Chapter 11: Introduction to Three Dimensional Geometry

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Exercise 11.1 — Exercise 11.1

Q1 A point is on the x-axis. What are its y-coordinate and z-coordinates?

SOLUTION

Any point on the x-axis lies on both the XY-plane and the XZ-plane simultaneously.

A point on the x-axis has the form (x, 0, 0): its perpendicular distance from the ZX-plane is 0 (so y = 0) and its perpendicular distance from the XY-plane is 0 (so z = 0).

Answer: The y-coordinate and z-coordinate are both 0.

Q2 A point is in the XZ-plane. What can you say about its y-coordinate?

SOLUTION

The XZ-plane is defined as the plane containing the x-axis and the z-axis.

The y-coordinate of a point measures its perpendicular distance from the ZX-plane. Any point lying in the XZ-plane has zero perpendicular distance from that plane.

Therefore, the y-coordinate of any point in the XZ-plane is 0.

Answer: The y-coordinate of a point in the XZ-plane is 0.

Q3 Name the octants in which the following points lie:
(1, 2, 3), (4, –2, 3), (4, –2, –5), (4, 2, –5), (–4, 2, –5), (–4, 2, 5), (–3, –1, 6), (–2, –4, –7).

SOLUTION

The eight octants are determined by the signs of the coordinates (x, y, z):
Octant I: (+, +, +), Octant II: (–, +, +), Octant III: (–, –, +), Octant IV: (+, –, +), Octant V: (+, +, –), Octant VI: (–, +, –), Octant VII: (–, –, –), Octant VIII: (+, –, –).

(1, 2, 3): $x > 0, y > 0, z > 0$ → Octant I

(4, –2, 3): $x > 0, y < 0, z > 0$ → Octant IV

(4, –2, –5): $x > 0, y < 0, z < 0$ → Octant VIII

(4, 2, –5): $x > 0, y > 0, z < 0$ → Octant V

(–4, 2, –5): $x < 0, y > 0, z < 0$ → Octant VI

(–4, 2, 5): $x < 0, y > 0, z > 0$ → Octant II

(–3, –1, 6): $x < 0, y < 0, z > 0$ → Octant III

(–2, –4, –7): $x < 0, y < 0, z < 0$ → Octant VII

Answer: (1,2,3) → I; (4,–2,3) → IV; (4,–2,–5) → VIII; (4,2,–5) → V; (–4,2,–5) → VI; (–4,2,5) → II; (–3,–1,6) → III; (–2,–4,–7) → VII.

Q4 Fill in the blanks:
(i) The x-axis and y-axis taken together determine a plane known as _____.
(ii) The coordinates of points in the XY-plane are of the form _____.
(iii) Coordinate planes divide the space into _____ octants.

SOLUTION

(i) The x-axis and y-axis both lie in the XY-plane; together they determine the XY-plane.

(ii) Every point in the XY-plane has its z-coordinate equal to 0 (zero perpendicular distance from the XY-plane). So the coordinates are of the form (x, y, 0).

(iii) The three coordinate planes (XY, YZ, ZX) intersect and divide all of three-dimensional space into 8 regions called octants.

Answer: (i) XY-plane. (ii) (x, y, 0). (iii) Eight.

Exercise 11.2 — Exercise 11.2

Q1 Find the distance between the following pairs of points:
(i) (2, 3, 5) and (4, 3, 1)
(ii) (–3, 7, 2) and (2, 4, –1)
(iii) (–1, 3, –4) and (1, –3, 4)
(iv) (2, –1, 3) and (–2, 1, 3).

