Chapter 6: Systems of Particles and Rotational Motion

Exercises — Exercises

Q1 Give the location of the centre of mass of a (i) sphere, (ii) cylinder, (iii) ring, and (iv) cube, each of uniform mass density. Does the centre of mass of a body necessarily lie inside the body?

SOLUTION

For any homogeneous body with a uniform mass density, the centre of mass coincides with the geometric centre (centroid) by symmetry — for every mass element dm at position r there is an equal element at −r, so $\sum m_i \cdot r_i = 0$ when the origin is at the geometric centre.

(i) Sphere: The centre of mass is at the geometric centre of the sphere.
(ii) Cylinder: The centre of mass is at the mid-point of the axis of the cylinder (geometric centre).
(iii) Ring: The centre of mass is at the centre of the ring, which is the geometric centre. Note this point lies on the axis of the ring, not on the ring itself.
(iv) Cube: The centre of mass is at the geometric centre of the cube, i.e. at the point of intersection of its main diagonals.

The centre of mass does NOT necessarily lie inside the body. For a ring or a hollow sphere, the centre of mass lies at the geometric centre, which is in the empty space inside — not on the material of the body itself. Similarly, for a boomerang or horseshoe shape, the CM lies outside the material of the object.

Answer: (i) Geometric centre of the sphere. (ii) Mid-point of the cylinder's axis. (iii) Centre of the ring (on its axis, not on the ring material). (iv) Geometric centre of the cube. The CM need not lie inside the body (e.g. for a ring it lies in empty space at the centre).

Q2 In the HCl molecule, the separation between the nuclei of the two atoms is about 1.27 Å (1 Å = 10^{-10} m). Find the approximate location of the CM of the molecule, given that a chlorine atom is about 35.5 times as massive as a hydrogen atom and nearly all the mass of an atom is concentrated in its nucleus.

SOLUTION

Given: separation between H and Cl nuclei = 1.27 Å; $m_{Cl} = 35.5 \times m_H$.

Let the H nucleus be at the origin (x = 0) and the Cl nucleus at x = 1.27 Å.

Let mass of H = m, so mass of Cl = 35.5 m. Total mass = m + 35.5 m = 36.5 m.

Position of CM:

$X_{CM} = (m \times 0 + 35.5m \times 1.27) / 36.5m$

$= (35.5 \times 1.27) / 36.5$

$= 45.085 / 36.5$

$= 1.235$ Å

So the CM is approximately 1.24 Å from the hydrogen nucleus (and 1.27 − 1.24 = 0.03 Å from the chlorine nucleus).

The CM lies very close to the Cl nucleus (which is much heavier), at about 1.24 Å from H.

Answer: The CM of HCl is approximately 1.24 Å from the hydrogen nucleus (and about 0.03 Å from the chlorine nucleus).

Q3 A child sits stationary at one end of a long trolley moving uniformly with a speed V on a smooth horizontal floor. If the child gets up and runs about on the trolley in any manner, what is the speed of the CM of the (trolley + child) system?

SOLUTION

Initially the child and trolley move together with speed V. The total linear momentum of the system is:

$P = (M_{trolley} + M_{child}) \times V$

When the child runs on the trolley, the forces involved (the child pushing off the trolley floor and vice versa) are all internal forces to the (child + trolley) system.

The floor is smooth (frictionless), so there is no external horizontal force on the system.

By Newton's second law for a system: $dP/dt = F_{ext} = 0$

Therefore P = constant, i.e. the total momentum of the system does not change.

The velocity of the CM is:
$V_{CM} = P / (M_{trolley} + M_{child}) = V$

No matter how the child runs on the trolley, the speed of the CM of the (trolley + child) system remains V.

Answer: The speed of the CM of the (trolley + child) system remains V throughout, because no external horizontal force acts on the system (smooth floor), so total linear momentum is conserved.

Q4 Show that the area of the triangle contained between the vectors a and b is one half of the magnitude of a × b.

