Chapter 12: Limits and Derivatives

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Exercise 12.1 — Exercise 12.1

Q1 Evaluate: $\lim_{x\to 3}(x + 3)$

SOLUTION

The function $f(x) = x + 3$ is a polynomial.

For any polynomial, the limit as $x \to a$ equals the value of the polynomial at $x = a$.

$\lim_{x\to 3}(x + 3) = 3 + 3 = 6$

Answer: 6

Q2 Evaluate: $\lim_{x\to \pi}(x - 22/7)$

SOLUTION

The function $f(x) = x - 22/7$ is a polynomial.

$\lim_{x\to \pi}(x - 22/7) = \pi - 22/7$

Answer: $\pi - 22/7$

Q3 Evaluate: $\lim_{r\to 1} \pi r^2$

SOLUTION

The function $f(r) = \pi r^2$ is a polynomial in r.

$\lim_{r\to 1} \pi r^2 = \pi(1)^2 = \pi$

Answer: π

Q4 Evaluate: $\lim_{x\to 4} \frac{4x + 3}{x - 2}$

SOLUTION

The function is a rational function. Check the denominator at $x = 4$: $4 - 2 = 2 \neq 0$.
So the limit is simply the value of the function at $x = 4$.
$\lim_{x\to 4} \frac{4x + 3}{x - 2} = \frac{16 + 3}{4 - 2} = \frac{19}{2}$

Answer: $\frac{19}{2}$

Q5 Evaluate: $\lim_{x\to -1} \frac{x^{10} + x^5 + 1}{x - 1}$

SOLUTION

The denominator at $x = -1$ is $-1 - 1 = -2 \neq 0$.
So the limit equals the value of the function at $x = -1$.
Numerator: $(-1)^{10} + (-1)^5 + 1 = 1 - 1 + 1 = 1$.
$\lim_{x\to -1} \frac{x^{10} + x^5 + 1}{x - 1} = \frac{1}{-2} = -\frac{1}{2}$

Answer: $-\frac{1}{2}$

Q6 Evaluate: $\lim_{x\to 0} [(x + 1)^5 - 1] / x$

SOLUTION

At $x = 0$ both numerator and denominator equal 0, giving the 0/0 form.
Expand $(x + 1)^5$ using the binomial theorem:
$(x + 1)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$
So $(x + 1)^5 - 1 = 5x + 10x^2 + 10x^3 + 5x^4 + x^5 = x(5 + 10x + 10x^2 + 5x^3 + x^4)$.

Therefore: $[(x + 1)^5 - 1]/x = 5 + 10x + 10x^2 + 5x^3 + x^4$.
Taking the limit as $x \to 0$: lim = $5 + 0 = 5$.

Alternatively, using the standard limit $\lim_{x\to a} (x^n - a^n)/(x - a) = na^{n-1}$:
Let $y = x + 1$; as $x \to 0$, $y \to 1$.
The expression becomes $(y^5 - 1^5)/(y - 1) \to 5(1)^4 = 5$.

Answer: 5

Q7 Evaluate: $\lim_{x\to 2} (3x^2 - x - 10)/(x^2 - 4)$

SOLUTION

At x = 2: numerator = 12 - 2 - 10 = 0, denominator = 4 - 4 = 0. This is 0/0 form.

Factor the numerator: $3x^2 - x - 10 = (3x + 5)(x - 2)$.
Factor the denominator: $x^2 - 4 = (x + 2)(x - 2)$.

For $x \neq 2$, cancel $(x - 2)$:
$(3x^2 - x - 10)/(x^2 - 4) = (3x + 5)/(x + 2)$.

$\lim_{x\to 2} (3x + 5)/(x + 2) = (6 + 5)/(2 + 2) = 11/4$

Answer: 11/4

Q8 Evaluate: $\lim_{x\to 3} (x^4 - 81)/(2x^2 - 5x - 3)$

SOLUTION

At x = 3: numerator = 81 - 81 = 0, denominator = 18 - 15 - 3 = 0. This is 0/0 form.

Factor the numerator: $x^4 - 81 = (x^2 - 9)(x^2 + 9) = (x - 3)(x + 3)(x^2 + 9)$.
Factor the denominator: $2x^2 - 5x - 3 = (2x + 1)(x - 3)$.

For $x \neq 3$, cancel $(x - 3)$:
$(x^4 - 81)/(2x^2 - 5x - 3) = (x + 3)(x^2 + 9)/(2x + 1)$.

$\lim_{x\to 3} = (3 + 3)(9 + 9)/(6 + 1) = 6 \times 18/7 = 108/7$

Answer: 108/7

Q9 Evaluate: $\lim_{x\to 0} (ax + b)/(cx + 1)$

SOLUTION

The function is a rational function. At x = 0, the denominator is $c(0) + 1 = 1 \neq 0$.
So the limit equals the value at x = 0.
$\lim_{x\to 0} (ax + b)/(cx + 1) = (0 + b)/(0 + 1) = b$

Answer: b

Q10 Evaluate: lim(z→1) (z^(1/3) - 1)/(z^(1/6) - 1)

SOLUTION

At z = 1 both numerator and denominator equal 0. Substitute y = z^(1/6), so z = y^6.

As z → 1, we have y → 1.

Then z^(1/3) = y^2 and z^(1/6) = y, so:

(z^(1/3) - 1)/(z^(1/6) - 1) = (y^2 - 1)/(y - 1) = (y - 1)(y + 1)/(y - 1) = y + 1 for y ≠ 1.

lim(y→1) (y + 1) = 1 + 1 = 2

Answer: 2

Q11 Evaluate: lim(x→1) (ax^2 + bx + c)/(cx^2 + bx + a), where a + b + c ≠ 0

SOLUTION

At x = 1: numerator = a + b + c, denominator = c + b + a = a + b + c.

Since a + b + c ≠ 0, the denominator is non-zero.

lim(x→1) (ax^2 + bx + c)/(cx^2 + bx + a) = (a + b + c)/(a + b + c) = 1

Answer: 1

Q12 Evaluate: lim(x→-2) [(1/x + 1/2)/(x + 2)]

SOLUTION

At x = -2: numerator = 1/(-2) + 1/2 = 0, denominator = -2 + 2 = 0. This is 0/0 form.

