Chapter 1: Relations and Functions

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Exercise 1.1 — Exercise 1.1

Q1 Determine whether each of the following relations are reflexive, symmetric and transitive:
(i) Relation R in the set A = {1, 2, 3, ..., 13, 14} defined as R = {(x, y) : 3x – y = 0}
(ii) Relation R in the set N of natural numbers defined as R = {(x, y) : y = x + 5 and x < 4}
(iii) Relation R in the set A = {1, 2, 3, 4, 5, 6} as R = {(x, y) : y is divisible by x}
(iv) Relation R in the set Z of all integers defined as R = {(x, y) : x – y is an integer}
(v) Relation R in the set A of human beings in a town at a particular time given by
(a) R = {(x, y) : x and y work at the same place}
(b) R = {(x, y) : x and y live in the same locality}
(c) R = {(x, y) : x is exactly 7 cm taller than y}
(d) R = {(x, y) : x is wife of y}
(e) R = {(x, y) : x is father of y}

SOLUTION

(i) R = {(x, y) : 3x − y = 0} in A = {1, 2, 3, ..., 14}.
The pairs satisfying 3x = y are: (1,3), (2,6), (3,9), (4,12).
Reflexive: Need (x, x) ∈ R, i.e., 3x = x => x = 0. Since 0 ∉ A, no element is related to itself. NOT reflexive.
Symmetric: (1, 3) ∈ R but 3×3 − 1 = 8 ≠ 0, so (3, 1) ∉ R. NOT symmetric.
Transitive: Check for chains. (1,3) ∈ R and (3,9) ∈ R, but (1,9) ∉ R (since 3×1=3≠9). So the transitivity condition fails. NOT transitive.

(ii) R = {(x, y) : y = x + 5 and x < 4} in N.
Since x ∈ N and x < 4, we have x ∈ {1, 2, 3}, giving R = {(1,6), (2,7), (3,8)}.
Reflexive: (x, x) would require x = x + 5, impossible. NOT reflexive.
Symmetric: (1, 6) ∈ R but (6, 1) requires y = 1 = 6 + 5 = 11, false. NOT symmetric.
Transitive: For a chain (x, y) and (y, z) both in R, y would need to be in {1,2,3} (as

first element) and also in {6,7,8} (as second element) — impossible. Vacuously transitive. Transitive.

(iii) R = {(x, y) : y is divisible by x} in A = {1, 2, 3, 4, 5, 6}.
Reflexive: x divides x for every x. (x, x) $\in$ R for all x $\in$ A. Reflexive.
Symmetric: (1, 2) $\in$ R since 2 is divisible by 1, but (2, 1) $\notin$ R since 1 is not divisible by 2. NOT symmetric.
Transitive: If y is divisible by x and z is divisible by y, then z = ky and y = mx for integers k, m, so z = kmx, meaning z is divisible by x. (x, z) $\in$ R. Transitive.

(iv) R = {(x, y) : x − y is an integer} in Z.
Reflexive: x − x = 0 is an integer. (x, x) $\in$ R for all x $\in$ Z. Reflexive.
Symmetric: If x − y is an integer, then y − x = −(x − y) is also an integer. (y, x) $\in$ R. Symmetric.
Transitive: If x − y and y − z are integers, then x − z = (x − y) + (y − z) is an integer. (x, z) $\in$ R. Transitive.
R is an equivalence relation.

(v)
(a) R = {(x, y) : x and y work at the same place}.
Reflexive: x works at the same place as x. Reflexive.
Symmetric: If x and y work at the same place, then y and x work at the same place. Symmetric.
Transitive: If x and y work at the same place, and y and z work at the same place, then x and z work at the same place. Transitive.
R is an equivalence relation.

(b) R = {(x, y) : x and y live in the same locality}.
By the same reasoning as (a): reflexive, symmetric, transitive. R is an equivalence relation.

(c) R = {(x, y) : x is exactly 7 cm taller than y}.
Reflexive: x cannot be 7 cm taller than itself. NOT reflexive.
Symmetric: If x is 7 cm taller than y, then y is 7 cm shorter than x, not 7 cm taller. NOT symmetric.
Transitive: If x is 7 cm taller than y and y is 7 cm taller than z, then x is 14 cm taller than z, not 7 cm. NOT transitive.