SOLUTION

Using the distance formula: $PQ = \sqrt{[(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2]}$

(i) P = (2, 3, 5), Q = (4, 3, 1):
$PQ = \sqrt{[(4-2)^2 + (3-3)^2 + (1-5)^2]}$
$= \sqrt{[4 + 0 + 16]}$
$= \sqrt{20} = 2\sqrt{5}$ units

(ii) P = (–3, 7, 2), Q = (2, 4, –1):
$PQ = \sqrt{[(2-(-3))^2 + (4-7)^2 + (-1-2)^2]}$
$= \sqrt{[25 + 9 + 9]}$
$= \sqrt{43}$ units

(iii) P = (–1, 3, –4), Q = (1, –3, 4):
$PQ = \sqrt{[(1-(-1))^2 + (-3-3)^2 + (4-(-4))^2]}$
$= \sqrt{[4 + 36 + 64]}$
$= \sqrt{104} = 2\sqrt{26}$ units

(iv) P = (2, –1, 3), Q = (–2, 1, 3):
$PQ = \sqrt{[(-2-2)^2 + (1-(-1))^2 + (3-3)^2]}$
$= \sqrt{[16 + 4 + 0]}$
$= \sqrt{20} = 2\sqrt{5}$ units

Answer: (i) $2\sqrt{5}$ units; (ii) $\sqrt{43}$ units; (iii) $2\sqrt{26}$ units; (iv) $2\sqrt{5}$ units.

Q2 Show that the points (–2, 3, 5), (1, 2, 3) and (7, 0, –1) are collinear.

SOLUTION

Let A = (–2, 3, 5), B = (1, 2, 3), C = (7, 0, –1).

$AB = \sqrt{[(1-(-2))^2 + (2-3)^2 + (3-5)^2]}$
$= \sqrt{[9 + 1 + 4]} = \sqrt{14}$

$BC = \sqrt{[(7-1)^2 + (0-2)^2 + (-1-3)^2]}$

= √[36 + 4 + 16] = √56 = 2√14

AC = √[(7–(–2))² + (0–3)² + (–1–5)²]
= √[81 + 9 + 36] = √126 = 3√14

Since AB + BC = √14 + 2√14 = 3√14 = AC,
point B lies between A and C on the segment AC.

Therefore, A, B and C are collinear.

Answer: AB = √14, BC = 2√14, AC = 3√14. Since AB + BC = AC, the three points are collinear.

Q3 Verify the following:
(i) (0, 7, –10), (1, 6, –6) and (4, 9, –6) are the vertices of an isosceles triangle.
(ii) (0, 7, 10), (–1, 6, 6) and (–4, 9, 6) are the vertices of a right angled triangle.
(iii) (–1, 2, 1), (1, –2, 5), (4, –7, 8) and (2, –3, 4) are the vertices of a parallelogram.

SOLUTION

(i) Let A = (0, 7, –10), B = (1, 6, –6), C = (4, 9, –6).

AB² = (1–0)² + (6–7)² + (–6–(–10))²
= 1 + 1 + 16 = 18, so AB = 3√2

BC² = (4–1)² + (9–6)² + (–6–(–6))²
= 9 + 9 + 0 = 18, so BC = 3√2

CA² = (0–4)² + (7–9)² + (–10–(–6))²
= 16 + 4 + 16 = 36, so CA = 6

Since AB = BC = 3√2 ≠ CA, triangle ABC is isosceles (two equal sides AB and BC).

(ii) Let A = (0, 7, 10), B = (–1, 6, 6), C = (–4, 9, 6).

AB² = (–1–0)² + (6–7)² + (6–10)²
= 1 + 1 + 16 = 18

BC² = (–4–(–1))² + (9–6)² + (6–6)²
= 9 + 9 + 0 = 18

CA² = (0–(–4))² + (7–9)² + (10–6)²
= 16 + 4 + 16 = 36

Check: AB² + BC² = 18 + 18 = 36 = CA².

By the converse of Pythagoras' theorem, the right angle is at B.
Therefore, triangle ABC is right-angled (at B).

(iii) Let A = (–1, 2, 1), B = (1, –2, 5), C = (4, –7, 8), D = (2, –3, 4).

$AB^2 = (1–(–1))^2 + (–2–2)^2 + (5–1)^2$
$= 4 + 16 + 16 = 36$

$CD^2 = (2–4)^2 + (–3–(–7))^2 + (4–8)^2$
$= 4 + 16 + 16 = 36$

$BC^2 = (4–1)^2 + (–7–(–2))^2 + (8–5)^2$
$= 9 + 25 + 9 = 43$

$DA^2 = (–1–2)^2 + (2–(–3))^2 + (1–4)^2$
$= 9 + 25 + 9 = 43$

Since AB = CD = 6 and BC = DA = $\sqrt{43}$, the opposite sides are equal.

