perpendicular from the origin to the line y = mx + c meets it at the point (–1, 2). Find the values of m and c.

SOLUTION

The perpendicular from the origin O(0, 0) meets y = mx + c at (–1, 2).

Slope of segment O(0,0) to (–1,2) = (2 – 0)/(–1 – 0) = –2.

Since this segment is perpendicular to y = mx + c (slope m):
m × (–2) = –1 => m = 1/2.

Since (–1, 2) lies on y = mx + c:
2 = (1/2)(–1) + c
2 = –1/2 + c

c = 5/2.

Answer: m = 1/2 and c = 5/2.

Q15 If p and q are the lengths of perpendiculars from the origin to the lines x cos θ – y sin θ = k cos 2θ and x sec θ + y cosec θ = k, respectively, prove that $p^2 + 4q^2 = k^2$.

SOLUTION

Line 1: x cos θ – y sin θ – k cos 2θ = 0.
p = distance from origin = |0 – 0 – k cos 2θ| / √(cos²θ + sin²θ) = |k cos 2θ| / 1 = |k cos 2θ|.

Line 2: x sec θ + y cosec θ = k, i.e., x/cos θ + y/sin θ = k.
Multiply by sin θ cos θ: x sin θ + y cos θ = k sin θ cos θ.
q = distance from origin = |0 + 0 – k sin θ cos θ| / √(sin²θ + cos²θ) = |k sin θ cos θ|.

Now compute $p^2 + 4q^2$:
$= k^2 \cos^2(2\theta) + 4k^2 \sin^2\theta \cos^2\theta$
$= k^2 \cos^2(2\theta) + 4k^2 (\sin 2\theta / 2)^2$ [since sin 2θ = 2 sin θ cos θ]
$= k^2 \cos^2(2\theta) + k^2 \sin^2(2\theta)$
$= k^2(\cos^2(2\theta) + \sin^2(2\theta))$
$= k^2$. Hence proved.

Answer: p = |k cos 2θ|, q = |k sin θ cos θ| = |k sin 2θ/2|. So $p^2 + 4q^2 = k^2\cos^2 2\theta + k^2\sin^2 2\theta = k^2$. Proved.

Q16 In the triangle ABC with vertices A(2, 3), B(4, –1) and C(1, 2), find the equation and length of altitude from the vertex A.

SOLUTION

The altitude from A(2, 3) is perpendicular to BC.

Slope of BC = (2 – (–1))/(1 – 4) = 3/(–3) = –1.

Slope of altitude from A = –1/(–1) = 1.

Equation of altitude through A(2, 3) with slope 1:
y – 3 = 1·(x – 2)
x – y + 1 = 0.

Length of altitude = perpendicular distance from A(2, 3) to line BC.
Equation of BC: line through B(4, –1) with slope –1:
$y + 1 = -1(x - 4) \Rightarrow x + y - 3 = 0$.

Distance from A(2, 3) to $x + y - 3 = 0$:
$d = |2 + 3 - 3| / \sqrt{(1 + 1)} = |2| / \sqrt{2} = 2/\sqrt{2} = \sqrt{2}$ units.

Answer: Equation of altitude: $x - y + 1 = 0$. Length = $\sqrt{2}$ units.

Q17 If p is the length of perpendicular from the origin to the line whose intercepts on the axes are a and b, then show that $1/p^2 = 1/a^2 + 1/b^2$.

SOLUTION

The line with x-intercept a and y-intercept b has equation $x/a + y/b = 1$, or $bx + ay - ab = 0$.

Perpendicular distance from the origin:
$p = |b(0) + a(0) - ab| / \sqrt{(b^2 + a^2)} = ab / \sqrt{(a^2 + b^2)}$.

So $p^2 = a^2b^2 / (a^2 + b^2)$.

Therefore:
$1/p^2 = (a^2 + b^2) / (a^2b^2) = a^2/(a^2b^2) + b^2/(a^2b^2) = 1/b^2 + 1/a^2$.