SOLUTION

Let vectors a and b originate from a common point O, with magnitudes a and b respectively, and let θ be the angle between them ($0 < \theta < 180°$).

The triangle formed by a and b has these two vectors as two of its sides.

Area of a parallelogram formed by a and b = base × height = a × (b sin θ) = ab sin θ.

The triangle formed by a and b is exactly half of this parallelogram (the diagonal of the parallelogram divides it into two congruent triangles).

Therefore, area of triangle = (1/2) ab sin θ.

Now, by definition, the magnitude of the cross product:
$|a \times b| = ab \sin \theta$.

Hence, area of triangle = $(1/2)|a \times b|$. QED.

Answer: Area of triangle formed by a and b = (1/2) ab sin θ = $(1/2)|a \times b|$. This follows because the triangle is half the parallelogram formed by the two vectors, and the parallelogram's area equals $|a \times b| = ab \sin \theta$.

Q5 Show that a·(b × c) is equal in magnitude to the volume of the parallelepiped formed on the three vectors a, b and c.

SOLUTION

Consider the parallelepiped formed by three vectors a, b, and c originating from a common point.

Step 1: b × c is a vector perpendicular to the plane of b and c, with magnitude |b × c| = bc sin θ, where θ is the angle between b and c. This magnitude equals the area of the base parallelogram (formed by b and c).

Step 2: The scalar triple product a·(b × c) = |a||b × c| cos φ, where φ is the angle between a and the vector (b × c).

Step 3: Since (b × c) is perpendicular to the base, φ is the angle between a and the normal to the base. The height of the parallelepiped is h = |a| cos φ.

Step 4: Volume of parallelepiped = base area × height

=	b × c	×	a	cos φ
=	a · (b × c)			

Hence a·(b × c) equals in magnitude the volume of the parallelepiped. QED.

Note: the sign of a·(b × c) indicates the handedness of the orientation of (a, b, c).

Answer: a·(b × c) = |a||b × c| cos φ = (base area of parallelogram formed by b,c) × (height h = |a| cos φ) = volume of the parallelepiped formed on a, b, c.

Q6 Find the components along the x, y, z axes of the angular momentum l of a particle, whose position vector is r with components x, y, z and momentum is p with components p_x, p_y and p_z. Show that if the particle moves only in the x-y plane the angular momentum has only a z-component.

SOLUTION

The angular momentum is l = r × p.

Using the determinant form:

	$\hat{i}$ $\hat{j}$ $\hat{k}$	
	x y z	
	p_x p_y p_z	

$l_x = y \cdot p_z - z \cdot p_y$
$l_y = z \cdot p_x - x \cdot p_z$
$l_z = x \cdot p_y - y \cdot p_x$

These are the three components of angular momentum along the x, y, z axes respectively.

Now, if the particle moves only in the x-y plane:
— Its z-coordinate is always zero: z = 0
— It has no z-component of momentum: $p_z = 0$

Substituting z = 0 and $p_z = 0$:
$l_x = y \cdot (0) - (0) \cdot p_y = 0$
$l_y = (0) \cdot p_x - x \cdot (0) = 0$
$l_z = x \cdot p_y - y \cdot p_x$ (generally non-zero)

Hence when the particle moves only in the x-y plane, $l_x = l_y = 0$, and the angular momentum has only a z-component: $l = l_z \hat{k}$. QED.

Answer: $l_x = y \cdot p_z - z \cdot p_y$, $l_y = z \cdot p_x - x \cdot p_z$, $l_z = x \cdot p_y - y \cdot p_x$. For motion in the x-y plane (z = 0, $p_z = 0$), $l_x = l_y = 0$, so l has only the z-component: $l = (x \cdot p_y - y \cdot p_x)\,\hat{k}$.

Q7 Two particles, each of mass m and speed v, travel in opposite directions along parallel lines separated by a distance d. Show that the angular momentum vector of the two particle system is the same whatever be the point about which the angular momentum is taken.

SOLUTION

Let the two parallel lines be in the x-direction, separated by distance d. Let particle 1 move with velocity $+v\,\hat{i}$ along the line y = 0 and particle 2 move with velocity $-v\,\hat{i}$ along the line y = d.

