Chapter 10: Conic Sections

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Exercise 10.1 — Exercise 10.1

Q1 Find the equation of the circle with centre (0, 2) and radius 2.

SOLUTION

The standard equation of a circle with centre (h, k) and radius r is $(x - h)^2 + (y - k)^2 = r^2$.

Here h = 0, k = 2, r = 2.

Substituting: $(x - 0)^2 + (y - 2)^2 = 4$

Answer: $x^2 + (y - 2)^2 = 4$

Q2 Find the equation of the circle with centre (–2, 3) and radius 4.

SOLUTION

Here h = -2, k = 3, r = 4.

Using $(x - h)^2 + (y - k)^2 = r^2$:

$(x - (-2))^2 + (y - 3)^2 = 16$

Answer: $(x + 2)^2 + (y - 3)^2 = 16$

Q3 Find the equation of the circle with centre (1/2, 1/4) and radius 1/12.

SOLUTION

Here h = 1/2, k = 1/4, r = 1/12.

Using $(x - h)^2 + (y - k)^2 = r^2$:

$(x - 1/2)^2 + (y - 1/4)^2 = (1/12)^2 = 1/144$

Answer: $(x - 1/2)^2 + (y - 1/4)^2 = 1/144$

Q4 Find the equation of the circle with centre (1, 1) and radius $\sqrt{2}$.

SOLUTION

Here $h = 1, k = 1, r = \sqrt{2}$.

Using $(x - h)^2 + (y - k)^2 = r^2$:

$(x - 1)^2 + (y - 1)^2 = (\sqrt{2})^2 = 2$

Answer: $(x - 1)^2 + (y - 1)^2 = 2$

Q5 Find the equation of the circle with centre (–a, –b) and radius $\sqrt{(a^2 – b^2)}$.

SOLUTION

Here $h = -a, k = -b, r = \sqrt{(a^2 - b^2)}$.

Using $(x - h)^2 + (y - k)^2 = r^2$:

$(x - (-a))^2 + (y - (-b))^2 = a^2 - b^2$

Answer: $(x + a)^2 + (y + b)^2 = a^2 – b^2$

Q6 Find the centre and radius of the circle $(x + 5)^2 + (y – 3)^2 = 36$.

SOLUTION

Rewrite in standard form: $(x - (-5))^2 + (y - 3)^2 = 6^2$.

Comparing with $(x - h)^2 + (y - k)^2 = r^2$, we get $h = -5, k = 3, r = 6$.

Answer: Centre (–5, 3), radius 6

Q7 Find the centre and radius of the circle $x^2 + y^2 – 4x – 8y – 45 = 0$.

SOLUTION

Rearrange: $(x^2 - 4x) + (y^2 - 8y) = 45$.

Complete the square for x: $x^2 - 4x + 4 = (x - 2)^2$.
Complete the square for y: $y^2 - 8y + 16 = (y - 4)^2$.

Add 4 and 16 to both sides:

$(x - 2)^2 + (y - 4)^2 = 45 + 4 + 16 = 65$.

So $h = 2$, $k = 4$, $r = \sqrt{65}$.

Answer: Centre (2, 4), radius $\sqrt{65}$

Q8 Find the centre and radius of the circle $x^2 + y^2 – 8x + 10y – 12 = 0$.

SOLUTION

Rearrange: $(x^2 - 8x) + (y^2 + 10y) = 12$.

Complete the square: $(x - 4)^2 - 16 + (y + 5)^2 - 25 = 12$.

$(x - 4)^2 + (y + 5)^2 = 12 + 16 + 25 = 53$.

So $h = 4$, $k = -5$, $r = \sqrt{53}$.

Answer: Centre (4, –5), radius $\sqrt{53}$

Q9 Find the centre and radius of the circle $2x^2 + 2y^2 – x = 0$.

SOLUTION

Divide throughout by 2: $x^2 + y^2 - x/2 = 0$.

Rearrange: $(x^2 - x/2) + y^2 = 0$.

Complete the square: $(x - 1/4)^2 - 1/16 + y^2 = 0$.

$(x - 1/4)^2 + y^2 = 1/16$.

So $h = 1/4$, $k = 0$, $r = 1/4$.

Answer: Centre (1/4, 0), radius 1/4

Q10 Find the equation of the circle passing through the points (4, 1) and (6, 5) and whose centre is on the line $4x + y = 16$.

SOLUTION

Let the equation of the circle be $(x - h)^2 + (y - k)^2 = r^2$.

Since the circle passes through (4, 1): $(4 - h)^2 + (1 - k)^2 = r^2$ ... (1)
Since the circle passes through (6, 5): $(6 - h)^2 + (5 - k)^2 = r^2$ ... (2)
Centre lies on $4x + y = 16$: $4h + k = 16$ ... (3)

From (1) = (2):
$(4 - h)^2 + (1 - k)^2 = (6 - h)^2 + (5 - k)^2$

$16 - 8h + h^2 + 1 - 2k + k^2 = 36 - 12h + h^2 + 25 - 10k + k^2$

$17 - 8h - 2k = 61 - 12h - 10k$

$4h + 8k = 44 \Rightarrow h + 2k = 11 \ldots (4)$

From (3): $k = 16 - 4h$. Substituting in (4):

$h + 2(16 - 4h) = 11 \Rightarrow h + 32 - 8h = 11 \Rightarrow -7h = -21 \Rightarrow h = 3$

So $k = 16 - 4(3) = 4$.

$r^2 = (4 - 3)^2 + (1 - 4)^2 = 1 + 9 = 10$.

Equation: $(x - 3)^2 + (y - 4)^2 = 10$.

Answer: $(x – 3)^2 + (y – 4)^2 = 10$

Q11 Find the equation of the circle passing through the points (2, 3) and (–1, 1) and whose centre is on the line $x – 3y – 11 = 0$.

SOLUTION

Let the circle be $(x - h)^2 + (y - k)^2 = r^2$.