(d) R = {(x, y) : x is wife of y}.
Reflexive: x cannot be wife of x. NOT reflexive.
Symmetric: If x is wife of y, then y is husband of x, not wife of x. NOT symmetric.

Transitive: If x is wife of y, then y is male and cannot be wife of any z. No chain (x,y),(y,z) exists in R. Vacuously transitive. Transitive.

(e) R = {(x, y) : x is father of y}.
Reflexive: x cannot be father of x. NOT reflexive.
Symmetric: If x is father of y, then y is child of x, not father. NOT symmetric.
Transitive: If x is father of y and y is father of z, then x is grandfather of z, not father. NOT transitive.

Answer: (i) Neither reflexive, nor symmetric, nor transitive. (ii) Neither reflexive nor symmetric, but transitive. (iii) Reflexive and transitive but not symmetric. (iv) Reflexive, symmetric and transitive (equivalence relation). (v)(a) equivalence, (b) equivalence, (c) none of the three, (d) transitive only, (e) none of the three.

Q2 Show that the relation R in the set R of real numbers, defined as R = {(a, b) : a ≤ b^2} is neither reflexive nor symmetric nor transitive.

SOLUTION

Not reflexive: We need a ≤ a^2 for all real a. Take a = 1/2: 1/2 ≤ $(1/2)^2$ = 1/4 is false. So (1/2, 1/2) ∉ R, and R is not reflexive.

Not symmetric: Take a = 1, b = 4. Then a ≤ b^2 since 1 ≤ 16, so (1, 4) ∈ R. But b ≤ a^2 requires 4 ≤ 1, which is false. So (4, 1) ∉ R, and R is not symmetric.

Not transitive: Take a = 25, b = −5, c = 2. Then a ≤ b^2: 25 ≤ $(-5)^2$ = 25, true, so (25, −5) ∈ R. Also b ≤ c^2: −5 ≤ 4, true, so (−5, 2) ∈ R. But a ≤ c^2: 25 ≤ 4 is false. So (25, 2) ∉ R, and R is not transitive.

Answer: R is neither reflexive (1/2 ≤ 1/4 fails), nor symmetric ((1,4) ∈ R but (4,1) ∉ R), nor transitive ((25,−5) ∈ R, (−5,2) ∈ R but (25,2) ∉ R).

Q3 Check whether the relation R defined in the set {1, 2, 3, 4, 5, 6} as R = {(a, b) : b = a + 1} is reflexive, symmetric or transitive.

SOLUTION

R = {(1,2), (2,3), (3,4), (4,5), (5,6)}.

Reflexive: (a, a) ∈ R requires a = a + 1, which is impossible. NOT reflexive.

Symmetric: (1, 2) ∈ R but (2, 1) ∉ R since 1 ≠ 2 + 1. NOT symmetric.

Transitive: (1, 2) ∈ R and (2, 3) ∈ R, but (1, 3) ∉ R since 3 ≠ 1 + 1. NOT transitive.

Answer: R is neither reflexive, nor symmetric, nor transitive.

Q4 Show that the relation R in R defined as R = {(a, b) : a ≤ b}, is reflexive and transitive but not symmetric.

SOLUTION

Reflexive: For any real number a, a ≤ a is true. So (a, a) ∈ R for all a ∈ R. R is reflexive.

Symmetric: Take a = 1, b = 2. Then 1 ≤ 2 so (1, 2) ∈ R, but 2 ≤ 1 is false, so (2, 1) ∉ R. R is not symmetric.

Transitive: If a ≤ b and b ≤ c, then a ≤ c. So (a, b) ∈ R and (b, c) ∈ R imply (a, c) ∈ R. R is transitive.

Answer: R is reflexive and transitive but not symmetric.

Q5 Check whether the relation R in R defined by R = {(a, b) : a ≤ b^3} is reflexive, symmetric or transitive.

SOLUTION

Reflexive: We need a ≤ a^3 for all real a. Take a = 1/2: 1/2 ≤ $(1/2)^3$ = 1/8 is false. NOT reflexive.