Simplify the numerator: 1/x + 1/2 = (2 + x)/(2x).

So (1/x + 1/2)/(x + 2) = (2 + x)/[2x(x + 2)] = 1/(2x), for x ≠ -2.

lim(x→-2) 1/(2x) = 1/(2 × (-2)) = -1/4

Answer: -1/4

Q13 Evaluate: lim(x→0) sin(ax)/(bx)

SOLUTION

Write sin(ax)/(bx) = (a/b) × sin(ax)/(ax).

As x → 0, we have ax → 0, and lim(ax→0) sin(ax)/(ax) = 1.

Therefore: lim(x→0) sin(ax)/(bx) = (a/b) × 1 = a/b

Answer: a/b

Q14 Evaluate: $\lim_{x\to 0} \sin(ax)/\sin(bx)$, where $a, b \neq 0$

SOLUTION

Write the expression as:
$\sin(ax)/\sin(bx) = [\sin(ax)/(ax)] \times (ax) \times 1/[(\sin(bx)/(bx)) \times (bx)]$
$= [\sin(ax)/(ax)] / [\sin(bx)/(bx)] \times (a/b)$.

As $x \to 0$: $\sin(ax)/(ax) \to 1$ and $\sin(bx)/(bx) \to 1$.

$\lim_{x\to 0} \sin(ax)/\sin(bx) = (1/1) \times (a/b) = a/b$

Answer: a/b

Q15 Evaluate: $\lim_{x\to \pi} \sin(\pi - x)/[\pi(\pi - x)]$

SOLUTION

Use the identity $\sin(\pi - x) = \sin x$.
So $\sin(\pi - x)/[\pi(\pi - x)] = \sin x/[\pi(\pi - x)]$.

Substitute $t = \pi - x$; as $x \to \pi$, $t \to 0$.
Then $\sin x = \sin(\pi - t) = \sin t$.

The expression becomes $\sin t/(\pi t) = (1/\pi) \times \sin t/t$.

$\lim_{t\to 0} (1/\pi) \times \sin t/t = (1/\pi) \times 1 = 1/\pi$

Answer: $1/\pi$

Q16 Evaluate: $\lim_{x\to 0} \cos x/(\pi - x)$

SOLUTION

The function $\cos x/(\pi - x)$ is defined at $x = 0$.
At $x = 0$: $\cos(0)/(\pi - 0) = 1/\pi$.
Since the function is continuous at $x = 0$:
$\lim_{x\to 0} \cos x/(\pi - x) = 1/\pi$

Answer: $1/\pi$

Q17 Evaluate: $\lim_{x\to 0} \frac{\cos 2x - 1}{\cos x - 1}$

SOLUTION

Use the identity $1 - \cos\theta = 2\sin^2(\theta/2)$:
$\cos 2x - 1 = -2\sin^2 x$ and $\cos x - 1 = -2\sin^2(x/2)$.

$\frac{\cos 2x - 1}{\cos x - 1} = \frac{-2\sin^2 x}{-2\sin^2(x/2)} = \frac{\sin^2 x}{\sin^2(x/2)}$.

Since $\sin x = 2\sin(x/2)\cos(x/2)$, we get $\sin^2 x = 4\sin^2(x/2)\cos^2(x/2)$.

So $\frac{\sin^2 x}{\sin^2(x/2)} = 4\cos^2(x/2)$.

$\lim_{x\to 0} 4\cos^2(x/2) = 4\cos^2(0) = 4 \times 1 = 4$

Answer: 4

Q18 Evaluate: $\lim_{x\to 0} \frac{ax + x\cos x}{b \sin x}$

SOLUTION

Factor x from the numerator: $\frac{ax + x\cos x}{b\sin x} = \frac{x(a + \cos x)}{b\sin x}$.

Rewrite as: $\frac{a + \cos x}{b} \times \frac{x}{\sin x}$.

As $x \to 0$: $a + \cos x \to a + 1$, and $\frac{x}{\sin x} \to 1$.

$\lim_{x\to 0} \frac{ax + x\cos x}{b\sin x} = \frac{a + 1}{b} \times 1 = \frac{a + 1}{b}$

Answer: $\frac{a + 1}{b}$

Q19 Evaluate: $\lim_{x\to 0} x \sec x$

SOLUTION

The function $x \sec x = \frac{x}{\cos x}$.
At $x = 0$: this equals $\frac{0}{\cos 0} = \frac{0}{1} = 0$.
$\lim_{x\to 0} x \sec x = 0$

Answer: 0

Q20 Evaluate: $\lim_{x\to 0} \frac{\sin ax + bx}{ax + \sin bx}$, where $a, b, a + b \neq 0$

SOLUTION

Divide numerator and denominator by x:
$\frac{\sin ax + bx}{ax + \sin bx} = \frac{\frac{\sin ax}{x} + b}{a + \frac{\sin bx}{x}}$.

Now $\frac{\sin ax}{x} = a \times \frac{\sin ax}{ax} \to a \times 1 = a$ as $x \to 0$.

Similarly, $(\sin bx)/x = b \times (\sin bx)/(bx) \to b \times 1 = b$ as $x \to 0$.

$\lim(x\to0) = (a + b)/(a + b) = 1$

Answer: 1

Q21 Evaluate: $\lim(x\to0)$ (cosec x - cot x)

SOLUTION

$\text{cosec } x - \cot x = 1/\sin x - \cos x/\sin x = (1 - \cos x)/\sin x$.

Multiply and divide by x:
$(1 - \cos x)/\sin x = [(1 - \cos x)/x] / [\sin x/x]$.

As $x \to 0$: $(1 - \cos x)/x \to 0$ and $\sin x/x \to 1$.

$\lim(x\to0)$ (cosec x - cot x) $= 0/1 = 0$

Answer: 0

Q22 Evaluate: $\lim(x\to\pi/2) \tan 2x / (x - \pi/2)$

SOLUTION

Substitute $t = x - \pi/2$, so $x = t + \pi/2$ and $t \to 0$ as $x \to \pi/2$.

$\tan 2x = \tan 2(t + \pi/2) = \tan(2t + \pi) = \tan 2t$ (since tan has period π).

The limit becomes: $\lim(t\to0) \tan 2t / t$.