Hence $1/p^2 = 1/a^2 + 1/b^2$. Proved.

Answer: $p = ab/\sqrt{(a^2+b^2)}$, so $1/p^2 = (a^2+b^2)/(a^2b^2) = 1/b^2 + 1/a^2 = 1/a^2 + 1/b^2$. Proved.

Misc — Miscellaneous Exercise

Q1 Find the values of k for which the line $(k-3)x - (4-k^2)y + k^2-7k+6 = 0$ is
(a) Parallel to the x-axis,
(b) Parallel to the y-axis,
(c) Passing through the origin.

SOLUTION

(a) Parallel to the x-axis means the coefficient of x is 0 (horizontal line):
$k - 3 = 0 \Rightarrow k = 3$.
Check: at k=3, the equation becomes $0\cdot x - (4-9)y + 9-21+6 = 0 \Rightarrow 5y - 6 = 0$, which is horizontal. ✓

(b) Parallel to the y-axis means the coefficient of y is 0 (vertical line):
$-(4 - k^2) = 0 \Rightarrow k^2 = 4 \Rightarrow k = \pm 2$.
Check: at k = 2: $(2-3)x + 4-14+6 = 0 \Rightarrow -x - 4 = 0 \Rightarrow x = -4$, vertical. ✓
At k = –2: $(-2-3)x + 4+14+6 = 0 \Rightarrow -5x + 24 = 0 \Rightarrow x = 24/5$, vertical. ✓

(c) Passing through the origin: substitute x = 0, y = 0:
$0 - 0 + k^2 - 7k + 6 = 0$
$k^2 - 7k + 6 = 0$
$(k - 1)(k - 6) = 0$
k = 1 or k = 6.

Answer: (a) k = 3. (b) k = ±2. (c) k = 1 or k = 6.

Q2 Find the equations of the lines, which cut-off intercepts on the axes whose sum and product are 1 and –6, respectively.

SOLUTION

Let the x-intercept be a and y-intercept be b.
Given: $a + b = 1$ and $a\cdot b = -6$.

So a and b are roots of $t^2 - (a+b)t + ab = 0$:
$t^2 - t - 6 = 0$
$(t - 3)(t + 2) = 0$
t = 3 or t = –2.

Case 1: a = 3, b = –2.
Intercept form: $x/3 + y/(-2) = 1 \Rightarrow 2x - 3y - 6 = 0$ (multiply by 6, then simplify).

Case 2: a = –2, b = 3.

Intercept form: $x/(-2) + y/3 = 1 \Rightarrow -3x + 2y - 6 = 0 \Rightarrow 3x - 2y + 6 = 0$.

Verify: $a + b = 3 + (-2) = 1$ ✓, $a \cdot b = 3 \times (-2) = -6$ ✓ for both cases.

Answer: $2x - 3y - 6 = 0$ and $3x - 2y + 6 = 0$.

Q3 What are the points on the y-axis whose distance from the line $x/3 + y/4 = 1$ is 4 units.

SOLUTION

Rewrite $x/3 + y/4 = 1$ as: $4x + 3y - 12 = 0$.

A point on the y-axis is $(0, b)$.

Distance from $(0, b)$ to the line:
$d = |4(0) + 3b - 12| / \sqrt{(16 + 9)} = |3b - 12| / 5 = 4$.

$|3b - 12| = 20$.

Case 1: $3b - 12 = 20 \Rightarrow b = 32/3$.
Case 2: $3b - 12 = -20 \Rightarrow b = -8/3$.

Answer: The points are $(0, 32/3)$ and $(0, -8/3)$.

Q4 Find perpendicular distance from the origin to the line joining the points $(\cos \theta, \sin \theta)$ and $(\cos \varphi, \sin \varphi)$.

SOLUTION

Let $A = (\cos \theta, \sin \theta)$ and $B = (\cos \varphi, \sin \varphi)$.