Passing through (2, 3): $(2 - h)^2 + (3 - k)^2 = r^2 \ldots (1)$

Passing through (-1, 1): $(-1 - h)^2 + (1 - k)^2 = r^2 \ldots (2)$

Centre on $x - 3y - 11 = 0$: $h - 3k = 11 \Rightarrow h = 3k + 11 \ldots (3)$

From (1) = (2):

$4 - 4h + h^2 + 9 - 6k + k^2 = 1 + 2h + h^2 + 1 - 2k + k^2$

$13 - 4h - 6k = 2 + 2h - 2k$

$11 = 6h + 4k \Rightarrow 6h + 4k = 11 \ldots (4)$

Substituting (3) into (4):

$6(3k + 11) + 4k = 11 \Rightarrow 18k + 66 + 4k = 11 \Rightarrow 22k = -55 \Rightarrow k = -5/2$

$h = 3(-5/2) + 11 = -15/2 + 22/2 = 7/2$

$r^2 = (2 - 7/2)^2 + (3 - (-5/2))^2 = (-3/2)^2 + (11/2)^2 = 9/4 + 121/4 = 130/4 = 65/2$

Equation: $(x - 7/2)^2 + (y + 5/2)^2 = 65/2$.

Answer: $(x – 7/2)^2 + (y + 5/2)^2 = 65/2$

Q12 Find the equation of the circle with radius 5 whose centre lies on x-axis and passes through the point (2, 3).

SOLUTION

Since the centre lies on the x-axis, k = 0. Let centre = (h, 0).

Circle passes through (2, 3) with radius 5:
$(2 - h)^2 + (3 - 0)^2 = 25$
$(2 - h)^2 + 9 = 25$
$(2 - h)^2 = 16$
$2 - h = \pm 4$

Case 1: $2 - h = 4 \Rightarrow h = -2$. Equation: $(x + 2)^2 + y^2 = 25$.
Case 2: $2 - h = -4 \Rightarrow h = 6$. Equation: $(x - 6)^2 + y^2 = 25$.

Answer: $(x + 2)^2 + y^2 = 25$ or $(x – 6)^2 + y^2 = 25$

Q13 Find the equation of the circle passing through (0, 0) and making intercepts a and b on the coordinate axes.

SOLUTION

Let the general equation be $x^2 + y^2 + Dx + Ey + F = 0$.

Since the circle passes through (0, 0): $0 + 0 + 0 + 0 + F = 0 \Rightarrow F = 0$.

The circle makes an intercept a on the x-axis, so it passes through (a, 0):
$a^2 + 0 + Da + 0 = 0 \Rightarrow D = -a$.

The circle makes an intercept b on the y-axis, so it passes through (0, b):
$0 + b^2 + 0 + Eb = 0 \Rightarrow E = -b$.

Substituting D = -a, E = -b, F = 0:

Answer: $x^2 + y^2 – ax – by = 0$

Q14 Find the equation of a circle with centre (2, 2) and passes through the point (4, 5).

SOLUTION

The radius is the distance from centre (2, 2) to the point (4, 5):
$r^2 = (4 - 2)^2 + (5 - 2)^2 = 4 + 9 = 13$.

Equation of the circle:

Answer: $(x - 2)^2 + (y - 2)^2 = 13$

Q15 Does the point (–2.5, 3.5) lie inside, outside or on the circle $x^2 + y^2 = 25$?

SOLUTION

For any point (x, y), compare $x^2 + y^2$ with 25:

- If $x^2 + y^2 < 25$, the point lies inside the circle.
- If $x^2 + y^2 = 25$, the point lies on the circle.
- If $x^2 + y^2 > 25$, the point lies outside the circle.

For (-2.5, 3.5):

$(-2.5)^2 + (3.5)^2 = 6.25 + 12.25 = 18.5$

Since $18.5 < 25$, the point lies inside the circle.

Answer: The point (–2.5, 3.5) lies inside the circle $x^2 + y^2 = 25$.

Exercise 10.2 — Exercise 10.2

Q1 Find the coordinates of the focus, axis of the parabola, the equation of the directrix and the length of the latus rectum: $y^2 = 12x$.

SOLUTION

The equation $y^2 = 12x$ is of the form $y^2 = 4ax$, where $4a = 12$, so $a = 3$.

Since the equation involves y^2 with positive coefficient of x, the parabola opens to the right.

Focus: (a, 0) = (3, 0).
Axis: x-axis (y = 0).
Directrix: x = -a => x = -3.
Latus rectum: 4a = 12.

Answer: Focus (3, 0); axis: x-axis; directrix x = –3; length of latus rectum = 12

Q2 Find the coordinates of the focus, axis of the parabola, the equation of the directrix and the length of the latus rectum: $x^2 = 6y$.

SOLUTION

The equation $x^2 = 6y$ is of the form $x^2 = 4ay$, where $4a = 6$, so $a = 3/2$.

Since the equation involves x^2 with positive coefficient of y, the parabola opens upward.

Focus: (0, a) = (0, 3/2).
Axis: y-axis (x = 0).
Directrix: y = -a => y = -3/2.
Latus rectum: 4a = 6.

Answer: Focus (0, 3/2); axis: y-axis; directrix y = –3/2; length of latus rectum = 6

Q3 Find the coordinates of the focus, axis of the parabola, the equation of the directrix and the length of the latus rectum: $y^2 = –8x$.

SOLUTION

The equation $y^2 = -8x$ is of the form $y^2 = -4ax$, where $4a = 8$, so $a = 2$.

The negative sign means the parabola opens to the left.

Focus: (-a, 0) = (-2, 0).

Axis: x-axis (y = 0).
Directrix: x = a => x = 2.
Latus rectum: 4a = 8.

Answer: Focus (–2, 0); axis: x-axis; directrix x = 2; length of latus rectum = 8

Q4 Find the coordinates of the focus, axis of the parabola, the equation of the directrix and the length of the latus rectum: $x^2 = -16y$.

SOLUTION

The equation $x^2 = -16y$ is of the form $x^2 = -4ay$, where $4a = 16$, so $a = 4$.

The negative sign means the parabola opens downward.

Focus: (0, -a) = (0, -4).
Axis: y-axis (x = 0).
Directrix: y = a => y = 4.
Latus rectum: 4a = 16.