Write $\tan 2t / t = (\sin 2t / \cos 2t) / t = (\sin 2t)/(2t) \times 2/\cos 2t$.

As $t \to 0$: $\sin 2t/(2t) \to 1$ and $1/\cos 2t \to 1/\cos 0 = 1$.

$\lim = 1 \times 2 \times 1 = 2$

Answer: 2

Q23 Find $\lim(x\to0) f(x)$ and $\lim(x\to1) f(x)$, where $f(x) = 2x + 3$ for $x \le 0$, and $f(x) = 3(x + 1)$ for $x > 0$.

SOLUTION

Finding $\lim(x\to0) f(x)$:
Left-hand limit ($x \to 0^-$): $f(x) = 2x + 3 \to 2(0) + 3 = 3$.
Right-hand limit ($x \to 0^+$): $f(x) = 3(x + 1) \to 3(0 + 1) = 3$.
Since LHL = RHL = 3, the limit exists and $\lim(x\to0) f(x) = 3$.

Finding $\lim_{x\to 1} f(x)$:
Since $1 > 0$, $f(x) = 3(x + 1)$ for all x near 1.
$\lim_{x\to 1} f(x) = 3(1 + 1) = 6$.

Answer: $\lim_{x\to 0} f(x) = 3$; $\lim_{x\to 1} f(x) = 6$

Q24 Find $\lim_{x\to 1} f(x)$, where $f(x) = x^2 - 1$ for $x \le 1$, and $f(x) = -x^2 - 1$ for $x > 1$.

SOLUTION

Left-hand limit ($x \to 1^-$): $f(x) = x^2 - 1 \to 1^2 - 1 = 0$.
Right-hand limit ($x \to 1^+$): $f(x) = -x^2 - 1 \to -1^2 - 1 = -2$.
Since LHL = 0 ≠ RHL = -2, the limit does not exist.

Answer: Limit does not exist (LHL = 0, RHL = -2)

Q25 Evaluate $\lim_{x\to 0} f(x)$, where $f(x) = |x|/x$ for $x \neq 0$ and $f(0) = 0$.

SOLUTION

For $x > 0$:	x	$= x$, so $f(x) = x/x = 1$.
For $x < 0$:	x	$= -x$, so $f(x) = -x/x = -1$.

Right-hand limit ($x \to 0^+$): $f(x) \to 1$.
Left-hand limit ($x \to 0^-$): $f(x) \to -1$.

Since LHL ≠ RHL, $\lim_{x\to 0} f(x)$ does not exist.

Answer: Limit does not exist (LHL = -1, RHL = 1)

Q26 Find $\lim_{x\to 0} f(x)$, where $f(x) = x/|x|$ for $x \neq 0$ and $f(0) = 0$.

SOLUTION

For $x > 0$:	x	$= x$, so $f(x) = x/x = 1$.
For $x < 0$:	x	$= -x$, so $f(x) = x/(-x) = -1$.

Right-hand limit ($x \to 0^+$): $f(x) \to 1$.
Left-hand limit ($x \to 0^-$): $f(x) \to -1$.

Since LHL = -1 ≠ RHL = 1, $\lim_{x\to 0} f(x)$ does not exist.

Answer: Limit does not exist (LHL = -1, RHL = 1)

Q27 Find $\lim_{x\to 5} f(x)$, where $f(x) = |x| - 5$.

SOLUTION

Near $x = 5$, we have $x > 0$, so $|x| = x$.
Therefore $f(x) = x - 5$ for x near 5.
$\lim_{x\to 5} (x - 5) = 5 - 5 = 0$.

Answer: 0

Q28 Suppose $f(x) = a + bx$ for $x < 1$, $f(1) = 4$, and $f(x) = b - ax$ for $x > 1$. If $\lim_{x\to 1} f(x) = f(1)$, what are the possible values of a and b?

SOLUTION

For the limit to exist, the left-hand and right-hand limits must both equal 4.

Left-hand limit ($x \to 1^-$): $\lim f(x) = a + b(1) = a + b$.
Right-hand limit ($x \to 1^+$): $\lim f(x) = b - a(1) = b - a$.

Setting both equal to $f(1) = 4$:
$a + b = 4$ ... (i)
$b - a = 4$ ... (ii)

Adding (i) and (ii): $2b = 8 \Rightarrow b = 4$.
Substituting into (i): $a + 4 = 4 \Rightarrow a = 0$.

Therefore $a = 0$ and $b = 4$.

Answer: $a = 0$, $b = 4$

Q29 Let $a_1, a_2, \ldots, a$ be fixed real numbers and define $f(x) = (x - a_1)(x - a_2)\cdots(x - a)$. What is $\lim_{x\to a_1} f(x)$? For some $a \neq a_1, a_2, \ldots, a$, compute $\lim_{x\to a} f(x)$.

SOLUTION

Since $f(x)$ is a polynomial, its limit at any point equals its value there.

$\lim_{x\to a_1} f(x) = f(a_1) = (a_1 - a_1)(a_1 - a_2)\cdots(a_1 - a) = 0 \times (\text{rest}) = 0$.

For $a \neq a_1, a_2, \ldots, a$ (so none of the factors vanish at $x = a$):
$\lim_{x\to a} f(x) = f(a) = (a - a_1)(a - a_2)\cdots(a - a)$.

Answer: $\lim_{x\to a_1} f(x) = 0$; for $a \neq$ any a: $\lim_{x\to a} f(x) = (a - a_1)(a - a_2)\cdots(a - a)$

Q30 If $f(x) = x + 1$ for $x < 0$, $f(0) = 0$, and $f(x) = x - 1$ for $x > 0$, for what value(s) of a does $\lim_{x\to a} f(x)$ exist?

SOLUTION

For $a \neq 0$: The function is a linear polynomial in a neighborhood of a, so it is continuous there, and the limit exists for all $a \neq 0$.

At $a = 0$:
Left-hand limit ($x \to 0^-$): $f(x) = x + 1 \to 0 + 1 = 1$.
Right-hand limit ($x \to 0^+$): $f(x) = x - 1 \to 0 - 1 = -1$.
Since LHL = 1 ≠ RHL = -1, the limit does not exist at $a = 0$.

Therefore the limit exists for all $a \neq 0$.