Slope of AB: $m = (\sin \varphi - \sin \theta)/(\cos \varphi - \cos \theta)$.

Using sum-to-product identities:
$\sin \varphi - \sin \theta = 2 \cos((\varphi+\theta)/2) \sin((\varphi-\theta)/2)$,
$\cos \varphi - \cos \theta = -2 \sin((\varphi+\theta)/2) \sin((\varphi-\theta)/2)$.

$m = -\cos((\varphi+\theta)/2) / \sin((\varphi+\theta)/2)$.

Let $s = (\varphi+\theta)/2$. The line through A with this slope:
$y - \sin \theta = -(\cos s/\sin s)(x - \cos \theta)$
$\sin s\,(y - \sin \theta) = -\cos s\,(x - \cos \theta)$
$\cos s \cdot x + \sin s \cdot y = \cos s \cos \theta + \sin s \sin \theta = \cos(s - \theta) = \cos((\varphi-\theta)/2)$.

Line equation: $(\cos s)x + (\sin s)y - \cos((\varphi-\theta)/2) = 0$.

Distance from origin:

d =	cos s·0 + sin s·0 – cos((φ–θ)/2)	/ √(cos²s + sin²s)
=	cos((φ–θ)/2)	/ 1
=	cos((φ–θ)/2)	.

Answer: Perpendicular distance = |cos((φ – θ)/2)|.

Q5 Find the equation of the line parallel to y-axis and drawn through the point of intersection of the lines x – 7y + 5 = 0 and 3x + y = 0.

SOLUTION

Find the intersection of x – 7y + 5 = 0 and 3x + y = 0.

From 3x + y = 0: y = –3x.
Substitute into x – 7y + 5 = 0:
x – 7(–3x) + 5 = 0
x + 21x + 5 = 0
22x = –5
x = –5/22.
y = –3(–5/22) = 15/22.

A line parallel to the y-axis through (–5/22, 15/22) has equation:
x = –5/22
or equivalently, 22x + 5 = 0.

Answer: 22x + 5 = 0.

Q6 Find the equation of a line drawn perpendicular to the line x/4 + y/6 = 1 through the point, where it meets the y-axis.

SOLUTION

Line x/4 + y/6 = 1 meets the y-axis when x = 0:
0 + y/6 = 1 => y = 6.
So the meeting point is (0, 6).

Rewrite x/4 + y/6 = 1 as 6x + 4y = 24 => 3x + 2y = 12.
Slope = –3/2.

The perpendicular slope = 2/3.

Line through (0, 6) with slope 2/3:

y – 6 = (2/3)(x – 0)
3y – 18 = 2x
2x – 3y + 18 = 0.

Answer: 2x – 3y + 18 = 0.

Q7 Find the area of the triangle formed by the lines y – x = 0, x + y = 0 and x – k = 0.

SOLUTION

The three lines are y = x, y = –x, and x = k.

Intersection of y = x and y = –x: x = –x => x = 0, y = 0. Vertex A = (0, 0).

Intersection of y = x and x = k: y = k. Vertex B = (k, k).

Intersection of y = –x and x = k: y = –k. Vertex C = (k, –k).

Area of triangle with vertices A(0,0), B(k,k), C(k,–k):

= (1/2)	x_A(y_B – y_C) + x_B(y_C – y_A) + x_C(y_A – y_B)	
= (1/2)	0(k–(–k)) + k(–k–0) + k(0–k)	
= (1/2)	0 – k² – k²	

= (1/2)(2k²) = k².

Answer: Area = k² square units.

Q8 Find the value of p so that the three lines 3x + y – 2 = 0, px + 2y – 3 = 0 and 2x – y – 3 = 0 may intersect at one point.

SOLUTION

Find the intersection of 3x + y – 2 = 0 and 2x – y – 3 = 0.

Adding: 5x – 5 = 0 => x = 1.
From 3x + y – 2 = 0: 3(1) + y – 2 = 0 => y = –1.
Intersection point = (1, –1).