Answer: Focus (0, –4); axis: y-axis; directrix y = 4; length of latus rectum = 16

Q5 Find the coordinates of the focus, axis of the parabola, the equation of the directrix and the length of the latus rectum: $y^2 = 10x$.

SOLUTION

The equation $y^2 = 10x$ is of the form $y^2 = 4ax$, where $4a = 10$, so $a = 5/2$.

The parabola opens to the right.

Focus: (a, 0) = (5/2, 0).
Axis: x-axis (y = 0).
Directrix: x = -a => x = -5/2.
Latus rectum: 4a = 10.

Answer: Focus (5/2, 0); axis: x-axis; directrix x = –5/2; length of latus rectum = 10

Q6 Find the coordinates of the focus, axis of the parabola, the equation of the directrix and the length of the latus rectum: $x^2 = -9y$.

SOLUTION

The equation $x^2 = -9y$ is of the form $x^2 = -4ay$, where $4a = 9$, so $a = 9/4$.

The parabola opens downward.

Focus: $(0, -a) = (0, -9/4)$.
Axis: y-axis ($x = 0$).
Directrix: $y = a \Rightarrow y = 9/4$.
Latus rectum: $4a = 9$.

Answer: Focus $(0, -9/4)$; axis: y-axis; directrix $y = 9/4$; length of latus rectum = 9

Q7 Find the equation of the parabola that satisfies the given conditions: Focus (6, 0); directrix $x = -6$.

SOLUTION

The focus (6, 0) lies on the x-axis, so the axis of the parabola is the x-axis.
The focus is to the right of the directrix ($x = -6$), so the parabola opens to the right.

The vertex is midway between focus and directrix: vertex = (0, 0), and $a = 6$.

Equation: $y^2 = 4ax = 4(6)x = 24x$.

Answer: $y^2 = 24x$

Q8 Find the equation of the parabola that satisfies the given conditions: Focus (0, –3); directrix $y = 3$.

SOLUTION

The focus (0, -3) lies on the y-axis, so the axis of the parabola is the y-axis.
The focus is below the directrix ($y = 3$), so the parabola opens downward.

Vertex at origin, $a = 3$.

Equation: $x^2 = -4ay = -4(3)y = -12y$.

Answer: $x^2 = -12y$

Q9 Find the equation of the parabola that satisfies the given conditions: Vertex (0, 0); focus (3, 0).

SOLUTION

Vertex at origin, focus at (3, 0) on the positive x-axis.
The parabola opens to the right with a = 3.

Equation: $y^2 = 4ax = 4(3)x = 12x$.

Answer: $y^2 = 12x$

Q10 Find the equation of the parabola that satisfies the given conditions: Vertex (0, 0); focus (–2, 0).

SOLUTION

Vertex at origin, focus at (-2, 0) on the negative x-axis.
The parabola opens to the left with a = 2.

Equation: $y^2 = -4ax = -4(2)x = -8x$.

Answer: $y^2 = –8x$

Q11 Find the equation of the parabola that satisfies the given conditions: Vertex (0, 0) passing through (2, 3) and axis is along x-axis.

SOLUTION

Since the axis is along the x-axis and vertex is at origin, the equation is $y^2 = 4ax$ or $y^2 = -4ax$.

The point (2, 3) has x > 0 and y > 0, so the parabola opens to the right: $y^2 = 4ax$.

Substituting (2, 3): $(3)^2 = 4a(2) \Rightarrow 9 = 8a \Rightarrow a = 9/8$.

Equation: $y^2 = 4(9/8)x = (9/2)x$.
Multiplying by 2: $2y^2 = 9x$.

Answer: $2y^2 = 9x$

Q12 Find the equation of the parabola that satisfies the given conditions: Vertex (0, 0), passing through (5, 2) and symmetric with respect to y-axis.

SOLUTION

Since the parabola is symmetric about the y-axis and vertex is at origin, the equation is $x^2 = 4ay$ or $x^2 = -4ay$.

The point (5, 2) has $y > 0$, so the parabola opens upward: $x^2 = 4ay$.

Substituting (5, 2): $(5)^2 = 4a(2) \Rightarrow 25 = 8a \Rightarrow a = 25/8$.

Equation: $x^2 = 4(25/8)y = (25/2)y$.
Multiplying by 2: $2x^2 = 25y$.

Answer: $2x^2 = 25y$

Exercise 10.3 — Exercise 10.3

Q1 Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse: $x^2/36 + y^2/16 = 1$.

SOLUTION

Here $a^2 = 36$, $b^2 = 16$, so $a = 6$, $b = 4$. Since $a > b$, the major axis is along the x-axis.

$c^2 = a^2 - b^2 = 36 - 16 = 20 \Rightarrow c = 2\sqrt{5}$.

Foci: $(\pm c, 0) = (\pm 2\sqrt{5}, 0)$.
Vertices: $(\pm a, 0) = (\pm 6, 0)$.
Major axis length: $2a = 12$.
Minor axis length: $2b = 8$.
Eccentricity: $e = c/a = 2\sqrt{5}/6 = \sqrt{5}/3$.
Latus rectum: $2b^2/a = 2(16)/6 = 32/6 = 16/3$.

Answer: Foci $(\pm 2\sqrt{5}, 0)$; vertices $(\pm 6, 0)$; major axis 12; minor axis 8; $e = \sqrt{5}/3$; latus rectum = 16/3

Q2 Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse: $x^2/4 + y^2/25 = 1$.

SOLUTION

Here the denominators are 4 and 25. Since $25 > 4$, $a^2 = 25$ (under y^2), $b^2 = 4$ (under x^2), so $a = 5$, $b = 2$. The major axis is along the y-axis.

$c^2 = a^2 - b^2 = 25 - 4 = 21 \Rightarrow c = \sqrt{21}$.

Foci: $(0, \pm c) = (0, \pm\sqrt{21})$.
Vertices: $(0, \pm a) = (0, \pm 5)$.
Major axis length: $2a = 10$.
Minor axis length: $2b = 4$.
Eccentricity: $e = c/a = \sqrt{21}/5$.
Latus rectum: $2b^2/a = 2(4)/5 = 8/5$.