Symmetric: Take a = 1, b = 2. Then 1 ≤ 8, so (1, 2) ∈ R. But 2 ≤ 1^3 = 1 is false, so (2, 1) ∉ R. NOT symmetric.

Transitive: Take a = 3, b = 3/2, c = 6/5. Then a ≤ b^3: 3 ≤ $(3/2)^3$ = 27/8 = 3.375, true. And b ≤ c^3: 3/2 ≤ $(6/5)^3$ = 216/125 = 1.728, true. But a ≤ c^3: 3 ≤ $(6/5)^3$ = 1.728 is false. NOT transitive.

R is neither reflexive, nor symmetric, nor transitive.

Answer: R is neither reflexive, nor symmetric, nor transitive.

Q6 Show that the relation R in the set {1, 2, 3} given by R = {(1, 2), (2, 1)} is symmetric but neither reflexive nor transitive.

SOLUTION

R = {(1, 2), (2, 1)} in A = {1, 2, 3}.

Not reflexive: (1, 1), (2, 2), (3, 3) are none of them in R. R is not reflexive.

Symmetric: The only pairs are (1,2) and (2,1), which are converses of each other. If (a,b) ∈ R then (b,a) ∈ R. R is symmetric.

Not transitive: (1, 2) ∈ R and (2, 1) ∈ R, but (1, 1) ∉ R. R is not transitive.

Answer: R is symmetric but neither reflexive nor transitive.

Q7 Show that the relation R in the set A of all the books in a library of a college, given by R = {(x, y) : x and y have same number of pages} is an equivalence relation.

SOLUTION

Reflexive: Every book x has the same number of pages as itself. So (x, x) ∈ R for all x ∈ A. R is reflexive.

Symmetric: If x and y have the same number of pages, then y and x have the same number of pages. So (x, y) ∈ R => (y, x) ∈ R. R is symmetric.

Transitive: If x and y have the same number of pages, and y and z have the same number of pages, then x and z have the same number of pages. So (x, y) ∈ R and (y, z) ∈ R => (x, z) ∈ R. R is transitive.

Since R is reflexive, symmetric, and transitive, R is an equivalence relation.

Answer: R is an equivalence relation (reflexive, symmetric, and transitive).

Q8 Show that the relation R in the set A = {1, 2, 3, 4, 5} given by R = {(a, b) : |a – b| is even}, is an equivalence relation. Show that all the elements of {1, 3, 5} are related to each other and all the elements of {2, 4} are related to each other. But no element of {1, 3, 5} is related to any element of {2, 4}.

SOLUTION

Reflexive: |a − a| = 0, which is even. So (a, a) ∈ R for all a ∈ A. R is reflexive.

Symmetric: If |a − b| is even, then |b − a| = |a − b| is also even. So (a, b) ∈ R => (b, a) ∈ R. R is symmetric.

Transitive: If |a − b| is even and |b − c| is even, then a − b and b − c are both even, so (a − c) = (a − b) + (b − c) is even, hence |a − c| is even. So (a, b) ∈ R and (b, c) ∈ R => (a, c) ∈ R. R is transitive.

R is an equivalence relation.

Elements of {1, 3, 5}: |1−3| = 2 (even), |1−5| = 4 (even), |3−5| = 2 (even). All pairs

satisfy the relation, so all elements of {1, 3, 5} are related to each other.

Elements of {2, 4}: |2−4| = 2 (even). All pairs satisfy the relation, so all elements of {2, 4} are related to each other.

Cross-comparison: |1−2| = 1 (odd), |1−4| = 3 (odd), |3−2| = 1 (odd), |3−4| = 1 (odd), |5−2| = 3 (odd), |5−4| = 1 (odd). No element of {1, 3, 5} is related to any element of {2, 4}.

Answer: R is an equivalence relation. {1,3,5} and {2,4} are the two equivalence classes; elements within each class are mutually related, and no cross-class pair belongs to R.

Q9 Show that each of the relation R in the set A = {x ∈ Z : 0 ≤ x ≤ 12}, given by
(i) R = {(a, b) : |a – b| is a multiple of 4}
(ii) R = {(a, b) : a = b}
is an equivalence relation. Find the set of all elements related to 1 in each case.