For the three lines to be concurrent, (1, –1) must satisfy px + 2y – 3 = 0:
p(1) + 2(–1) – 3 = 0
p – 2 – 3 = 0
p = 5.

Answer: p = 5.

Q9 If three lines whose equations are y = m1x + c1, y = m2x + c2 and y = m3x + c3 are concurrent, then show that m1(c2 – c3) + m2(c3 – c1) + m3(c1 – c2) = 0.

SOLUTION

Find the intersection of y = m1x + c1 and y = m2x + c2 (assuming m1 ≠ m2):
m1x + c1 = m2x + c2
x(m1 – m2) = c2 – c1
x = (c2 – c1)/(m1 – m2).
y = m1(c2 – c1)/(m1 – m2) + c1 = (m1c2 – m2c1)/(m1 – m2).

For the third line y = m3x + c3 to pass through this point:
(m1c2 – m2c1)/(m1 – m2) = m3 · (c2 – c1)/(m1 – m2) + c3.

Multiply through by (m1 – m2):
m1c2 – m2c1 = m3(c2 – c1) + c3(m1 – m2).

Rearrange:
m1c2 – m2c1 – m3c2 + m3c1 – m1c3 + m2c3 = 0
m1(c2 – c3) + m2(c3 – c1) + m3(c1 – c2) = 0. Proved.

Answer: Substituting the intersection of lines 1 and 2 into line 3 yields m1(c2–c3) + m2(c3–c1) + m3(c1–c2) = 0. Proved.

Q10 Find the equation of the lines through the point (3, 2) which make an angle of 45° with the line x – 2y = 3.

SOLUTION

Slope of x – 2y = 3 is m0 = 1/2.

Let slope of the required line be m. With θ = 45°:
tan 45° = |(m – 1/2)/(1 + m/2)| = 1.

Case 1: (m – 1/2)/(1 + m/2) = 1
m – 1/2 = 1 + m/2
m/2 = 3/2 => m = 3.

Case 2: (m – 1/2)/(1 + m/2) = –1
m – 1/2 = –1 – m/2
3m/2 = –1/2 => m = –1/3.

Line 1 (m = 3) through (3, 2):
y – 2 = 3(x – 3) => y – 2 = 3x – 9 => 3x – y – 7 = 0.

Line 2 (m = –1/3) through (3, 2):
y – 2 = (–1/3)(x – 3) => 3y – 6 = –x + 3 => x + 3y – 9 = 0.

Answer: 3x – y – 7 = 0 and x + 3y – 9 = 0.

Q11 Find the equation of the line passing through the point of intersection of the lines 4x + 7y – 3 = 0 and 2x – 3y + 1 = 0 that has equal intercepts on the axes.

SOLUTION

Find the intersection of 4x + 7y – 3 = 0 and 2x – 3y + 1 = 0.

From 2x – 3y + 1 = 0: 4x – 6y + 2 = 0.
Subtract from 4x + 7y – 3 = 0: 13y – 5 = 0 => y = 5/13.
2x = 3(5/13) – 1 = 15/13 – 13/13 = 2/13 => x = 1/13.
Intersection point = (1/13, 5/13).

Case 1: Equal intercepts (a = b), so the line is x + y = a (x-intercept = y-intercept).
Through (1/13, 5/13): 1/13 + 5/13 = a => a = 6/13.
Line: x + y = 6/13, i.e. 13x + 13y – 6 = 0.

Case 2: Both intercepts are 0 (line through origin).
Slope = (5/13)/(1/13) = 5, line: y = 5x.

Both are valid. The non-degenerate case: 13x + 13y – 6 = 0.

Answer: 13x + 13y – 6 = 0 (or equivalently x + y = 6/13), and also y = 5x.

Q12 Show that the equation of the line passing through the origin and making an angle θ with the line y = mx + c is y/x = (m ± tan θ)/(1 ∓ m tan θ).