Answer: Foci $(0, \pm\sqrt{21})$; vertices $(0, \pm 5)$; major axis 10; minor axis 4; $e = \sqrt{21}/5$; latus rectum = 8/5

Q3 Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse: $x^2/16 + y^2/9 = 1$.

SOLUTION

Here $a^2 = 16$, $b^2 = 9$ (since $16 > 9$), $a = 4$, $b = 3$. Major axis along x-axis.

$c^2 = 16 - 9 = 7 \Rightarrow c = \sqrt{7}$.

Foci: $(\pm\sqrt{7}, 0)$.
Vertices: $(\pm 4, 0)$.
Major axis length: $2a = 8$.
Minor axis length: $2b = 6$.
Eccentricity: $e = \sqrt{7}/4$.
Latus rectum: $2b^2/a = 2(9)/4 = 18/4 = 9/2$.

Answer: Foci $(\pm\sqrt{7}, 0)$; vertices $(\pm 4, 0)$; major axis 8; minor axis 6; $e = \sqrt{7}/4$; latus rectum = 9/2

Q4 Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse: $x^2/25 + y^2/100 = 1$.

SOLUTION

Here the denominators are 25 and 100. Since $100 > 25$, $a^2 = 100$ (under y^2), $b^2 = 25$ (under x^2), $a = 10$, $b = 5$. Major axis along y-axis.

$c^2 = 100 - 25 = 75 \Rightarrow c = 5\sqrt{3}$.

Foci: $(0, \pm 5\sqrt{3})$.
Vertices: $(0, \pm 10)$.
Major axis length: $2a = 20$.
Minor axis length: $2b = 10$.
Eccentricity: $e = 5\sqrt{3}/10 = \sqrt{3}/2$.
Latus rectum: $2b^2/a = 2(25)/10 = 50/10 = 5$.

Answer: Foci $(0, \pm 5\sqrt{3})$; vertices $(0, \pm 10)$; major axis 20; minor axis 10; $e = \sqrt{3}/2$; latus rectum = 5

Q5 Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse: $x^2/49 + y^2/36 = 1$.

SOLUTION

Here $a^2 = 49$, $b^2 = 36$ (since $49 > 36$), $a = 7$, $b = 6$. Major axis along x-axis.

$c^2 = 49 - 36 = 13 \Rightarrow c = \sqrt{13}$.

Foci: $(\pm\sqrt{13}, 0)$.
Vertices: $(\pm 7, 0)$.
Major axis length: $2a = 14$.
Minor axis length: $2b = 12$.
Eccentricity: $e = \sqrt{13}/7$.
Latus rectum: $2b^2/a = 2(36)/7 = 72/7$.

Answer: Foci $(\pm\sqrt{13}, 0)$; vertices $(\pm 7, 0)$; major axis 14; minor axis 12; $e = \sqrt{13}/7$; latus rectum = 72/7

Q6 Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse: $x^2/100 + y^2/400 = 1$.

SOLUTION

Here the denominators are 100 and 400. Since $400 > 100$, $a^2 = 400$ (under y^2), $b^2 = 100$ (under x^2), $a = 20$, $b = 10$. Major axis along y-axis.

$c^2 = 400 - 100 = 300 \Rightarrow c = 10\sqrt{3}$.

Foci: $(0, \pm 10\sqrt{3})$.
Vertices: $(0, \pm 20)$.
Major axis length: $2a = 40$.
Minor axis length: $2b = 20$.
Eccentricity: $e = 10\sqrt{3}/20 = \sqrt{3}/2$.
Latus rectum: $2b^2/a = 2(100)/20 = 200/20 = 10$.

Answer: Foci $(0, \pm 10\sqrt{3})$; vertices $(0, \pm 20)$; major axis 40; minor axis 20; $e = \sqrt{3}/2$; latus rectum = 10

Q7 Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse: $36x^2 + 4y^2 = 144$.

SOLUTION

Divide by 144: $x^2/4 + y^2/36 = 1$.

Here $a^2 = 36$ (under y^2), $b^2 = 4$ (under x^2) since $36 > 4$, $a = 6$, $b = 2$. Major axis along y-axis.

$c^2 = 36 - 4 = 32 \Rightarrow c = 4\sqrt{2}$.

Foci: $(0, \pm 4\sqrt{2})$.
Vertices: $(0, \pm 6)$.
Major axis length: $2a = 12$.
Minor axis length: $2b = 4$.
Eccentricity: $e = 4\sqrt{2}/6 = 2\sqrt{2}/3$.
Latus rectum: $2b^2/a = 2(4)/6 = 8/6 = 4/3$.

Answer: Foci $(0, \pm 4\sqrt{2})$; vertices $(0, \pm 6)$; major axis 12; minor axis 4; $e = 2\sqrt{2}/3$; latus rectum = 4/3

Q8 Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse: $16x^2 + y^2 = 16$.

SOLUTION

Divide by 16: $x^2/1 + y^2/16 = 1$.

Here $a^2 = 16$ (under y^2), $b^2 = 1$ (under x^2), $a = 4$, $b = 1$. Major axis along y-axis.

$c^2 = 16 - 1 = 15 \Rightarrow c = \sqrt{15}$.

Foci: $(0, \pm\sqrt{15})$.
Vertices: $(0, \pm 4)$.
Major axis length: $2a = 8$.
Minor axis length: $2b = 2$.
Eccentricity: $e = \sqrt{15}/4$.
Latus rectum: $2b^2/a = 2(1)/4 = 1/2$.

Answer: Foci $(0, \pm\sqrt{15})$; vertices $(0, \pm 4)$; major axis 8; minor axis 2; $e = \sqrt{15}/4$; latus rectum = 1/2

Q9 Find the coordinates of the foci, the vertices, the length of major axis, the minor axis, the eccentricity and the length of the latus rectum of the ellipse: $4x^2 + 9y^2 = 36$.

SOLUTION

Divide by 36: $x^2/9 + y^2/4 = 1$.