SOLUTION

(i) R = {(a, b) :	$a - b$	is a multiple of 4} in A = {0, 1, 2, ..., 12}.										
Reflexive:	$a - a$	$= 0 = 4 \times 0$, a multiple of 4. $(a, a) \in R$. Reflexive.										
Symmetric: If	$a - b$	is a multiple of 4, then	$b - a$	=	$a - b$	is also a multiple of 4. Symmetric.						

(i) R = {(a, b) :	a − b	is a multiple of 4} in A = {0, 1, 2, ..., 12}.										
Transitive: If	a − b	= 4k and	b − c	= 4m, then	a − c	=	(a − b) + (b − c)	≤	a − b	+	b − c	but more precisely a − c = (a − b) + (b − c) so it is divisible by 4. Transitive.

R is an equivalence relation.

Elements related to 1: need	1 − b	to be a multiple of 4, with b ∈ A.
	1 − b	∈ {0, 4, 8, 12, ...} => b ∈ {1, 5, 9} (since 1−(−3) = 4 gives b = −3 ∉ A and 1−13 out of range).

Set of elements related to 1 = {1, 5, 9}.

(ii) R = {(a, b) : a = b} in A.
This is the identity relation.
Reflexive: a = a for all a. Reflexive.
Symmetric: If a = b then b = a. Symmetric.
Transitive: If a = b and b = c then a = c. Transitive.
R is an equivalence relation.
Elements related to 1: need a = 1, so only b = 1.
Set of elements related to 1 = {1}.

Answer: (i) R is an equivalence relation; set of elements related to 1 = {1, 5, 9}. (ii) R is an equivalence relation; set of elements related to 1 = {1}.

Q10 Give an example of a relation. Which is
(i) Symmetric but neither reflexive nor transitive.
(ii) Transitive but neither reflexive nor symmetric.
(iii) Reflexive and symmetric but not transitive.
(iv) Reflexive and transitive but not symmetric.
(v) Symmetric and transitive but not reflexive.

SOLUTION

(i) Symmetric but neither reflexive nor transitive.
Let A = {1, 2, 3} and R = {(1, 2), (2, 1)}.
Symmetric: (1,2) $\in$ R and (2,1) $\in$ R. Symmetric.
Not reflexive: (1,1) $\notin$ R.
Not transitive: (1,2) $\in$ R and (2,1) $\in$ R but (1,1) $\notin$ R.

(ii) Transitive but neither reflexive nor symmetric.
Let A = {1, 2, 3} and R = {(1, 2)}.
Transitive: No pair (a,b),(b,c) both in R exists, so vacuously transitive.
Not reflexive: (1,1) $\notin$ R.
Not symmetric: (1,2) $\in$ R but (2,1) $\notin$ R.

(iii) Reflexive and symmetric but not transitive.
Let A = {1, 2, 3} and R = {(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2)}.
Reflexive: (1,1),(2,2),(3,3) $\in$ R.
Symmetric: For every (a,b) $\in$ R, (b,a) $\in$ R.
Not transitive: (1,2) $\in$ R and (2,3) $\in$ R but (1,3) $\notin$ R.

(iv) Reflexive and transitive but not symmetric.
Let A = {1, 2, 3} and R = {(a,b): a $\leq$ b} = {(1,1),(2,2),(3,3),(1,2),(2,3),(1,3)}.
Reflexive: (a,a) $\in$ R for all a.
Transitive: If a $\leq$ b and b $\leq$ c then a $\leq$ c.
Not symmetric: (1,2) $\in$ R but (2,1) $\notin$ R.

(v) Symmetric and transitive but not reflexive.
Let A = {1, 2, 3} and R = {(1,2),(2,1),(1,1),(2,2)}.
Symmetric: For every (a,b) $\in$ R, (b,a) $\in$ R.
Transitive: Check all chains — (1,2),(2,1) => (1,1) $\in$ R; (2,1),(1,2) => (2,2) $\in$ R; (1,2),(2,2) => (1,2) $\in$ R; etc. Transitive.
Not reflexive: (3,3) $\notin$ R.