Answer: The limit exists for all $a \neq 0$

Q31 If the function f(x) satisfies $\lim_{x\to 1} [f(x) - 2]/(x^2 - 1) = \pi$, evaluate $\lim_{x\to 1} f(x)$.

SOLUTION

Let $L = \lim_{x\to 1} f(x)$. As $x \to 1$, the denominator $x^2 - 1 \to 0$.
For the given limit to equal π (a finite value), the numerator $f(x) - 2$ must also tend to 0.
Therefore $\lim_{x\to 1} [f(x) - 2] = 0$, which gives $L - 2 = 0$, so $L = 2$.

Verification: $\lim_{x\to 1} f(x) = 2$.

Answer: $\lim_{x\to 1} f(x) = 2$

Q32 If $f(x) = mx^2 + n$ for $x < 0$, $f(x) = nx + m$ for $0 \leq x \leq 1$, and $f(x) = nx^3 + m$ for $x > 1$, for what integers m and n do both $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$ exist?

SOLUTION

Condition for $\lim_{x\to 0} f(x)$ to exist:
Left-hand limit ($x \to 0^-$): $f(x) = mx^2 + n \to m(0) + n = n$.
Right-hand limit ($x \to 0^+$): $f(x) = nx + m \to n(0) + m = m$.
For the limit to exist: $n = m$.

Condition for $\lim_{x\to 1} f(x)$ to exist:
Left-hand limit ($x \to 1^-$): $f(x) = nx + m \to n + m$.
Right-hand limit ($x \to 1^+$): $f(x) = nx^3 + m \to n + m$.

The two one-sided limits are both n + m for any integers m and n, so this condition imposes no additional restriction.

Therefore both limits exist if and only if m = n.

Answer: m = n (any integer value satisfying m = n)

Exercise 12.2 — Exercise 12.2

Q1 Find the derivative of $x^2 - 2$ at $x = 10$.

SOLUTION

Let $f(x) = x^2 - 2$. Using the definition of derivative:

$f'(x) = \lim_{h\to 0} [f(x+h) - f(x)]/h = \lim_{h\to 0} [(x+h)^2 - 2 - (x^2-2)]/h$

$= \lim_{h\to 0} [x^2+2xh+h^2-x^2]/h = \lim_{h\to 0} (2x+h) = 2x.$

At $x = 10$: $f'(10) = 2(10) = 20$.

Answer: 20

Q2 Find the derivative of x at $x = 1$.

SOLUTION

Let $f(x) = x$. Then $f'(x) = \lim_{h\to 0} [(x+h) - x]/h = \lim_{h\to 0} h/h = 1$.

The derivative is the constant function 1.

At $x = 1$: $f'(1) = 1$.

Answer: 1

Q3 Find the derivative of $99x$ at $x = 100$.

SOLUTION

Let $f(x) = 99x$. Then $f'(x) = \lim_{h\to 0} [99(x+h) - 99x]/h = \lim_{h\to 0} 99h/h = 99$.

The derivative is the constant 99.

At $x = 100$: $f'(100) = 99$.

Answer: 99

Q4 Find the derivative of the following functions from first principle: (i) $x^3 - 27$ (ii) $(x - 1)(x - 2)$ (iii) $1/x^2$ (iv) $(x + 1)/(x - 1)$

SOLUTION

(i) Let $f(x) = x^3 - 27$.

$f'(x) = \lim_{h\to 0} [(x+h)^3 - 27 - (x^3-27)]/h$

$= \lim_{h\to 0} [(x+h)^3 - x^3]/h$

$= \lim_{h\to 0} [3x^2h + 3xh^2 + h^3]/h$

$= \lim_{h\to 0} (3x^2 + 3xh + h^2) = 3x^2.$

(ii) Let $f(x) = (x-1)(x-2) = x^2-3x+2$.
$f'(x) = \lim(h\to0)\ [(x+h)^2-3(x+h)+2 - (x^2-3x+2)]/h$
$= \lim(h\to0)\ [2xh + h^2 - 3h]/h$
$= \lim(h\to0)\ (2x + h - 3) = 2x - 3$.

(iii) Let $f(x) = 1/x^2$.
$f'(x) = \lim(h\to0)\ [1/(x+h)^2 - 1/x^2]/h$
$= \lim(h\to0)\ [x^2 - (x+h)^2] / [h\cdot x^2\cdot(x+h)^2]$
$= \lim(h\to0)\ [-2xh - h^2] / [h\cdot x^2\cdot(x+h)^2]$
$= \lim(h\to0)\ (-2x - h) / [x^2(x+h)^2]$
$= -2x/x^4 = -2/x^3$.

(iv) Let $f(x) = (x+1)/(x-1)$.
$f'(x) = \lim(h\to0)\ [(x+h+1)/(x+h-1) - (x+1)/(x-1)]/h$
Numerator of combined fraction:
$(x+h+1)(x-1) - (x+1)(x+h-1)$
$= (x^2-x+hx-h+x-1) - (x^2+xh-x+x+h-1)$
$= (x^2+hx-h-1) - (x^2+xh+h-1)$
$= -2h$.
So $f'(x) = \lim(h\to0)\ (-2h)/[h(x+h-1)(x-1)] = -2/(x-1)^2$.

Answer: (i) $3x^2$ (ii) $2x - 3$ (iii) $-2/x^3$ (iv) $-2/(x-1)^2$

Q5 For the function $f(x) = x^{100}/100 + x^{99}/99 + \dots + x^2/2 + x + 1$, prove that $f'(1) = 100\ f'(0)$.

SOLUTION

Differentiate term by term using d/dx(x^n) = nx^(n-1):
$f'(x) = x^{99} + x^{98} + x^{97} + \dots + x + 1$ (99 + 1 = 100 terms starting from x^{99} down to 1).

At x = 0:
$f'(0) = 0^{99} + 0^{98} + \dots + 0 + 1 = 1$.

At x = 1:
$f'(1) = 1 + 1 + 1 + \dots + 1$ (100 terms) = 100.

Therefore $f'(1) = 100 = 100 \times 1 = 100\ f'(0)$. ∎

Answer: $f'(1) = 100$ and $f'(0) = 1$, so $f'(1) = 100\ f'(0)$

Q6 Find the derivative of x^n + ax^(n-1) + a²x^(n-2) + … + a^(n-1)x + a^n for some fixed real number a.