Also, midpoint of diagonal AC = ((-1+4)/2, (2–7)/2, (1+8)/2) = (3/2, –5/2, 9/2), and midpoint of diagonal BD = ((1+2)/2, (–2–3)/2, (5+4)/2) = (3/2, –5/2, 9/2).

The diagonals bisect each other, confirming ABCD is a parallelogram.

Answer: (i) AB = BC = $3\sqrt{2}$, CA = 6: isosceles triangle. (ii) $AB^2 + BC^2 = CA^2$: right-angled at B. (iii) AB = CD, BC = DA, diagonals bisect each other: parallelogram.

Q4 Find the equation of the set of points which are equidistant from the points (1, 2, 3) and (3, 2, –1).

SOLUTION

Let P(x, y, z) be a point equidistant from A(1, 2, 3) and B(3, 2, –1).

Condition: PA = PB, i.e., $PA^2 = PB^2$.

$PA^2 = (x–1)^2 + (y–2)^2 + (z–3)^2$
$PB^2 = (x–3)^2 + (y–2)^2 + (z+1)^2$

Setting $PA^2 = PB^2$:
$(x–1)^2 + (y–2)^2 + (z–3)^2 = (x–3)^2 + (y–2)^2 + (z+1)^2$

The $(y–2)^2$ terms cancel:
$(x–1)^2 + (z–3)^2 = (x–3)^2 + (z+1)^2$

Expanding:

$x^2 - 2x + 1 + z^2 - 6z + 9 = x^2 - 6x + 9 + z^2 + 2z + 1$

$-2x - 6z + 10 = -6x + 2z + 10$

$4x - 8z = 0$

$x - 2z = 0$

Answer: The required equation is $x - 2z = 0$.

Q5 Find the equation of the set of points P, the sum of whose distances from A(4, 0, 0) and B(–4, 0, 0) is equal to 10.

SOLUTION

Let P(x, y, z) be any point such that PA + PB = 10.

$PA = \sqrt{[(x-4)^2 + y^2 + z^2]}$, $PB = \sqrt{[(x+4)^2 + y^2 + z^2]}$

So: $\sqrt{[(x-4)^2 + y^2 + z^2]} = 10 - \sqrt{[(x+4)^2 + y^2 + z^2]}$

Squaring both sides:

$(x-4)^2 + y^2 + z^2 = 100 - 20\sqrt{[(x+4)^2 + y^2 + z^2]} + (x+4)^2 + y^2 + z^2$

$x^2 - 8x + 16 = 100 - 20\sqrt{[(x+4)^2 + y^2 + z^2]} + x^2 + 8x + 16$

$-8x = 100 - 20\sqrt{[(x+4)^2 + y^2 + z^2]} + 8x$

$20\sqrt{[(x+4)^2 + y^2 + z^2]} = 100 + 16x$

$5\sqrt{[(x+4)^2 + y^2 + z^2]} = 25 + 4x$

Squaring again:

$25[(x+4)^2 + y^2 + z^2] = (25 + 4x)^2$

$25(x^2 + 8x + 16 + y^2 + z^2) = 625 + 200x + 16x^2$

$25x^2 + 200x + 400 + 25y^2 + 25z^2 = 625 + 200x + 16x^2$

$9x^2 + 25y^2 + 25z^2 = 225$

Answer: The required equation is $9x^2 + 25y^2 + 25z^2 = 225$.

Misc — Miscellaneous Exercise

Q1 Three vertices of a parallelogram ABCD are A(3, –1, 2), B(1, 2, –4) and C(–1, 1, 2). Find the coordinates of the fourth vertex.

SOLUTION

In a parallelogram, the diagonals bisect each other.
So the midpoint of diagonal AC equals the midpoint of diagonal BD.