For an arbitrary origin O at position (x_o, y_o):

Position vector of particle 1 relative to O: $r_1 = (x_1 - x_o)\,\hat{i} + (0 - y_o)\,\hat{j}$
Momentum of particle 1: $p_1 = mv\,\hat{i}$

Angular momentum of particle 1 about O:
$l_1 = r_1 \times p_1 = [(x_1 - x_o)\,\hat{i} - y_o\,\hat{j}] \times (mv\,\hat{i})$
$= mv[(x_1 - x_o)(\hat{i} \times \hat{i}) - y_o(\hat{j} \times \hat{i})]$
$= mv[0 - y_o(-\hat{k})]$

$= mv \cdot y_0 \hat{k}$

Position vector of particle 2 relative to O: $r_2 = (x_2 - x_0)\,\hat{i} + (d - y_0)\,\hat{j}$
Momentum of particle 2: $p_2 = -mv\,\hat{i}$

Angular momentum of particle 2 about O:
$l_2 = r_2 \times p_2 = [(x_2 - x_0)\,\hat{i} + (d - y_0)\,\hat{j}] \times (-mv\,\hat{i})$
$= -mv[(x_2 - x_0)(\hat{i} \times \hat{i}) + (d - y_0)(\hat{j} \times \hat{i})]$
$= -mv[0 + (d - y_0)(-\hat{k})]$
$= mv(d - y_0)\,\hat{k}$

Total angular momentum:
$L = l_1 + l_2 = mv \cdot y_0\,\hat{k} + mv(d - y_0)\,\hat{k}$
$= mv[y_0 + d - y_0]\,\hat{k}$
$= mvd\,\hat{k}$

The result $L = mvd\,\hat{k}$ is independent of x_0 and y_0 (the position of the chosen origin). Therefore the angular momentum of the two-particle system is the same about any point. QED.

Answer: $L = mvd\,\hat{k}$, which is independent of the choice of origin. The y_0-dependent terms cancel exactly, leaving only the separation d between the lines. So the total angular momentum is the same whatever point it is taken about.

Q8 A non-uniform bar of weight W is suspended at rest by two strings of negligible weight as shown in Fig. 6.33. The angles made by the strings with the vertical are 36.9° and 53.1° respectively. The bar is 2 m long. Calculate the distance d of the centre of gravity of the bar from its left end.

SOLUTION

Given: bar length = 2 m; left string makes 36.9° with vertical; right string makes 53.1° with vertical.
Using $\sin(36.9°) \approx 0.60$, $\cos(36.9°) \approx 0.80$, $\sin(53.1°) \approx 0.80$, $\cos(53.1°) \approx 0.60$.

Let T_1 = tension in left string, T_2 = tension in right string.

Translational equilibrium:
Horizontal: $T_1 \sin(36.9°) = T_2 \sin(53.1°)$
$\Rightarrow T_1 \times 0.60 = T_2 \times 0.80$
$\Rightarrow T_1 = (4/3)\,T_2$ ... (i)

Vertical: $T_1 \cos(36.9°) + T_2 \cos(53.1°) = W$

=> $T_1 \times 0.80 + T_2 \times 0.60 = W$ ... (ii)

From (i) and (ii):

$(4/3)T_2 \times 0.80 + T_2 \times 0.60 = W$

$T_2(3.2/3 + 0.6) = W$

$T_2 \times (5/3) = W$

$T_2 = 3W/5 = 0.60W$

$T_1 = (4/3)(0.60W) = 0.80W$

Rotational equilibrium about the left end (where T_1 acts):

$T_2 \cos(53.1°) \times 2 = W \times d$

$(0.60W)(0.60) \times 2 = W \times d$

$0.72W = W \times d$

$d = 0.72$ m

Answer: d = 0.72 m from the left end.

Q9 A car weighs 1800 kg. The distance between its front and back axles is 1.8 m. Its centre of gravity is 1.05 m behind the front axle. Determine the force exerted by the level ground on each front wheel and each back wheel.