Here $a^2 = 9$ (under x^2), $b^2 = 4$ (under y^2) since $9 > 4$, $a = 3$, $b = 2$. Major axis along x-axis.

$c^2 = 9 - 4 = 5 \Rightarrow c = \sqrt{5}$.

Foci: $(\pm\sqrt{5}, 0)$.
Vertices: $(\pm 3, 0)$.
Major axis length: $2a = 6$.
Minor axis length: $2b = 4$.
Eccentricity: $e = \sqrt{5}/3$.
Latus rectum: $2b^2/a = 2(4)/3 = 8/3$.

Answer: Foci $(\pm\sqrt{5}, 0)$; vertices $(\pm 3, 0)$; major axis 6; minor axis 4; $e = \sqrt{5}/3$; latus rectum = 8/3

Q10 Find the equation for the ellipse that satisfies the given conditions: Vertices $(\pm 5, 0)$, foci $(\pm 4, 0)$.

SOLUTION

Vertices on x-axis, so major axis is along x-axis: $a = 5$, $c = 4$.

$b^2 = a^2 - c^2 = 25 - 16 = 9$.

Answer: $x^2/25 + y^2/9 = 1$

Q11 Find the equation for the ellipse that satisfies the given conditions: Vertices $(0, \pm 13)$, foci $(0, \pm 5)$.

SOLUTION

Vertices on y-axis, so major axis is along y-axis: $a = 13$, $c = 5$.

$b^2 = a^2 - c^2 = 169 - 25 = 144$.

Standard form (major axis along y-axis): $x^2/b^2 + y^2/a^2 = 1$.

Answer: $x^2/144 + y^2/169 = 1$

Q12 Find the equation for the ellipse that satisfies the given conditions: Vertices (±6, 0), foci (±4, 0).

SOLUTION

Vertices on x-axis, so major axis is along x-axis: a = 6, c = 4.

$b^2 = a^2 - c^2 = 36 - 16 = 20$.

Answer: $x^2/36 + y^2/20 = 1$

Q13 Find the equation for the ellipse that satisfies the given conditions: Ends of major axis (±3, 0), ends of minor axis (0, ±2).

SOLUTION

Major axis along x-axis: a = 3, b = 2.

Standard equation: $x^2/a^2 + y^2/b^2 = 1$.

Answer: $x^2/9 + y^2/4 = 1$

Q14 Find the equation for the ellipse that satisfies the given conditions: Ends of major axis (0, ±√5), ends of minor axis (±1, 0).

SOLUTION

Major axis along y-axis: a = √5, b = 1.

Standard equation (major axis along y-axis): $x^2/b^2 + y^2/a^2 = 1 \Rightarrow x^2/1 + y^2/5 = 1$.

Answer: $x^2 + y^2/5 = 1$

Q15 Find the equation for the ellipse that satisfies the given conditions: Length of major axis 26, foci (±5, 0).

SOLUTION

Foci on x-axis: $2a = 26 \Rightarrow a = 13, c = 5$.

$b^2 = a^2 - c^2 = 169 - 25 = 144$.

Answer: $x^2/169 + y^2/144 = 1$

Q16 Find the equation for the ellipse that satisfies the given conditions: Length of minor axis 16, foci (0, ±6).

SOLUTION

Foci on y-axis: 2b = 16 => b = 8, c = 6.

$a^2 = b^2 + c^2 = 64 + 36 = 100$.

Standard equation (major axis along y-axis): $x^2/b^2 + y^2/a^2 = 1$.

Answer: $x^2/64 + y^2/100 = 1$

Q17 Find the equation for the ellipse that satisfies the given conditions: Foci (±3, 0), a = 4.

SOLUTION

Foci on x-axis: c = 3, a = 4.

$b^2 = a^2 - c^2 = 16 - 9 = 7$.

Answer: $x^2/16 + y^2/7 = 1$

Q18 Find the equation for the ellipse that satisfies the given conditions: b = 3, c = 4, centre at the origin; foci on the x-axis.

SOLUTION

Foci on x-axis: c = 4, b = 3.

$a^2 = b^2 + c^2 = 9 + 16 = 25$.

Answer: $x^2/25 + y^2/9 = 1$

Q19 Find the equation for the ellipse that satisfies the given conditions: Centre at (0, 0), major axis on the y-axis and passes through the points (3, 2) and (1, 6).

SOLUTION

Since the major axis is on the y-axis, the equation is $x^2/b^2 + y^2/a^2 = 1$ (with $a > b$).

Substituting (3, 2): $9/b^2 + 4/a^2 = 1$ ... (1)
Substituting (1, 6): $1/b^2 + 36/a^2 = 1$ ... (2)

Let $u = 1/b^2$, $v = 1/a^2$.
$9u + 4v = 1$ ... (1)

$u + 36v = 1$ ... (2)

From (2): $u = 1 - 36v$. Substitute in (1):
$9(1 - 36v) + 4v = 1 \Rightarrow 9 - 324v + 4v = 1 \Rightarrow -320v = -8 \Rightarrow v = 1/40$.

$u = 1 - 36(1/40) = 1 - 9/10 = 1/10$.

So $b^2 = 1/u = 10$, $a^2 = 1/v = 40$.

Answer: $x^2/10 + y^2/40 = 1$

Q20 Find the equation for the ellipse that satisfies the given conditions: Major axis on the x-axis and passes through the points (4, 3) and (6, 2).

SOLUTION

Equation: $x^2/a^2 + y^2/b^2 = 1$.

Substituting (4, 3): $16/a^2 + 9/b^2 = 1$ ... (1)
Substituting (6, 2): $36/a^2 + 4/b^2 = 1$ ... (2)

Let $p = 1/a^2$, $q = 1/b^2$.
$16p + 9q = 1$ ... (1)
$36p + 4q = 1$ ... (2)

Multiply (1) by 4: $64p + 36q = 4$ ... (3)
Multiply (2) by 9: $324p + 36q = 9$ ... (4)
Subtract (3) from (4): $260p = 5 \Rightarrow p = 1/52$.