SOLUTION

Let f(x) = x^n + ax^(n-1) + a²x^(n-2) + … + a^(n-1)x + a^n.

Differentiate each term using d/dx(x^k) = kx^(k-1) and constants have zero derivative:

d/dx(x^n) = nx^(n-1)

d/dx(ax^(n-1)) = (n-1)ax^(n-2)

d/dx(a²x^(n-2)) = (n-2)a²x^(n-3)

⋮

d/dx(a^(n-1)x) = a^(n-1)

d/dx(a^n) = 0

f'(x) = nx^(n-1) + (n-1)ax^(n-2) + (n-2)a²x^(n-3) + … + 2a^(n-2)x + a^(n-1)

Answer: f'(x) = nx^(n-1) + (n-1)ax^(n-2) + (n-2)a²x^(n-3) + … + 2a^(n-2)x + a^(n-1)

Q7 For some constants a and b, find the derivative of: (i) (x - a)(x - b) (ii) (ax² + b)² (iii) (x - a)/(x - b)

SOLUTION

(i) f(x) = (x - a)(x - b) = x² - (a+b)x + ab.

f'(x) = 2x - (a + b).

(ii) f(x) = (ax² + b)².

Expand: f(x) = a²x⁴ + 2abx² + b².

f'(x) = 4a²x³ + 4abx = 4ax(ax² + b).

(iii) f(x) = (x - a)/(x - b). Using the quotient rule:

f'(x) = [(x - b)·1 - (x - a)·1]/(x - b)²

= (x - b - x + a)/(x - b)²

= (a - b)/(x - b)².

Answer: (i) 2x - (a+b) (ii) 4ax(ax²+b) (iii) (a-b)/(x-b)²

Q8 Find the derivative of (x^n - a^n)/(x - a) for some constant a.

SOLUTION

Let f(x) = (x^n - a^n)/(x - a). Apply the quotient rule with u = x^n - a^n and v = x - a.

u' = nx^(n-1), v' = 1.

f'(x) = [nx^(n-1)(x - a) - (x^n - a^n)·1] / (x - a)²

Expanding the numerator: nx^n - nax^(n-1) - x^n + a^n = (n-1)x^n - nax^(n-1) + a^n.

f'(x) = [nx^(n-1)(x - a) - (x^n - a^n)] / (x - a)²

Answer: [nx^(n-1)(x-a) - (x^n-a^n)]/(x-a)²

Q9 Find the derivative of: (i) 2x - 3/4 (ii) (5x³ + 3x - 1)(x - 1) (iii) x^(-3)(5 + 3x) (iv) x⁵(3 - 6x^(-9)) (v) x^(-4)(3 - 4x^(-5)) (vi) 2/(x+1) - x²/(3x-1)

SOLUTION

(i) d/dx(2x - 3/4) = 2.

(ii) f(x) = (5x³ + 3x - 1)(x - 1). Use product rule:
f'(x) = (15x² + 3)(x - 1) + (5x³ + 3x - 1)(1)
= 15x³ - 15x² + 3x - 3 + 5x³ + 3x - 1
= 20x³ - 15x² + 6x - 4.

(iii) f(x) = x^(-3)(5 + 3x) = 5x^(-3) + 3x^(-2).
f'(x) = -15x^(-4) + (-6)x^(-3) = -15/x⁴ - 6/x³.

(iv) f(x) = x⁵(3 - 6x^(-9)) = 3x⁵ - 6x^(-4).
f'(x) = 15x⁴ + 24x^(-5) = 15x⁴ + 24/x⁵.

(v) f(x) = x^(-4)(3 - 4x^(-5)) = 3x^(-4) - 4x^(-9).
f'(x) = -12x^(-5) + 36x^(-10) = -12/x⁵ + 36/x¹⁰.

(vi) f(x) = 2/(x+1) - x²/(3x-1).
d/dx[2/(x+1)] = -2/(x+1)².
d/dx[x²/(3x-1)] = [2x(3x-1) - x²·3]/(3x-1)² = (6x²-2x-3x²)/(3x-1)² = (3x²-2x)/(3x-1)².
f'(x) = -2/(x+1)² - (3x²-2x)/(3x-1)².

Answer: (i) 2 (ii) 20x³-15x²+6x-4 (iii) -15/x⁴-6/x³ (iv) 15x⁴+24/x⁵ (v) -12/x⁵+36/x¹⁰ (vi) -2/(x+1)²-(3x²-2x)/(3x-1)²

Q10 Find the derivative of cos x from first principle.

SOLUTION

Let f(x) = cos x. By definition:
f'(x) = lim(h→0) [cos(x+h) - cos x]/h.

Using the identity cos A - cos B = -2 sin((A+B)/2) sin((A-B)/2):
cos(x+h) - cos x = -2 sin(x + h/2) sin(h/2).

f'(x) = lim(h→0) [-2 sin(x+h/2) sin(h/2)]/h
= lim(h→0) -sin(x+h/2) × [sin(h/2)/(h/2)]
= -sin x × 1 = -sin x.

Answer: d/dx(cos x) = -sin x

Q11 Find the derivative of the following functions: (i) sin x cos x (ii) sec x (iii) 5 sec x + 4 cos x (iv) cosec x (v) 3 cot x + 5 cosec x (vi) 5 sin x - 6 cos x + 7 (vii) 2 tan x - 7 sec x

SOLUTION

(i) f(x) = sin x cos x = (1/2) sin 2x.
f'(x) = (1/2) × 2 cos 2x = cos 2x.
Alternatively, using product rule: (cos x)(cos x) + (sin x)(-sin x) = $\cos^2 x - \sin^2 x$ = cos 2x.

(ii) f(x) = sec x = 1/cos x.
f'(x) = [0·cos x - 1·(-sin x)]/$\cos^2 x$ = sin x/$\cos^2 x$ = sec x tan x.

(iii) f(x) = 5 sec x + 4 cos x.
f'(x) = 5 sec x tan x + 4(-sin x) = 5 sec x tan x - 4 sin x.