Answer: (i) {(1,2),(2,1)} in {1,2,3}. (ii) {(1,2)} in {1,2,3}. (iii) {(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2)} in {1,2,3}. (iv) {(a,b): a ≤ b} in {1,2,3}. (v) {(1,1),(2,2),(1,2),(2,1)} in {1,2,3}.

Q11 Show that the relation R in the set A of points in a plane given by R = {(P, Q) : distance of the point P from the origin is same as the distance of the point Q from the origin}, is an equivalence relation. Further, show that the set of all points related to a point P ≠ (0, 0) is the circle passing through P with origin as centre.

SOLUTION

Let O denote the origin. R = {(P, Q) : |OP| = |OQ|}.

Reflexive: |OP| = |OP| for every point P. So (P, P) ∈ R. R is reflexive.

Symmetric: If |OP| = |OQ|, then |OQ| = |OP|, so (Q, P) ∈ R. R is symmetric.

Transitive: If |OP| = |OQ| and |OQ| = |OR|, then |OP| = |OR|, so (P, R) ∈ R. R is transitive.

R is an equivalence relation.

The equivalence class of a point P ≠ (0,0): it consists of all points Q such that |OQ| = |OP|, i.e., all points at the same distance r = |OP| from the origin. This is the circle centred at the origin with radius |OP|, which passes through P.

Answer: R is an equivalence relation. The set of all points related to P ≠ (0,0) is the circle centred at the origin passing through P (radius = |OP|).

Q12 Show that the relation R defined in the set A of all triangles as R = {(T1, T2) : T1 is similar to T2}, is equivalence relation. Consider three right angle triangles T1 with sides 3, 4, 5, T2 with sides 5, 12, 13 and T3 with sides 6, 8, 10. Which triangles among T1, T2 and T3 are related?

SOLUTION

Reflexive: Every triangle is similar to itself. (T, T) ∈ R for all T. R is reflexive.

Symmetric: If T1 is similar to T2, then T2 is similar to T1 (similarity is a symmetric relation by definition). R is symmetric.

Transitive: If T1 ~ T2 and T2 ~ T3 (sides in proportion), then T1 ~ T3 (ratios compose). R is transitive.

R is an equivalence relation.

Now check which triangles are related:
T1 has sides 3, 4, 5. T3 has sides 6, 8, 10 = 2×(3, 4, 5). The ratios are 6/3 = 8/4 = 10/5 = 2. So T1 ~ T3; hence (T1, T3) ∈ R.
T2 has sides 5, 12, 13. Compare with T1 (3,4,5): 5/3 ≠ 12/4 (1.67 ≠ 3), so T1 is not similar to T2.
Compare T2 with T3: 6/5 ≠ 8/12, so T2 is not similar to T3.
T1 and T3 are related to each other. T2 is not related to T1 or T3.

Answer: R is an equivalence relation. T1 and T3 are related (similar); T2 is not related to either T1 or T3.

Q13 Show that the relation R defined in the set A of all polygons as R = {(P1, P2) : P1 and P2 have same number of sides}, is an equivalence relation. What is the set of all elements in A related to the right angle triangle T with sides 3, 4 and 5?

SOLUTION

Reflexive: Every polygon P has the same number of sides as itself. (P, P) ∈ R. R is reflexive.

Symmetric: If P1 and P2 have the same number of sides, then P2 and P1 have the same number of sides. R is symmetric.

Transitive: If P1 and P2 have the same number of sides, and P2 and P3 have the same number of sides, then P1 and P3 have the same number of sides. R is transitive.

R is an equivalence relation.

The triangle T has 3 sides. The set of all elements related to T is the set of all triangles (polygons with 3 sides) in A.

Answer: R is an equivalence relation. The set of all elements related to triangle T is the set of all triangles in A.

Q14 Let L be the set of all lines in XY plane and R be the relation in L defined as R = {(L1, L2) : L1 is parallel to L2}. Show that R is an equivalence relation. Find the set of all lines related to the line y = 2x + 4.

SOLUTION

Convention: We consider each line to be parallel to itself (a line is parallel to itself in the usual geometric sense used in this context).

Reflexive: L1 is parallel to L1 (a line is parallel to itself). (L1, L1) $\in$ R. R is reflexive.