Substituting back in (1): $16/52 + 9q = 1 \Rightarrow 9q = 1 - 4/13 = 9/13 \Rightarrow q = 1/13$.

So $a^2 = 52$, $b^2 = 13$.

Answer: $x^2/52 + y^2/13 = 1$

Exercise 10.4 — Exercise 10.4

Q1 Find the coordinates of the foci and the vertices, the eccentricity and the length of the latus rectum of the hyperbola: $x^2/16 - y^2/9 = 1$.

SOLUTION

Here $a^2 = 16$, $b^2 = 9$, so $a = 4$, $b = 3$. The transverse axis is along the x-axis.

$c^2 = a^2 + b^2 = 16 + 9 = 25 \Rightarrow c = 5$.

Foci: $(\pm c, 0) = (\pm 5, 0)$.
Vertices: $(\pm a, 0) = (\pm 4, 0)$.
Eccentricity: $e = c/a = 5/4$.
Latus rectum: $2b^2/a = 2(9)/4 = 18/4 = 9/2$.

Answer: Foci $(\pm 5, 0)$; vertices $(\pm 4, 0)$; $e = 5/4$; latus rectum $= 9/2$

Q2 Find the coordinates of the foci and the vertices, the eccentricity and the length of the latus rectum of the hyperbola: $y^2/9 - x^2/27 = 1$.

SOLUTION

Here $a^2 = 9$, $b^2 = 27$, so $a = 3$, $b = 3\sqrt{3}$. The transverse axis is along the y-axis.

$c^2 = a^2 + b^2 = 9 + 27 = 36 \Rightarrow c = 6$.

Foci: $(0, \pm c) = (0, \pm 6)$.
Vertices: $(0, \pm a) = (0, \pm 3)$.
Eccentricity: $e = c/a = 6/3 = 2$.
Latus rectum: $2b^2/a = 2(27)/3 = 18$.

Answer: Foci $(0, \pm 6)$; vertices $(0, \pm 3)$; $e = 2$; latus rectum $= 18$

Q3 Find the coordinates of the foci and the vertices, the eccentricity and the length of the latus rectum of the hyperbola: $9y^2 - 4x^2 = 36$.

SOLUTION

Divide by 36: $y^2/4 - x^2/9 = 1$.

Here $a^2 = 4$, $b^2 = 9$, $a = 2$, $b = 3$. Transverse axis along y-axis.

$c^2 = 4 + 9 = 13 \Rightarrow c = \sqrt{13}$.

Foci: $(0, \pm\sqrt{13})$.
Vertices: $(0, \pm 2)$.

Eccentricity: $e = c/a = \sqrt{13}/2$.
Latus rectum: $2b^2/a = 2(9)/2 = 9$.

Answer: Foci $(0, \pm\sqrt{13})$; vertices $(0, \pm 2)$; $e = \sqrt{13}/2$; latus rectum = 9

Q4 Find the coordinates of the foci and the vertices, the eccentricity and the length of the latus rectum of the hyperbola: $16x^2 - 9y^2 = 576$.

SOLUTION

Divide by 576: $x^2/36 - y^2/64 = 1$.

Here $a^2 = 36$, $b^2 = 64$, $a = 6$, $b = 8$. Transverse axis along x-axis.

$c^2 = 36 + 64 = 100 \Rightarrow c = 10$.

Foci: $(\pm 10, 0)$.
Vertices: $(\pm 6, 0)$.
Eccentricity: $e = 10/6 = 5/3$.
Latus rectum: $2b^2/a = 2(64)/6 = 128/6 = 64/3$.

Answer: Foci $(\pm 10, 0)$; vertices $(\pm 6, 0)$; $e = 5/3$; latus rectum = 64/3

Q5 Find the coordinates of the foci and the vertices, the eccentricity and the length of the latus rectum of the hyperbola: $5y^2 - 9x^2 = 36$.

SOLUTION

Divide by 36: $y^2/(36/5) - x^2/4 = 1$.

Here $a^2 = 36/5$, $b^2 = 4$, $a = 6/\sqrt{5} = 6\sqrt{5}/5$, $b = 2$. Transverse axis along y-axis.

$c^2 = a^2 + b^2 = 36/5 + 4 = 36/5 + 20/5 = 56/5 \Rightarrow c = \sqrt{(56/5)} = 2\sqrt{(14/5)} = 2\sqrt{70}/5$.

Foci: $(0, \pm 2\sqrt{70}/5)$.
Vertices: $(0, \pm 6\sqrt{5}/5)$.
Eccentricity: $e = c/a = (2\sqrt{70}/5)/(6\sqrt{5}/5) = 2\sqrt{70}/(6\sqrt{5}) = \sqrt{70}/(3\sqrt{5}) = \sqrt{14}/3$.
Latus rectum: $2b^2/a = 2(4)/(6/\sqrt{5}) = 8\sqrt{5}/6 = 4\sqrt{5}/3$.

Answer: Foci $(0, \pm 2\sqrt{70}/5)$; vertices $(0, \pm 6\sqrt{5}/5)$; $e = \sqrt{14}/3$; latus rectum = $4\sqrt{5}/3$

Q6 Find the coordinates of the foci and the vertices, the eccentricity and the length of the latus rectum of the hyperbola: $49y^2 - 16x^2 = 784$.

SOLUTION

Divide by 784: $y^2/16 - x^2/49 = 1$.

Here $a^2 = 16$, $b^2 = 49$, $a = 4$, $b = 7$. Transverse axis along y-axis.

$c^2 = 16 + 49 = 65 \Rightarrow c = \sqrt{65}$.

Foci: $(0, \pm\sqrt{65})$.
Vertices: $(0, \pm 4)$.
Eccentricity: $e = \sqrt{65}/4$.
Latus rectum: $2b^2/a = 2(49)/4 = 98/4 = 49/2$.

Answer: Foci $(0, \pm\sqrt{65})$; vertices $(0, \pm 4)$; $e = \sqrt{65}/4$; latus rectum $= 49/2$

Q7 Find the equations of the hyperbola satisfying the given conditions: Vertices $(\pm 2, 0)$, foci $(\pm 3, 0)$.