SOLUTION

Let the required line through the origin have slope M, so y = Mx.

The angle between y = Mx and y = mx + c (slope m) satisfies:
tan θ = |(M – m)/(1 + mM)|.

Case 1: (M – m)/(1 + mM) = tan θ
M – m = tan θ (1 + mM)
M – mM tan θ = m + tan θ
M(1 – m tan θ) = m + tan θ

$M = (m + \tan\theta)/(1 - m\tan\theta)$.

Case 2: $(M - m)/(1 + mM) = -\tan\theta$
$M - m = -\tan\theta\,(1 + mM)$
$M + mM\tan\theta = m - \tan\theta$
$M(1 + m\tan\theta) = m - \tan\theta$
$M = (m - \tan\theta)/(1 + m\tan\theta)$.

Since $M = y/x$ (line through origin), both cases give:
$y/x = (m + \tan\theta)/(1 - m\tan\theta)$ or $y/x = (m - \tan\theta)/(1 + m\tan\theta)$.

Combining: $y/x = (m \pm \tan\theta)/(1 \mp m\tan\theta)$. Proved.

Answer: Both cases of the angle formula give $y/x = (m \pm \tan\theta)/(1 \mp m\tan\theta)$. Proved.

Q13 In what ratio, the line joining $(-1, 1)$ and $(5, 7)$ is divided by the line $x + y = 4$?

SOLUTION

Parametrise the segment from A(–1, 1) to B(5, 7):
Point $P = (-1 + 6t, 1 + 6t)$ for $t \in [0, 1]$.

P lies on $x + y = 4$:
$(-1 + 6t) + (1 + 6t) = 4$
$12t = 4$
$t = 1/3$.

Division point $= (-1 + 6\cdot(1/3), 1 + 6\cdot(1/3)) = (1, 3)$.

Ratio AP:PB $= t:(1-t) = (1/3):(2/3) = 1:2$.

Answer: The line $x + y = 4$ divides the segment in the ratio 1:2 at the point (1, 3).

Q14 Find the distance of the line $4x + 7y + 5 = 0$ from the point (1, 2) along the line $2x - y = 0$.

SOLUTION

The line $2x - y = 0$ (i.e. $y = 2x$) passes through the origin with slope 2.

We need the distance from (1, 2) to $4x + 7y + 5 = 0$ measured along the line $2x - y = 0$.

Find the point Q where the line through (1, 2) with slope 2 meets $4x + 7y + 5 = 0$.

The line through (1, 2) parallel to 2x – y = 0 has slope 2: y – 2 = 2(x – 1) => y = 2x. Since 2(1) – 2 = 0, the point (1, 2) already lies on 2x – y = 0, so the measuring line is 2x – y = 0 itself.

Intersection of 2x – y = 0 and 4x + 7y + 5 = 0:
From y = 2x: 4x + 7(2x) + 5 = 0 => 4x + 14x + 5 = 0 => 18x = –5 => x = –5/18.
y = 2(–5/18) = –5/9.
So Q = (–5/18, –5/9).

Distance from (1, 2) to Q(–5/18, –5/9):
d = $\sqrt{((1 - (-5/18))^2 + (2 - (-5/9))^2)}$
= $\sqrt{((23/18)^2 + (23/9)^2)}$
= $\sqrt{(529/324 + 529/81)}$
= $\sqrt{(529/324 + 2116/324)}$
= $\sqrt{(2645/324)}$
= $(23\sqrt{5})/18$.

Answer: Distance = $23\sqrt{5}/18$ units.

Q15 Find the direction in which a straight line must be drawn through the point (–1, 2) so that its point of intersection with the line x + y = 4 may be at a distance of 3 units from this point.

SOLUTION

Let the line through P(–1, 2) make angle α with the x-axis.
Parametric form: (x, y) = (–1 + t cos α, 2 + t sin α).