(iv) f(x) = cosec x = 1/sin x.
f'(x) = [0·sin x - 1·cos x]/$\sin^2 x$ = -cos x/$\sin^2 x$ = -cosec x cot x.

(v) f(x) = 3 cot x + 5 cosec x.
d/dx(cot x) = -$\text{cosec}^2 x$, d/dx(cosec x) = -cosec x cot x.
f'(x) = -3 $\text{cosec}^2 x$ - 5 cosec x cot x.

(vi) f(x) = 5 sin x - 6 cos x + 7.
f'(x) = 5 cos x - 6(-sin x) + 0 = 5 cos x + 6 sin x.

(vii) f(x) = 2 tan x - 7 sec x.
f'(x) = 2 $\sec^2 x$ - 7 sec x tan x.

Answer: (i) cos 2x (ii) sec x tan x (iii) 5 sec x tan x - 4 sin x (iv) -cosec x cot x (v) $-3\,\text{cosec}^2x - 5\,\text{cosec}\,x\cot x$ (vi) 5 cos x + 6 sin x (vii) $2\sec^2x - 7\sec x\tan x$

Misc — Miscellaneous Exercise

Q1 Find the derivative of the following functions from first principle: (i) -x (ii) (-x)^(-1) (iii) sin(x + 1) (iv) cos(x - π/8)

SOLUTION

(i) f(x) = -x.
f'(x) = lim(h→0) [-(x+h) - (-x)]/h = lim(h→0) (-h)/h = -1.

(ii) f(x) = (-x)^(-1) = -1/x.
f'(x) = lim(h→0) [-1/(x+h) - (-1/x)]/h
= lim(h→0) [1/x - 1/(x+h)]/h
= lim(h→0) [(x+h) - x]/[h·x(x+h)]
= lim(h→0) h/[h·x(x+h)]
= lim(h→0) 1/[x(x+h)] = 1/x².
So f'(x) = 1/x².

(iii) f(x) = sin(x + 1).
f'(x) = lim(h→0) [sin(x+h+1) - sin(x+1)]/h.
Using sin A - sin B = 2cos((A+B)/2)sin((A-B)/2):
= lim(h→0) [2cos(x+1+h/2)sin(h/2)]/h
= lim(h→0) cos(x+1+h/2) × [sin(h/2)/(h/2)]
= cos(x+1) × 1 = cos(x+1).

(iv) f(x) = cos(x - π/8).
f'(x) = lim(h→0) [cos(x+h-π/8) - cos(x-π/8)]/h.
Using cos A - cos B = -2sin((A+B)/2)sin((A-B)/2):
= lim(h→0) [-2sin(x-π/8+h/2)sin(h/2)]/h
= lim(h→0) -sin(x-π/8+h/2) × [sin(h/2)/(h/2)]
= -sin(x - π/8).

Answer: (i) -1 (ii) 1/x² (iii) cos(x+1) (iv) -sin(x-π/8)

Q2 Find the derivative of (x + a).

SOLUTION

d/dx(x + a) = d/dx(x) + d/dx(a) = 1 + 0 = 1.

Answer: 1

Q3 Find the derivative of (px + q)(r/x + s).

SOLUTION

Expand: (px + q)(r/x + s) = pr + psx + qr/x + qs.

Differentiate term by term:
d/dx(pr) = 0, d/dx(psx) = ps, d/dx(qr/x) = -qr/x², d/dx(qs) = 0.

f'(x) = ps - qr/x².

Answer: ps - qr/x²

Q4 Find the derivative of (ax + b)(cx + d)².

SOLUTION

Let u = ax + b and v = (cx + d)².
u' = a, v' = 2c(cx + d).

Using product rule: f'(x) = u'v + uv'
= a(cx + d)² + (ax + b)·2c(cx + d)
= (cx + d)[a(cx + d) + 2c(ax + b)]
= (cx + d)[acx + ad + 2acx + 2bc]
= (cx + d)(3acx + ad + 2bc).

Answer: (cx+d)(3acx + ad + 2bc)

Q5 Find the derivative of (ax + b)/(cx + d).

SOLUTION

Using the quotient rule with u = ax + b, v = cx + d; u' = a, v' = c:
f'(x) = (u'v - uv')/v² = [a(cx + d) - c(ax + b)]/(cx + d)²
= [acx + ad - acx - bc]/(cx + d)²
= (ad - bc)/(cx + d)².

Answer: (ad - bc)/(cx + d)²

Q6 Find the derivative of (1 + 1/x)/(1 - 1/x).

SOLUTION

Simplify by multiplying numerator and denominator by x:
(1 + 1/x)/(1 - 1/x) = (x + 1)/(x - 1).

Using the quotient rule with u = x+1, v = x-1; u'=1, v'=1:
f'(x) = [(x-1) - (x+1)]/(x-1)² = (-2)/(x-1)² = -2/(x-1)².

Answer: -2/(x-1)²

Q7 Find the derivative of 1/(ax² + bx + c).

SOLUTION

Let f(x) = (ax² + bx + c)^(-1).
Using quotient rule (or chain rule): f'(x) = -1/(ax²+bx+c)² × d/dx(ax²+bx+c)
= -(2ax + b)/(ax² + bx + c)².

Answer: -(2ax + b)/(ax² + bx + c)²

Q8 Find the derivative of (ax + b)/(px² + qx + r).

SOLUTION

Let u = ax+b, v = px²+qx+r; u'=a, v'=2px+q.

f'(x) = (u'v - uv')/v²
= [a(px²+qx+r) - (ax+b)(2px+q)] / (px²+qx+r)².

Expand the numerator:
apx² + aqx + ar - (2apx²+aqx+2bpx+bq)
= apx² + aqx + ar - 2apx² - aqx - 2bpx - bq
= -apx² - 2bpx + ar - bq.

f'(x) = (-apx² - 2bpx + ar - bq)/(px² + qx + r)².

Answer: (-apx² - 2bpx + ar - bq)/(px² + qx + r)²

Q9 Find the derivative of $(px^2 + qx + r)/(ax + b)$.

SOLUTION

Let $u = px^2+qx+r$, $v = ax+b$; $u'=2px+q$, $v'=a$.

$f'(x) = (u'v - uv')/v^2$

$= [(2px+q)(ax+b) - (px^2+qx+r)\cdot a]/(ax+b)^2$.