Symmetric: If L1 is parallel to L2, then L2 is parallel to L1. R is symmetric.

Transitive: If L1 is parallel to L2 and L2 is parallel to L3, then L1 is parallel to L3 (all have the same slope). R is transitive.

R is an equivalence relation.

The line y = 2x + 4 has slope 2. A line is parallel to it if and only if it has the same slope 2. The set of all lines related to y = 2x + 4 is {y = 2x + c : c $\in$ R}.

Answer: R is an equivalence relation. The set of all lines related to y = 2x + 4 is the family {y = 2x + c : c $\in$ R}, i.e., all lines with slope 2.

Q15 Let R be the relation in the set {1, 2, 3, 4} given by R = {(1, 2), (2, 2), (1, 1), (4,4), (1, 3), (3, 3), (3, 2)}. Choose the correct answer.
(A) R is reflexive and symmetric but not transitive.
(B) R is reflexive and transitive but not symmetric.
(C) R is symmetric and transitive but not reflexive.
(D) R is an equivalence relation.

SOLUTION

R = {(1,2), (2,2), (1,1), (4,4), (1,3), (3,3), (3,2)} in {1,2,3,4}.

Reflexive: Check (1,1),(2,2),(3,3),(4,4) $\in$ R — all present. R is reflexive.

Symmetric: (1,2) $\in$ R but (2,1) $\notin$ R. R is NOT symmetric.

Transitive: Check all possible chains:
(1,2) and (2,2) => need (1,2) $\in$ R. Yes.
(1,3) and (3,2) => need (1,2) $\in$ R. Yes.
(1,3) and (3,3) => need (1,3) $\in$ R. Yes.
(3,2) and (2,2) => need (3,2) $\in$ R. Yes.

All chains are satisfied. R is transitive.

R is reflexive and transitive but not symmetric. Answer: (B).

Answer: (B) R is reflexive and transitive but not symmetric.

Q16 Let R be the relation in the set N given by R = {(a, b) : a = b – 2, b > 6}. Choose the correct answer.
(A) (2, 4) ∈ R
(B) (3, 8) ∈ R
(C) (6, 8) ∈ R
(D) (8, 7) ∈ R

SOLUTION

R = {(a, b) : a = b − 2 and b > 6} in N.

Check (A): (2, 4). b = 4, b > 6? No. (2,4) ∉ R.
Check (B): (3, 8). b = 8 > 6 ✓. a = b − 2 = 8 − 2 = 6 ≠ 3. (3,8) ∉ R.
Check (C): (6, 8). b = 8 > 6 ✓. a = b − 2 = 8 − 2 = 6 ✓. (6,8) ∈ R.
Check (D): (8, 7). b = 7 > 6 ✓. a = b − 2 = 7 − 2 = 5 ≠ 8. (8,7) ∉ R.

Answer: (C).

Answer: (C) (6, 8) ∈ R.

Exercise 1.2 — Exercise 1.2

Q1 Show that the function $f : R^* \to R^*$ defined by $f(x) = 1/x$ is one-one and onto, where R^* is the set of all non-zero real numbers. Is the result true, if the domain R^* is replaced by N with co-domain being same as R^*?

SOLUTION

$f : R^* \to R^*$, $f(x) = 1/x$.

One-one: Suppose $f(x1) = f(x2)$, i.e., $1/x1 = 1/x2$. Then $x1 = x2$. So f is one-one.

Onto: For any $y \in R^*$, we need $x \in R^*$ with $f(x) = y$, i.e., $1/x = y$, so $x = 1/y$. Since $y \neq 0$, $1/y \in R^*$, and $f(1/y) = y$. So f is onto.

f is bijective (one-one and onto) as a map $R^* \to R^*$.

Now consider $f : N \to R^*$, $f(n) = 1/n$.
One-one: If $1/n1 = 1/n2$, then $n1 = n2$. Still one-one.
Onto: Is every element of R^* achieved? We need, for every $y \in R^*$, some $n \in N$ with $1/n = y$, i.e., $n = 1/y$. But $1/2 \in R^*$ requires $n = 2 \in N$ (that works), yet 1/3 requires $n = 3$ (works). However, consider $y = 1/2$: $n = 2 \in N$, works. But what about $y = 2$? Then $n = 1/2 \notin N$. So $2 \in R^*$ has no pre-image in N.
Therefore $f : N \to R^*$ is one-one but NOT onto.