At distance 3 from P: t = 3.
The intersection point is at (–1 + 3 cos α, 2 + 3 sin α).

This point lies on x + y = 4:
(–1 + 3 cos α) + (2 + 3 sin α) = 4
1 + 3(cos α + sin α) = 4
3(cos α + sin α) = 3
cos α + sin α = 1.

Squaring: $\cos^2\alpha + 2 \sin \alpha \cos \alpha + \sin^2\alpha = 1$
1 + sin 2α = 1
sin 2α = 0
2α = 0° or 180°
α = 0° or 90°.

So the line must be drawn horizontally (parallel to x-axis) or vertically (parallel to y-axis).

Answer: The line must be drawn either parallel to the x-axis (angle 0°) or parallel to the y-axis (angle 90°).

Q16 The hypotenuse of a right angled triangle has its ends at the points (1, 3) and (–4, 1). Find an equation of the legs (perpendicular sides) of the triangle which are parallel to the axes.

SOLUTION

The legs are parallel to the axes, so they are horizontal and vertical lines.

The right-angle vertex C is at the intersection of a horizontal and vertical leg.
If one leg is horizontal through one endpoint and the other leg is vertical through the other endpoint, they meet at a corner.

Possibility 1: C = (1, 1). Legs: x = 1 (vertical from C to (1,3)) and y = 1 (horizontal from C to (–4,1)).
Verify: CA is vertical (x=1, from (1,1) to (1,3)), CB is horizontal (y=1, from (1,1) to (–4,1)). Right angle at C ✓.

Possibility 2: C = (–4, 3). Legs: x = –4 (vertical from C to (–4,1)) and y = 3 (horizontal from C to (1,3)).
Verify: CA is horizontal (y=3, from (–4,3) to (1,3)), CB is vertical (x=–4, from (–4,3) to (–4,1)). Right angle at C ✓.

Answer: x = 1 and y = 1, or x = –4 and y = 3.

Q17 Find the image of the point (3, 8) with respect to the line x + 3y = 7 assuming the line to be a plane mirror.

SOLUTION

Let Q(h, k) be the image of P(3, 8) in the line x + 3y = 7.

Condition 1: PQ is perpendicular to x + 3y = 7 (slope of line = –1/3, so slope of PQ = 3):
(k – 8)/(h – 3) = 3 => k – 8 = 3(h – 3) = 3h – 9 => 3h – k = 1. ...(1)

Condition 2: Midpoint of PQ lies on x + 3y = 7:
((h+3)/2) + 3·((k+8)/2) = 7

(h + 3 + 3k + 24)/2 = 7
h + 3k + 27 = 14
h + 3k = –13. ...(2)

From (1): k = 3h – 1. Substitute in (2):
h + 3(3h – 1) = –13
h + 9h – 3 = –13
10h = –10 => h = –1.
k = 3(–1) – 1 = –4.

Answer: Image of (3, 8) = (–1, –4).

Q18 If the lines y = 3x + 1 and 2y = x + 3 are equally inclined to the line y = mx + 4, find the value of m.

SOLUTION

Slopes: m1 = 3 (from y = 3x + 1), m2 = 1/2 (from 2y = x + 3).

Both lines are equally inclined to y = mx + 4 (slope m), meaning the acute angles that y = 3x+1 and y = (1/2)x+3 make with y = mx+4 are equal:
|(3 – m)/(1 + 3m)| = |(1/2 – m)/(1 + m/2)| = |(1 – 2m)/(2 + m)|.

Case 1 (same sign): (3 – m)/(1 + 3m) = (1 – 2m)/(2 + m).
(3 – m)(2 + m) = (1 – 2m)(1 + 3m)
6 + 3m – 2m – m² = 1 + 3m – 2m – 6m²
6 + m – m² = 1 + m – 6m²
5m² + 5 = 0. No real solutions.