SOLUTION

Given: M = 1800 kg, axle separation L = 1.8 m, CG is 1.05 m behind front axle.
Weight W = Mg = 1800 × 9.8 = 17640 N.

Let N_f = total reaction on front axle, N_r = total reaction on rear axle.

Translational equilibrium (vertical):
N_f + N_r = W = 17640 N ... (i)

Rotational equilibrium about the front axle:
N_r × 1.8 = W × 1.05
N_r = 17640 × 1.05 / 1.8 = 18522 / 1.8 = 10290 N ... (ii)

From (i):
N_f = 17640 − 10290 = 7350 N

The car has two front wheels and two rear wheels, so:
Force per front wheel = N_f / 2 = 7350 / 2 = 3675 N
Force per rear wheel = N_r / 2 = 10290 / 2 = 5145 N

Answer: Force on each front wheel = 3675 N; Force on each rear wheel = 5145 N.

Q10 Torques of equal magnitude are applied to a hollow cylinder and a solid sphere, both having the same mass and radius. The cylinder is free to rotate about its standard axis of symmetry, and the sphere is free to rotate about an axis passing through its centre. Which of the two will acquire a greater angular speed after a given time?

SOLUTION

Given: same mass M, same radius R, same torque τ applied for the same time t.

Moment of inertia of hollow cylinder about its symmetry axis: $I_{cyl} = MR^2$
Moment of inertia of solid sphere about a diameter: $I_{sph} = (2/5)MR^2$

Since $I_{sph} < I_{cyl}$, and torque $\tau = I\alpha$, angular acceleration $\alpha = \tau/I$:
$\alpha_{cyl} = \tau/(MR^2)$
$\alpha_{sph} = \tau/[(2/5)MR^2] = 5\tau/(2MR^2)$

Ratio: $\alpha_{sph}/\alpha_{cyl} = (5/2) = 2.5$

After time t, angular speed $\omega = \alpha \times t$ (starting from rest):
$\omega_{sph}/\omega_{cyl} = \alpha_{sph}/\alpha_{cyl} = 5/2$

The solid sphere acquires 2.5 times the angular speed of the hollow cylinder in the same time.

Answer: The solid sphere acquires the greater angular speed. Its moment of inertia ($2MR^2/5$) is smaller than that of the hollow cylinder (MR^2), so for the same torque it has a larger angular acceleration ($\alpha_{sph} = 5\tau/2MR^2$ vs $\alpha_{cyl} = \tau/MR^2$), giving $\omega_{sph} = 2.5\ \omega_{cyl}$ after the same time.

Q11 A solid cylinder of mass 20 kg rotates about its axis with angular speed 100 rad s^{-1}. The radius of the cylinder is 0.25 m. What is the kinetic energy associated with the rotation of the cylinder? What is the magnitude of angular momentum of the cylinder about its axis?

SOLUTION

Given: M = 20 kg, R = 0.25 m, ω = 100 rad s^{-1}.

Moment of inertia of solid cylinder about its axis:
$I = MR^2/2 = (20 \times 0.25^2)/2 = (20 \times 0.0625)/2 = 1.25/2 = 0.625$ kg m^2

Kinetic energy of rotation:
$KE = (1/2)I\omega^2 = (1/2) \times 0.625 \times (100)^2$
$= 0.5 \times 0.625 \times 10000$

= 3125 J

Angular momentum about the axis:
$L = I\omega = 0.625 \times 100 = 62.5$ kg m² s⁻¹

Answer: Kinetic energy = 3125 J; Angular momentum = 62.5 kg m² s⁻¹.

Q12 (a) A child stands at the centre of a turntable with his two arms outstretched. The turntable is set rotating with an angular speed of 40 rev/min. How much is the angular speed of the child if he folds his hands back and thereby reduces his moment of inertia to 2/5 times the initial value? Assume that the turntable rotates without friction. (b) Show that the child's new kinetic energy of rotation is more than the initial kinetic energy of rotation. How do you account for this increase in kinetic energy?