Answer: $f : R^* \to R^*$ is one-one and onto (bijective). When domain is N (co-domain R^*), f is one-one but not onto.

Q2 Check the injectivity and surjectivity of the following functions:
(i) $f : N \to N$ given by $f(x) = x^2$
(ii) $f : Z \to Z$ given by $f(x) = x^2$
(iii) $f : R \to R$ given by $f(x) = x^2$
(iv) $f : N \to N$ given by $f(x) = x^3$
(v) $f : Z \to Z$ given by $f(x) = x^3$

SOLUTION

(i) $f : N \to N$, $f(x) = x^2$.
Injective: If $x1^2 = x2^2$ with $x1, x2 \in N$ (positive), then $x1 = x2$. Injective.
Surjective: $2 \in N$ but $x^2 = 2$ has no solution in N. Not surjective.

(ii) $f : Z \to Z$, $f(x) = x^2$.
Injective: $f(-1) = 1 = f(1)$ but $-1 \neq 1$. Not injective.
Surjective: $-1 \in Z$ but $x^2 = -1$ has no real solution. Not surjective.

(iii) $f : R \to R$, $f(x) = x^2$.
Injective: $f(-1) = 1 = f(1)$. Not injective.
Surjective: $-1 \in R$ but $x^2 = -1$ has no real solution. Not surjective.

(iv) $f : N \to N$, $f(x) = x^3$.
Injective: If $x1^3 = x2^3$ with $x1, x2 \in N$, then $x1 = x2$ (cube root is unique and positive). Injective.
Surjective: $2 \in N$ but $x^3 = 2$ has no solution in N. Not surjective.

(v) $f : Z \to Z$, $f(x) = x^3$.
Injective: $x1^3 = x2^3$ for real numbers implies $x1 = x2$ (cube function is strictly monotone). Injective.
Surjective: $2 \in Z$ but $x^3 = 2$ has no integer solution (2^(1/3) is irrational). Not surjective.

Answer: (i) Injective, not surjective. (ii) Neither. (iii) Neither. (iv) Injective, not surjective. (v) Injective, not surjective.

Q3 Prove that the Greatest Integer Function $f : R \to R$, given by $f(x) = [x]$, is neither one-one nor onto, where $[x]$ denotes the greatest integer less than or equal to x.

SOLUTION

Not one-one: $f(1.2) = [1.2] = 1$ and $f(1.7) = [1.7] = 1$, but $1.2 \neq 1.7$. So f is not one-one.

Not onto: For any $x \in R$, $[x]$ is an integer. So the range of f is $Z \subset R$. An element like $1/2 \in R$ is not in the range of f. So f is not onto.

Answer: f is neither one-one ($f(1.2) = f(1.7) = 1$) nor onto ($1/2 \in R$ has no pre-image).

Q4 Show that the Modulus Function $f : R \to R$, given by $f(x) = |x|$, is neither one-one nor onto, where $|x|$ is x, if x is positive or 0 and $|x|$ is –x, if x is negative.

SOLUTION

Not one-one: $f(1) = |1| = 1$ and $f(-1) = |-1| = 1$, but $1 \neq -1$. So f is not one-one.

Not onto: $|x| \geq 0$ for all $x \in R$, so the range of f is $[0, \infty) \subset R$. The element $-1 \in R$ is not in the range of f. So f is not onto.

Answer: f is neither one-one (f(1) = f(−1) = 1) nor onto (negative reals are not in the range).

Q5 Show that the Signum Function f : R → R, given by
f(x) = 1 if x > 0, f(x) = 0 if x = 0, f(x) = −1 if x < 0
is neither one-one nor onto.

SOLUTION

Not one-one: f(1) = 1 and f(2) = 1, but 1 ≠ 2. So f is not one-one (many positive numbers map to 1).

Not onto: The range of f is {−1, 0, 1}, which is a proper subset of R. For example, 2 ∈ R is not in the range. So f is not onto.

Answer: f is neither one-one nor onto. Range of f = {−1, 0, 1} ≠ R.