Expand the numerator:

$2apx^2+2bpx+aqx+bq - apx^2-aqx-ar$

$= apx^2 + 2bpx + bq - ar$.

$f'(x) = (apx^2 + 2bpx + bq - ar)/(ax+b)^2$.

Answer: $(apx^2 + 2bpx + bq - ar)/(ax + b)^2$

Q10 Find the derivative of $a/x^4 - b/x^2 + \cos x$.

SOLUTION

Rewrite as f(x) = ax^(-4) - bx^(-2) + cos x.

Differentiate term by term:

d/dx(ax^(-4)) = -4ax^(-5) = $-4a/x^5$.

d/dx(-bx^(-2)) = 2bx^(-3) = $2b/x^3$.

d/dx(cos x) = -sin x.

$f'(x) = -4a/x^5 + 2b/x^3 - \sin x$.

Answer: $-4a/x^5 + 2b/x^3 - \sin x$

Q11 Find the derivative of $4\sqrt{x} - 2$.

SOLUTION

Rewrite as f(x) = 4x^(1/2) - 2.

$f'(x) = 4 \times (1/2)x^{(-1/2)} - 0 = 2x^{(-1/2)} = 2/\sqrt{x}$.

Answer: $2/\sqrt{x}$

Q12 Find the derivative of (ax + b)^n.

SOLUTION

Let u = ax + b, so f(x) = u^n.
By the chain rule (or by the power rule and product rule):
f'(x) = n(ax+b)^(n-1) × d/dx(ax+b) = n(ax+b)^(n-1) × a = na(ax+b)^(n-1).

Answer: na(ax + b)^(n-1)

Q13 Find the derivative of (ax + b)^n (cx + d)^m.

SOLUTION

Let u = (ax+b)^n and v = (cx+d)^m.
u' = na(ax+b)^(n-1), v' = mc(cx+d)^(m-1).

By product rule: f'(x) = u'v + uv'
= na(ax+b)^(n-1)(cx+d)^m + mc(cx+d)^(m-1)(ax+b)^n
= (ax+b)^(n-1)(cx+d)^(m-1)[na(cx+d) + mc(ax+b)].

Answer: (ax+b)^(n-1)(cx+d)^(m-1)[na(cx+d) + mc(ax+b)]

Q14 Find the derivative of sin(x + a).

SOLUTION

Using the addition formula: sin(x + a) = sin x cos a + cos x sin a.
d/dx[sin x cos a + cos x sin a] = cos x cos a - sin x sin a = cos(x + a).

Therefore d/dx[sin(x+a)] = cos(x + a).

Answer: cos(x + a)

Q15 Find the derivative of cosec x cot x.

SOLUTION

Let f(x) = cosec x cot x. Use the product rule with u = cosec x and v = cot x.
u' = -cosec x cot x, v' = -cosec²x.

f'(x) = u'v + uv'
= (-cosec x cot x)(cot x) + (cosec x)(-cosec²x)
= -cosec x cot²x - cosec³x
= -cosec x (cot²x + cosec²x).

Answer: -cosec x ($\cot^2 x + \operatorname{cosec}^2 x$)

Q16 Find the derivative of cos x/(1 + sin x).

SOLUTION

Using quotient rule with u = cos x, v = 1 + sin x; u' = -sin x, v' = cos x:

$f'(x) = (-\sin x(1+\sin x) - \cos x \cdot \cos x)/(1+\sin x)^2$

$= (-\sin x - \sin^2 x - \cos^2 x)/(1+\sin x)^2$

$= (-\sin x - 1)/(1+\sin x)^2$ [since $\sin^2 x + \cos^2 x = 1$]

$= -(1+\sin x)/(1+\sin x)^2$

$= -1/(1+\sin x)$.

Answer: -1/(1 + sin x)

Q17 Find the derivative of (sin x + cos x)/(sin x - cos x).

SOLUTION

Using quotient rule with u = sin x + cos x, v = sin x - cos x:

u' = cos x - sin x, v' = cos x + sin x.

$f'(x) = (u'v - uv')/v^2$

$= [(\cos x - \sin x)(\sin x - \cos x) - (\sin x + \cos x)(\cos x + \sin x)]/(\sin x - \cos x)^2$.

Numerator:

$(\cos x - \sin x)(\sin x - \cos x) = -(\sin x - \cos x)^2 = -(1 - \sin 2x)$.

$(\sin x + \cos x)^2 = 1 + \sin 2x$.

So numerator = $-(1 - \sin 2x) - (1 + \sin 2x) = -1 + \sin 2x - 1 - \sin 2x = -2$.

$f'(x) = -2/(\sin x - \cos x)^2$.

Answer: $-2/(\sin x - \cos x)^2$

Q18 Find the derivative of (sec x - 1)/(sec x + 1).

SOLUTION

Rewrite by multiplying numerator and denominator by cos x:

$(\sec x - 1)/(\sec x + 1) = (1 - \cos x)/(1 + \cos x)$.

Using quotient rule with u = 1 - cos x, v = 1 + cos x; u' = sin x, v' = -sin x:

$f'(x) = [\sin x(1+\cos x) - (1-\cos x)(-\sin x)]/(1+\cos x)^2$

$= [\sin x + \sin x \cos x + \sin x - \sin x \cos x]/(1+\cos x)^2$

= 2 sin x/(1+cos x)².

Answer: 2 sin x/(1 + cos x)²

Q19 Find the derivative of sin^n x.

SOLUTION

Write f(x) = (sin x)^n.
Using the product/power rule: f'(x) = n(sin x)^(n-1) × d/dx(sin x)
= n(sin x)^(n-1) cos x
= n sin^(n-1)x cos x.

Answer: n sin^(n-1)x cos x

Q20 Find the derivative of (a + b sin x)/(c + d cos x).

SOLUTION

Using quotient rule with u = a + b sin x, v = c + d cos x:
u' = b cos x, v' = -d sin x.

f'(x) = [b cos x(c + d cos x) - (a + b sin x)(-d sin x)]/(c + d cos x)²
= [bc cos x + bd cos²x + ad sin x + bd sin²x]/(c + d cos x)²
= [bc cos x + bd(cos²x + sin²x) + ad sin x]/(c + d cos x)²
= (bc cos x + bd + ad sin x)/(c + d cos x)².