SOLUTION

Given: $\omega_o = 40$ rev/min; $I_{new} = (2/5)\, I_o$.

Part (a):
The turntable is frictionless, so no external torque acts. Angular momentum is conserved:
$I_o \times \omega_o = I_{new} \times \omega_{new}$
$I_o \times 40 = (2/5) I_o \times \omega_{new}$
$\omega_{new} = 40 \times (5/2) = 100$ rev/min

Part (b):
Initial KE: $KE_o = (1/2)\, I_o\, \omega_o^2$

New KE: $KE_{new} = (1/2)\, I_{new}\, \omega_{new}^2$
$= (1/2) \times (2/5) I_o \times (5/2 \times \omega_o)^2$
$= (1/2) \times (2/5) I_o \times (25/4) \omega_o^2$
$= (1/2)\, I_o\, \omega_o^2 \times (2/5)(25/4)$
$= (1/2)\, I_o\, \omega_o^2 \times (50/20)$
$= (1/2)\, I_o\, \omega_o^2 \times (5/2)$
$= (5/2)\, KE_o$

Since $KE_{new} = (5/2)\, KE_o > KE_o$, the new kinetic energy is 2.5 times the initial.

The increase in kinetic energy comes from the work done by the child's muscles in pulling his arms inward against the centrifugal tendency. The child does internal work on the system by exerting force inward while the arms move inward, and this

muscular energy is converted into additional rotational kinetic energy.

Answer: (a) The child's angular speed increases to 100 rev/min. (b) $KE_{new} = (5/2)\,KE_0$ — it is 2.5 times the initial KE. The increase is supplied by the child's muscular effort in folding the arms inward (internal work done against centrifugal effect).

Q13 A rope of negligible mass is wound round a hollow cylinder of mass 3 kg and radius 40 cm. What is the angular acceleration of the cylinder if the rope is pulled with a force of 30 N? What is the linear acceleration of the rope? Assume that there is no slipping.

SOLUTION

Given: $M = 3$ kg, $R = 40$ cm $= 0.40$ m, $F = 30$ N.

Moment of inertia of hollow cylinder about its axis:
$I = MR^2 = 3 \times (0.40)^2 = 3 \times 0.16 = 0.48$ kg m^2

Torque due to the rope:
$\tau = F \times R = 30 \times 0.40 = 12.0$ N m

Angular acceleration:
$\alpha = \tau / I = 12.0/0.48 = 25$ rad s^{-2}

Linear acceleration of the rope (no slipping):
$a = \alpha \times R = 25 \times 0.40 = 10$ m s^{-2}

Answer: Angular acceleration $\alpha = 25$ rad s^{-2}; Linear acceleration of rope $a = 10$ m s^{-2}.

Q14 To maintain a rotor at a uniform angular speed of 200 rad s^{-1}, an engine needs to transmit a torque of 180 N m. What is the power required by the engine? (Note: uniform angular velocity in the absence of friction implies zero torque. In practice, applied torque is needed to counter frictional torque). Assume that the engine is 100% efficient.

SOLUTION

Given: $\omega = 200$ rad s^{-1}, $\tau = 180$ N m.

The power required to maintain a constant angular speed against friction:
$P = \tau \times \omega$
$= 180 \times 200$

= 36000 W
= 36 kW

Answer: Power required = 36 kW.

Q15 From a uniform disk of radius R, a circular hole of radius R/2 is cut out. The centre of the hole is at R/2 from the centre of the original disc. Locate the centre of gravity of the resulting flat body.

SOLUTION

Let the centre of the original disc be the origin. The hole is centred at x = R/2.

Surface mass density of disk: $\sigma = M/(\pi R^2)$, where M is the mass of the full disk.