Midpoint of AC = ((3 + (–1))/2, (–1 + 1)/2, (2 + 2)/2)
= (2/2, 0/2, 4/2) = (1, 0, 2)

Let D = (x, y, z). Midpoint of BD = ((1 + x)/2, (2 + y)/2, (–4 + z)/2).

Setting midpoint of BD = (1, 0, 2):
(1 + x)/2 = 1 => x = 1
(2 + y)/2 = 0 => y = –2
(–4 + z)/2 = 2 => z = 8

Therefore, D = (1, –2, 8).

Verification: Midpoint of BD = ((1+1)/2, (2–2)/2, (–4+8)/2) = (1, 0, 2) = Midpoint of AC. ✓

Answer: The fourth vertex D is (1, –2, 8).

Q2 Find the lengths of the medians of the triangle with vertices A(0, 0, 6), B(0, 4, 0) and C(6, 0, 0).

SOLUTION

Let D, E, F be the midpoints of BC, CA, AB respectively.

D = midpoint of BC = ((0+6)/2, (4+0)/2, (0+0)/2) = (3, 2, 0)
E = midpoint of CA = ((6+0)/2, (0+0)/2, (0+6)/2) = (3, 0, 3)
F = midpoint of AB = ((0+0)/2, (0+4)/2, (6+0)/2) = (0, 2, 3)

Median from A to D:
AD = $\sqrt{[(3–0)^2 + (2–0)^2 + (0–6)^2]}$
= $\sqrt{[9 + 4 + 36]} = \sqrt{49} = 7$

Median from B to E:
BE = $\sqrt{[(3–0)^2 + (0–4)^2 + (3–0)^2]}$
= $\sqrt{[9 + 16 + 9]} = \sqrt{34}$

Median from C to F:
CF = $\sqrt{[(0–6)^2 + (2–0)^2 + (3–0)^2]}$
= $\sqrt{[36 + 4 + 9]} = \sqrt{49} = 7$

Answer: The lengths of the medians are: from A = 7, from B = $\sqrt{34}$, from C = 7.

Q3 If the origin is the centroid of the triangle PQR with vertices P(2a, 2, 6), Q(–4, 3b, –10) and R(8, 14, 2c), then find the values of a, b and c.

SOLUTION

The centroid of triangle PQR has coordinates:
G = ((x_P + x_Q + x_R)/3, (y_P + y_Q + y_R)/3, (z_P + z_Q + z_R)/3)

Given G = (0, 0, 0):

x-coordinate: (2a – 4 + 8)/3 = 0
=> 2a + 4 = 0
=> a = –2

y-coordinate: (2 + 3b + 14)/3 = 0
=> 3b + 16 = 0
=> b = –16/3

z-coordinate: (6 – 10 + 2c)/3 = 0
=> 2c – 4 = 0
=> c = 2

Answer: a = –2, b = –16/3, c = 2.

Q4 If A and B be the points (3, 4, 5) and (–1, 3, –7), respectively, find the equation of the set of points P such that $PA^2 + PB^2 = k^2$, where k is a constant.

SOLUTION

Let P(x, y, z) be any point satisfying the given condition.

$PA^2 = (x–3)^2 + (y–4)^2 + (z–5)^2$
$= x^2 – 6x + 9 + y^2 – 8y + 16 + z^2 – 10z + 25$
$= x^2 + y^2 + z^2 – 6x – 8y – 10z + 50$

$PB^2 = (x+1)^2 + (y–3)^2 + (z+7)^2$
$= x^2 + 2x + 1 + y^2 – 6y + 9 + z^2 + 14z + 49$
$= x^2 + y^2 + z^2 + 2x – 6y + 14z + 59$

$PA^2 + PB^2 = 2x^2 + 2y^2 + 2z^2 – 4x – 14y + 4z + 109$

Setting $PA^2 + PB^2 = k^2$:

$2x^2 + 2y^2 + 2z^2 – 4x – 14y + 4z + 109 = k^2$

Therefore:

$2(x^2 + y^2 + z^2) – 4x – 14y + 4z = k^2 – 109$

Answer: The required equation is $2(x^2 + y^2 + z^2) – 4x – 14y + 4z = k^2 – 109$.