Case 2 (opposite signs): (3 – m)/(1 + 3m) = –(1 – 2m)/(2 + m) = (2m – 1)/(2 + m).
(3 – m)(2 + m) = (2m – 1)(1 + 3m)
6 + 3m – 2m – m² = 2m + 6m² – 1 – 3m
6 + m – m² = 6m² – m – 1
7m² – 2m – 7 = 0
m = (2 ± √(4 + 196))/14 = (2 ± √200)/14 = (2 ± 10√2)/14 = (1 ± 5√2)/7.

Answer: m = (1 + 5√2)/7 or m = (1 – 5√2)/7.

Q19 If sum of the perpendicular distances of a variable point P(x, y) from the lines $x + y – 5 = 0$ and $3x – 2y + 7 = 0$ is always 10. Show that P must move on a line.

SOLUTION

Perpendicular distances from P(x, y):

d1 =	$x + y – 5$	/ √2,
d2 =	$3x – 2y + 7$	/ √13.

Given: $d1 + d2 = 10$.

Depending on the signs of the expressions inside the absolute values, we get four possible cases, each giving a linear equation in x and y.

For example, if $x + y – 5 > 0$ and $3x – 2y + 7 > 0$:
$(x + y – 5)/\sqrt{2} + (3x – 2y + 7)/\sqrt{13} = 10$,
which, after multiplying by $\sqrt{2}\cdot\sqrt{13}$, gives a linear equation in x and y.

Similarly for other sign combinations, each case yields a linear equation.

Therefore, in every case, the locus of P is a straight line. Hence proved.

Answer: Each choice of signs for the absolute values yields a linear equation in x and y, so P's locus is always a straight line. Proved.

Q20 Find equation of the line which is equidistant from parallel lines $9x + 6y – 7 = 0$ and $3x + 2y + 6 = 0$.

SOLUTION

Write both lines with the same coefficients for comparison.
$3x + 2y + 6 = 0$ can be rewritten as $9x + 6y + 18 = 0$ (multiply by 3).

Now the two parallel lines are $9x + 6y – 7 = 0$ and $9x + 6y + 18 = 0$.
Here $A = 9$, $B = 6$, $C1 = –7$, $C2 = 18$.

The line equidistant from both parallel lines has constant $C = (C1 + C2)/2 = (–7 + 18)/2 = 11/2$.

So the line is: $9x + 6y + 11/2 = 0$, i.e. $18x + 12y + 11 = 0$.

Verify: distance from 18x+12y+11=0 to 18x+12y–14=0:	11–(–14)	/√(324+144) = 25/√468.
Distance to 18x+12y+36=0:	11–36	/√468 = 25/√468. Equal. ✓

Answer: 18x + 12y + 11 = 0.

Q21 A ray of light passing through the point (1, 2) reflects on the x-axis at point A and the reflected ray passes through the point (5, 3). Find the coordinates of A.

SOLUTION

When a ray reflects off the x-axis, the angle of incidence equals the angle of reflection.
Equivalently, the reflected ray appears to come from the image of (1, 2) in the x-axis, which is (1, –2).

The reflected ray passes through A (on x-axis) and (5, 3).
The incident ray from (1, 2) passes through A, so A lies on the line from (1, –2) to (5, 3).

Slope of the line through (1, –2) and (5, 3):
m = (3 – (–2))/(5 – 1) = 5/4.

Equation of this line:
y – (–2) = (5/4)(x – 1)
4(y + 2) = 5(x – 1)
4y + 8 = 5x – 5
5x – 4y – 13 = 0.

A is on the x-axis, so y = 0:
5x – 0 – 13 = 0 => x = 13/5.

A = (13/5, 0).

Answer: A = (13/5, 0).

Q22 Prove that the product of the lengths of the perpendiculars drawn from the points $(\sqrt{(a^2-b^2)}, 0)$ and $(-\sqrt{(a^2-b^2)}, 0)$ to the line $(x/a)\cos\theta + (y/b)\sin\theta = 1$ is b^2.