Mass of the cut-out circle (radius R/2):
$m_{hole} = \sigma \times \pi(R/2)^2 = [M/(\pi R^2)] \times \pi R^2/4 = M/4$

Mass of the remaining body:
$m_{rem} = M - M/4 = 3M/4$

CG of original disk: x = 0
CG of cut-out (at centre of hole): x = R/2

Using: $M \times 0 = m_{rem} \times x_{CG} + m_{hole} \times (R/2)$
$0 = (3M/4) \times x_{CG} + (M/4) \times (R/2)$
$(3M/4) \times x_{CG} = -MR/8$
$x_{CG} = -(MR/8) / (3M/4) = -R/6$

The CG of the remaining body is at x = −R/6, i.e. at a distance R/6 from the centre of the original disc on the side opposite to the hole.

Answer: The CG of the resulting body is at R/6 from the centre of the original disc, on the side diametrically opposite to the hole (i.e. shifted away from the hole by R/6).

Q16 A metre stick is balanced on a knife edge at its centre. When two coins, each of mass 5 g are put one on top of the other at the 12.0 cm mark, the stick is found to be balanced at 45.0 cm. What is the mass of the metre stick?

SOLUTION

Given: Two coins each of 5 g (total 10 g) placed at the 12.0 cm mark.
New balance point (pivot): 45.0 cm mark.
The metre stick is uniform, so its CM is at the 50.0 cm mark.

For rotational equilibrium about the new pivot at 45.0 cm:

Anticlockwise torque (coins on left of pivot):
$\tau_{coins} = 10\ g \times (45.0 - 12.0)\ cm = 10 \times 33.0 = 330\ g\cdot cm$

Clockwise torque (weight of stick acts at 50.0 cm, right of pivot):
$\tau_{stick} = m \times (50.0 - 45.0)\ cm = m \times 5.0\ g\cdot cm$

Setting them equal:
$m \times 5.0 = 330$
$m = 66\ g$

Answer: The mass of the metre stick is 66 g.

Q17 The oxygen molecule has a mass of 5.30×10^{-26} kg and a moment of inertia of 1.94×10^{-46} kg m² about an axis through its centre perpendicular to the lines joining the two atoms. Suppose the mean speed of such a molecule in a gas is 500 m/s and that its kinetic energy of rotation is two thirds of its kinetic energy of translation. Find the average angular velocity of the molecule.

SOLUTION

Given: $m = 5.30 \times 10^{-26}$ kg, $I = 1.94 \times 10^{-46}$ kg m², $v = 500$ m/s.
Condition: $KE_{rot} = (2/3)\ KE_{trans}$.

Translational KE:
$KE_{trans} = (1/2)mv^2 = (1/2) \times 5.30 \times 10^{-26} \times (500)^2$
$= (1/2) \times 5.30 \times 10^{-26} \times 250000$
$= 6.625 \times 10^{-21}$ J

Rotational KE:
$KE_{rot} = (2/3) \times KE_{trans} = (2/3) \times 6.625 \times 10^{-21} = 4.417 \times 10^{-21}$ J

Also $KE_{rot} = (1/2)I\omega^2$, so:
$\omega^2 = 2 \times KE_{rot} / I$

$= 2 \times 4.417 \times 10^{-21} / 1.94 \times 10^{-46}$

$= 8.833 \times 10^{-21} / 1.94 \times 10^{-46}$

$= 4.553 \times 10^{25}\ \text{rad}^2\ \text{s}^{-2}$

Equivalently, using the condition directly:

$(1/2)I\omega^2 = (2/3)(1/2)mv^2$

$\omega^2 = (2mv^2)/(3I)$

$= (2 \times 5.30 \times 10^{-26} \times 250000) / (3 \times 1.94 \times 10^{-46})$

$= (2.65 \times 10^{-20}) / (5.82 \times 10^{-46})$

$= 4.553 \times 10^{25}\ \text{rad}^2\ \text{s}^{-2}$

$\omega = \sqrt{(4.553 \times 10^{25})} \approx 6.75 \times 10^{12}\ \text{rad s}^{-1}$

Answer: The average angular velocity of the oxygen molecule is approximately $6.75 \times 10^{12}\ \text{rad s}^{-1}$.