SOLUTION

Vertices and foci on x-axis: $a = 2$, $c = 3$.

$b^2 = c^2 - a^2 = 9 - 4 = 5$.

Standard form: $x^2/a^2 - y^2/b^2 = 1$.

Answer: $x^2/4 - y^2/5 = 1$

Q8 Find the equations of the hyperbola satisfying the given conditions: Vertices $(0, \pm 5)$, foci $(0, \pm 8)$.

SOLUTION

Vertices and foci on y-axis: $a = 5$, $c = 8$.

$b^2 = c^2 - a^2 = 64 - 25 = 39$.

Standard form: $y^2/a^2 - x^2/b^2 = 1$.

Answer: $y^2/25 - x^2/39 = 1$

Q9 Find the equations of the hyperbola satisfying the given conditions: Vertices (0, ±3), foci (0, ±5).

SOLUTION

Vertices and foci on y-axis: a = 3, c = 5.

$b^2 = c^2 - a^2 = 25 - 9 = 16$.

Standard form: $y^2/a^2 – x^2/b^2 = 1$.

Answer: $y^2/9 – x^2/16 = 1$

Q10 Find the equations of the hyperbola satisfying the given conditions: Foci (±5, 0), the transverse axis is of length 8.

SOLUTION

Foci on x-axis: c = 5, transverse axis 2a = 8 => a = 4.

$b^2 = c^2 - a^2 = 25 - 16 = 9$.

Answer: $x^2/16 – y^2/9 = 1$

Q11 Find the equations of the hyperbola satisfying the given conditions: Foci (0, ±13), the conjugate axis is of length 24.

SOLUTION

Foci on y-axis: c = 13, conjugate axis 2b = 24 => b = 12.

$a^2 = c^2 - b^2 = 169 - 144 = 25$.

Standard form: $y^2/a^2 – x^2/b^2 = 1$.

Answer: $y^2/25 – x^2/144 = 1$

Q12 Find the equations of the hyperbola satisfying the given conditions: Foci (±3√5, 0), the latus rectum is of length 8.

SOLUTION

Foci on x-axis: $c = 3\sqrt{5}$, so $c^2 = 45$.
Latus rectum = $2b^2/a = 8 \Rightarrow b^2 = 4a$.

Using $c^2 = a^2 + b^2$:
$45 = a^2 + 4a \Rightarrow a^2 + 4a - 45 = 0$.

Factoring: $(a + 9)(a - 5) = 0 \Rightarrow a = 5$ (taking positive value).

$b^2 = 4(5) = 20$.

Verify: $c^2 = 25 + 20 = 45$ ✓

Answer: $x^2/25 – y^2/20 = 1$

Q13 Find the equations of the hyperbola satisfying the given conditions: Foci (±4, 0), the latus rectum is of length 12.

SOLUTION

Foci on x-axis: $c = 4$, so $c^2 = 16$.
Latus rectum $= 2b^2/a = 12 \Rightarrow b^2 = 6a$.

Using $c^2 = a^2 + b^2$:
$16 = a^2 + 6a \Rightarrow a^2 + 6a - 16 = 0$.

Factoring: $(a + 8)(a - 2) = 0 \Rightarrow a = 2$ (taking positive value).

$b^2 = 6(2) = 12$.

Verify: $c^2 = 4 + 12 = 16$ ✓

Answer: $x^2/4 – y^2/12 = 1$

Q14 Find the equations of the hyperbola satisfying the given conditions: Vertices (±7, 0), e = 4/3.

SOLUTION

Vertices on x-axis: $a = 7$.
$e = c/a \Rightarrow 4/3 = c/7 \Rightarrow c = 28/3$.

$b^2 = c^2 - a^2 = (28/3)^2 - 49 = 784/9 - 441/9 = 343/9$.

Answer: $x^2/49 – y^2/(343/9) = 1$, i.e., $x^2/49 – 9y^2/343 = 1$

Q15 Find the equations of the hyperbola satisfying the given conditions: Foci (0, ±√10), passing through (2, 3).

SOLUTION

Foci on y-axis: $c = \sqrt{10}$, $c^2 = 10$. Form: $y^2/a^2 – x^2/b^2 = 1$.

Also $b^2 = c^2 - a^2 = 10 - a^2$.

The hyperbola passes through (2, 3):
$9/a^2 – 4/(10 - a^2) = 1$.

Let $u = a^2$:
$9(10 - u) - 4u = u(10 - u)$
$90 - 9u - 4u = 10u - u^2$
$u^2 - 23u + 90 = 0$.

$u = (23 \pm \sqrt{(529 - 360)})/2 = (23 \pm 13)/2 \Rightarrow u = 18$ or $u = 5$.

If $a^2 = 18$: $b^2 = 10 - 18 = -8$ (invalid).
So $a^2 = 5$, $b^2 = 10 - 5 = 5$.

Verify: $9/5 - 4/5 = 5/5 = 1$ ✓

Answer: $y^2/5 – x^2/5 = 1$, i.e., $y^2 – x^2 = 5$

Misc — Miscellaneous Exercise

Q1 If a parabolic reflector is 20 cm in diameter and 5 cm deep, find the focus.

SOLUTION

Place the vertex of the parabola at the origin with the axis along the positive x-axis.
The opening of the reflector has diameter 20 cm, so at depth x = 5 cm, the radius is y = 10 cm.

Equation of parabola: $y^2 = 4ax$.
Substituting (5, 10): $100 = 4a(5) \Rightarrow 100 = 20a \Rightarrow a = 5$.

The focus is at (a, 0) = (5, 0), i.e., 5 cm from the vertex inside the reflector.

Answer: Focus is at a distance of 5 cm from the vertex of the reflector.

Q2 An arch is in the form of a parabola with its axis vertical. The arch is 10 m high and 5 m wide at the base. How wide is it 2 m from the vertex of the parabola?

SOLUTION

Set up coordinates with vertex at the top (highest point) of the arch, y-axis pointing downward.
The parabola opens downward (toward the base): $x^2 = 4ay$ (where positive y is downward).