SOLUTION

Let $c = \sqrt{(a^2 - b^2)}$. The two points are $(c, 0)$ and $(-c, 0)$.

Rewrite the line in standard form by multiplying through by ab:
$(b\cos\theta)x + (a\sin\theta)y - ab = 0$.

Distance p1 from (c, 0):

$p1 =$	$b\cos\theta \cdot c + a\sin\theta \cdot 0 - ab$	$/ \sqrt{(b^2\cos^2\theta + a^2\sin^2\theta)}$
$=$	$bc\cos\theta - ab$	$/ \sqrt{(b^2\cos^2\theta + a^2\sin^2\theta)}$
$= b$	$c\cos\theta - a$	$/ \sqrt{(b^2\cos^2\theta + a^2\sin^2\theta)}.$

Distance p2 from (–c, 0):

$p2 =$	$b\cos\theta \cdot (-c) - ab$	$/ \sqrt{(b^2\cos^2\theta + a^2\sin^2\theta)}$
$= b$	$c\cos\theta + a$	$/ \sqrt{(b^2\cos^2\theta + a^2\sin^2\theta)}.$

Product:

$p1 \cdot p2 = b^2$	$c\cos\theta - a$		$c\cos\theta + a$	$/ (b^2\cos^2\theta + a^2\sin^2\theta)$
$= b^2$	$c^2\cos^2\theta - a^2$	$/ (b^2\cos^2\theta + a^2\sin^2\theta)$		
$= b^2$	$(a^2-b^2)\cos^2\theta - a^2$	$/ (b^2\cos^2\theta + a^2\sin^2\theta)$ [since $c^2 = a^2-b^2$]		
$= b^2$	$a^2\cos^2\theta - b^2\cos^2\theta - a^2$	$/ (b^2\cos^2\theta + a^2\sin^2\theta)$		
$= b^2$	$-b^2\cos^2\theta - a^2(1 - \cos^2\theta)$	$/ (b^2\cos^2\theta + a^2\sin^2\theta)$		
$= b^2$	$-(b^2\cos^2\theta + a^2\sin^2\theta)$	$/ (b^2\cos^2\theta + a^2\sin^2\theta)$		

$= b^2 \cdot (b^2\cos^2\theta + a^2\sin^2\theta) / (b^2\cos^2\theta + a^2\sin^2\theta)$

$= b^2$. Hence proved.

Answer: The product of perpendicular distances = b^2. Proved.

Q23 A person standing at the junction (crossing) of two straight paths represented by the equations 2x – 3y + 4 = 0 and 3x + 4y – 5 = 0 wants to reach the path whose equation is 6x – 7y + 8 = 0 in the least time. Find equation of the path that he should follow.

SOLUTION

Find the person's position (intersection of the two paths).

2x – 3y + 4 = 0 ...(1)
3x + 4y – 5 = 0 ...(2)

Multiply (1) by 4 and (2) by 3:
8x – 12y + 16 = 0
9x + 12y – 15 = 0
Adding: 17x + 1 = 0 => x = –1/17.

From (1): 2(–1/17) – 3y + 4 = 0 => –2/17 + 4 = 3y => 3y = 66/17 => y = 22/17.

Person's position: P = (–1/17, 22/17).

To reach path 6x – 7y + 8 = 0 in least time, follow the shortest (perpendicular) path.

Slope of 6x – 7y + 8 = 0: m = 6/7.
Slope of perpendicular: m_perp = –7/6.

Line through P(–1/17, 22/17) with slope –7/6:
y – 22/17 = (–7/6)(x – (–1/17))
6(17y – 22) = –7(17x + 1)·(17/17)... Multiply both sides by 6·17:
6·17(y – 22/17) = –7·17(x + 1/17)
6(17y – 22) = –7(17x + 1)
102y – 132 = –119x – 7
119x + 102y – 125 = 0.

Answer: 119x + 102y – 125 = 0.