Answer: (bc cos x + ad sin x + bd)/(c + d cos x)²

Q21 Find the derivative of sin(x + a)/cos x.

SOLUTION

Expand using addition formula: sin(x+a) = sin x cos a + cos x sin a.
So sin(x+a)/cos x = sin x cos a/cos x + sin a = tan x cos a + sin a.

d/dx = cos a × sec²x.

Alternatively, using the quotient rule with u = sin(x+a), v = cos x:
u' = cos(x+a), v' = -sin x.
f'(x) = [cos(x+a) cos x + sin(x+a) sin x]/cos²x
= cos(x+a-x)/cos²x = cos a/cos²x = cos a · sec²x.

Answer: cos a · sec²x

Q22 Find the derivative of $x^4(5 \sin x - 3 \cos x)$.

SOLUTION

Let $u = x^4$ and $v = 5 \sin x - 3 \cos x$.
$u' = 4x^3$, $v' = 5 \cos x + 3 \sin x$.

$f'(x) = u'v + uv'$
$= 4x^3(5 \sin x - 3 \cos x) + x^4(5 \cos x + 3 \sin x)$
$= x^3[4(5 \sin x - 3 \cos x) + x(5 \cos x + 3 \sin x)]$
$= x^3[20 \sin x - 12 \cos x + 5x \cos x + 3x \sin x]$
$= x^3[(20 + 3x) \sin x + (5x - 12) \cos x]$.

Answer: $x^3[(20+3x) \sin x + (5x-12) \cos x]$

Q23 Find the derivative of $(x^2 + 1) \cos x$.

SOLUTION

Let $u = x^2+1$, $v = \cos x$; $u' = 2x$, $v' = -\sin x$.

$f'(x) = 2x \cos x + (x^2+1)(-\sin x)$
$= 2x \cos x - (x^2+1) \sin x$.

Answer: $2x \cos x - (x^2+1) \sin x$

Q24 Find the derivative of $(ax + \sin x)(p + q \cos x)$.

SOLUTION

Let $u = ax + \sin x$, $v = p + q \cos x$.
$u' = a + \cos x$, $v' = -q \sin x$.

$f'(x) = (a + \cos x)(p + q \cos x) + (ax + \sin x)(-q \sin x)$
$= (a+\cos x)(p+q \cos x) - q(ax+\sin x) \sin x$.

Expanding: $= ap + aq \cos x + p \cos x + q \cos^2 x - aqx \sin x - q \sin^2 x$
$= ap + (aq+p) \cos x + q(\cos^2 x - \sin^2 x) - aqx \sin x$
$= ap + (aq+p) \cos x + q \cos 2x - aqx \sin x$.

Answer: $(a+\cos x)(p+q \cos x) - q(ax+\sin x) \sin x$

Q25 Find the derivative of (x + cos x)(x - tan x).

SOLUTION

Let u = x + cos x, v = x - tan x.
u' = 1 - sin x, v' = 1 - sec²x.

f'(x) = (1 - sin x)(x - tan x) + (x + cos x)(1 - sec²x).

Answer: (1-sin x)(x-tan x) + (x+cos x)(1-sec²x)

Q26 Find the derivative of (4x + 5 sin x)/(3x + 7 cos x).

SOLUTION

Let u = 4x+5 sin x, v = 3x+7 cos x.
u' = 4+5 cos x, v' = 3-7 sin x.

f'(x) = (u'v - uv')/v²
= [(4+5 cos x)(3x+7 cos x) - (4x+5 sin x)(3-7 sin x)]/(3x+7 cos x)².

Expand the numerator:
(4+5 cos x)(3x+7 cos x) = 12x+28 cos x+15x cos x+35 cos²x.
(4x+5 sin x)(3-7 sin x) = 12x-28x sin x+15 sin x-35 sin²x.
Numerator = 12x+28 cos x+15x cos x+35 cos²x - 12x+28x sin x-15 sin x+35 sin²x
= 28 cos x+15x cos x+35(cos²x+sin²x)+28x sin x-15 sin x
= 35 + 28 cos x - 15 sin x + x(15 cos x + 28 sin x).

f'(x) = [35 + 28 cos x - 15 sin x + x(15 cos x + 28 sin x)]/(3x+7 cos x)².

Answer: [35+28 cos x-15 sin x+x(15 cos x+28 sin x)]/(3x+7 cos x)²

Q27 Find the derivative of x² cos(π/4)/sin x.

SOLUTION

Note cos(π/4) = 1/√2 is a constant. Write f(x) = cos(π/4) × x²/sin x.

Using quotient rule on x²/sin x with u = x², v = sin x; u'=2x, v'=cos x:
d/dx(x²/sin x) = (2x sin x - x² cos x)/sin²x.

f'(x) = cos(π/4) × (2x sin x - x² cos x)/sin²x
= (1/√2) × x(2 sin x - x cos x)/sin²x.

Answer: (x(2 sin x - x cos x))/(√2 sin²x)

Q28 Find the derivative of x/(1 + tan x).

SOLUTION

Using quotient rule with u = x, v = 1+tan x; u'=1, v'=sec²x:
f'(x) = [(1)(1+tan x) - x·sec²x]/(1+tan x)²
= (1 + tan x - x sec²x)/(1+tan x)².

Answer: (1 + tan x - x sec²x)/(1+tan x)²

Q29 Find the derivative of (x + sec x)(x - tan x).

SOLUTION

Let u = x + sec x, v = x - tan x.
u' = 1 + sec x tan x, v' = 1 - sec²x.

f'(x) = (1+sec x tan x)(x-tan x) + (x+sec x)(1-sec²x).

Answer: (1+sec x tan x)(x-tan x) + (x+sec x)(1-sec²x)

Q30 Find the derivative of x/sin^n x.

SOLUTION

Using quotient rule with u = x, v = sin^n x; u'=1, v'=n sin^(n-1)x cos x:
f'(x) = [1·sin^n x - x·n sin^(n-1)x cos x]/(sin^n x)²
= sin^(n-1)x [sin x - nx cos x]/sin^(2n)x
= (sin x - nx cos x)/sin^(n+1)x.

Answer: (sin x - nx cos x)/sin^(n+1)x