The base is 10 m below the vertex and 5 m wide, so the base point is (2.5, 10).
Substituting: $(2.5)^2 = 4a(10) \Rightarrow 6.25 = 40a \Rightarrow a = 6.25/40 = 5/32$.

At 2 m from the vertex (y = 2):
$x^2 = 4(5/32)(2) = 40/32 = 5/4$
$x = \sqrt{5}/2$.

Width = $2x = 2 \times (\sqrt{5}/2) = \sqrt{5}$ m.

Answer: Width at 2 m from vertex = $\sqrt{5}$ m ≈ 2.23 m

Q3 The cable of a uniformly loaded suspension bridge hangs in the form of a parabola. The roadway which is horizontal and 100 m long is supported by vertical wires attached to the cable, the longest wire being 30 m and the shortest being 6 m. Find the length of a supporting wire attached to the roadway 18 m from the middle.

SOLUTION

The cable is lowest at the midpoint of the bridge. The shortest wire is 6 m (at the middle) and the longest wire is 30 m (at each end, 50 m from middle).

Set up: place origin at the midpoint of the roadway. Let the y-axis point upward, so the cable is at height y above the road.
Cable equation (vertex at lowest point): $x^2 = 4a(y - 6)$, where $y = 6$ at $x = 0$.

At $x = 50$, $y = 30$: $50^2 = 4a(30 - 6) = 4a(24) = 96a \Rightarrow a = 2500/96 = 625/24$.

At $x = 18$:
$18^2 = 4(625/24)(y - 6)$
$324 = (625/6)(y - 6)$
$y - 6 = 324 \times 6/625 = 1944/625$.

Length of wire = $y = 6 + 1944/625 = 3750/625 + 1944/625 = 5694/625 \approx 9.11$ m.

Answer: Length of wire = 5694/625 m ≈ 9.11 m

Q4 An arch is in the form of a semi-ellipse. It is 8 m wide and 2 m high at the centre. Find the height of the arch at a point 1.5 m from one end.

SOLUTION

Take the centre of the ellipse at the origin. The arch is 8 m wide, so the semi-major axis $a = 4$ m along x-axis. The arch is 2 m high, so semi-minor axis $b = 2$ m along y-axis.

Equation of the ellipse: $x^2/16 + y^2/4 = 1$.

A point 1.5 m from one end of the arch (end is at $x = 4$) is at $x = 4 - 1.5 = 2.5$.

Substituting $x = 2.5$:
$y^2/4 = 1 - (2.5)^2/16 = 1 - 6.25/16 = 1 - 25/64 = 39/64$.
$y^2 = 4 \times 39/64 = 39/16$.
$y = \sqrt{39}/4$.

Answer: Height of the arch at 1.5 m from one end = $\sqrt{39}/4$ m ≈ 1.56 m

Q5 A rod of length 12 cm moves with its ends always touching the coordinate axes. Determine the equation of the locus of a point P on the rod, which is 3 cm from the end in contact with the x-axis.

SOLUTION

Let A be the end on the x-axis and B be the end on the y-axis. AB = 12 cm.
P is on the rod with AP = 3 cm, so PB = 9 cm.

Let the rod make angle θ with the x-axis.

In triangle PBQ (Q is foot of perpendicular from P to y-axis):
cos θ = x/PB = x/9.

In triangle PRA (R is foot of perpendicular from P to x-axis):
sin θ = y/AP = y/3.

Using $\cos^2\theta + \sin^2\theta = 1$:
$(x/9)^2 + (y/3)^2 = 1$
$x^2/81 + y^2/9 = 1$.

Answer: $x^2/81 + y^2/9 = 1$ (an ellipse)

Q6 Find the area of the triangle formed by the lines joining the vertex of the parabola $x^2 = 12y$ to the ends of its latus rectum.

SOLUTION

Given parabola: $x^2 = 12y$, so $4a = 12 \Rightarrow a = 3$.

Vertex: O = (0, 0).
Latus rectum is at y = a = 3, with end points where $x^2 = 12(3) = 36$, so x = ±6.
End points of latus rectum: A = (6, 3) and B = (-6, 3).

The triangle OAB has:
Base AB = 12 (horizontal segment at y = 3).
Height = perpendicular distance from O(0,0) to line y = 3 = 3.

Area = (1/2) × base × height = (1/2) × 12 × 3 = 18 sq units.

Answer: Area = 18 sq units

Q7 A man running a racecourse notes that the sum of the distances from the two flag posts from him is always 10 m and the distance between the flag posts is 8 m. Find the equation of the path traced by the man.

SOLUTION

The sum of distances from two fixed points (foci) is constant — this is the definition of an ellipse.

Here: $2a = 10 \Rightarrow a = 5$.
Distance between flag posts = $2c = 8 \Rightarrow c = 4$.

$b^2 = a^2 - c^2 = 25 - 16 = 9$.

Taking the flag posts on the x-axis with centre at origin:

Answer: $x^2/25 + y^2/9 = 1$

Q8 An equilateral triangle is inscribed in the parabola $y^2 = 4ax$, where one vertex is at the vertex of the parabola. Find the length of the side of the triangle.

SOLUTION

Let O = (0, 0) be the vertex of the parabola. The parabola is symmetric about the x-axis.
Let the other two vertices of the equilateral triangle be $P = (at^2, 2at)$ and $P' = (at^2, -2at)$ (using the parametric form of $y^2 = 4ax$).

PP' = 4at (vertical distance).

$OP = \sqrt{((at^2)^2 + (2at)^2)} = at\sqrt{(t^2 + 4)}$.

For equilateral triangle: OP = PP'
$at\sqrt{(t^2 + 4)} = 4at$
$\sqrt{(t^2 + 4)} = 4$
$t^2 + 4 = 16 \Rightarrow t^2 = 12 \Rightarrow t = 2\sqrt{3}$.

Side length = PP' = $4at = 4a(2\sqrt{3}) = 8\sqrt{3}\ a$.

Answer: Length of side = $8\sqrt{3}\ a$