Q6 Let A = {1, 2, 3}, B = {4, 5, 6, 7} and let f = {(1, 4), (2, 5), (3, 6)} be a function from A to B. Show that f is one-one.

SOLUTION

The function f maps 1 → 4, 2 → 5, 3 → 6. Each element of A is mapped to a distinct element of B.

Formally: f(1) = 4, f(2) = 5, f(3) = 6 are all different. If f(x1) = f(x2), then since the three output values 4, 5, 6 are distinct, we must have x1 = x2.

Therefore f is one-one.

Answer: f is one-one since f(1) = 4, f(2) = 5, f(3) = 6 are all distinct.

Q7 In each of the following cases, state whether the function is one-one, onto or bijective. Justify your answer.
(i) f : R → R defined by f(x) = 3 – 4x
(ii) f : R → R defined by f(x) = 1 + x²

SOLUTION

(i) f(x) = 3 − 4x.
One-one: Suppose f(x1) = f(x2). Then 3 − 4x1 = 3 − 4x2 => −4x1 = −4x2 => x1 = x2. f is one-one.
Onto: For any y ∈ R, we solve y = 3 − 4x => x = (3 − y)/4, which is a real number.

So every $y \in R$ has a pre-image. f is onto.
f is bijective.

(ii) $f(x) = 1 + x^2$.
One-one: $f(1) = 1 + 1 = 2$ and $f(-1) = 1 + 1 = 2$, but $1 \neq -1$. Not one-one.
Onto: $f(x) = 1 + x^2 \geq 1$ for all $x \in R$. So the range $\subseteq [1, \infty) \subsetneq R$. For example, $0 \in R$ has no pre-image. Not onto.
f is neither one-one nor onto.

Answer: (i) f is bijective. (ii) f is neither one-one nor onto.

Q8 Let A and B be sets. Show that $f : A \times B \to B \times A$ such that $f(a, b) = (b, a)$ is bijective function.

SOLUTION

One-one: Suppose $f(a1, b1) = f(a2, b2)$. Then $(b1, a1) = (b2, a2)$, which means $b1 = b2$ and $a1 = a2$. So $(a1, b1) = (a2, b2)$. f is one-one.

Onto: For any $(b, a) \in B \times A$, the element $(a, b) \in A \times B$ satisfies $f(a, b) = (b, a)$. So every element of $B \times A$ has a pre-image in $A \times B$. f is onto.

f is bijective.

Answer: f is bijective: one-one (since (b1,a1)=(b2,a2) => a1=a2,b1=b2) and onto (every (b,a) is the image of (a,b)).

Q9 Let $f : N \to N$ be defined by $f(n) = (n+1)/2$ if n is odd, and $f(n) = n/2$ if n is even, for all $n \in N$. State whether the function f is bijective. Justify your answer.

SOLUTION

Compute f for small values: $f(1) = (1+1)/2 = 1$, $f(2) = 2/2 = 1$, $f(3) = (3+1)/2 = 2$, $f(4) = 4/2 = 2$, $f(5) = 3$, $f(6) = 3$, ...

Not injective: $f(1) = 1 = f(2)$ but $1 \neq 2$. So f is not one-one.

Surjective: For any $n \in N$, the even number $2n$ satisfies $f(2n) = 2n/2 = n$. So every natural number n has a pre-image (namely $2n$). f is onto.

f is not bijective (it is onto but not one-one).

Answer: f is not bijective. It is onto (surjective) but not one-one (injective): f(1) = f(2) = 1.

Q10 Let A = R – {3} and B = R – {1}. Consider the function f : A → B defined by f(x) = (x – 2)/(x – 3). Is f one-one and onto? Justify your answer.

SOLUTION

One-one: Suppose f(x1) = f(x2), i.e., (x1 − 2)/(x1 − 3) = (x2 − 2)/(x2 − 3).
Cross-multiplying: (x1 − 2)(x2 − 3) = (x2 − 2)(x1 − 3).
Expanding: x1*x2 − 3x1 − 2x2 + 6 = x1*x2 − 3x2 − 2x1 + 6.
Simplifying: −3x1 − 2x2 = −3x2 − 2x1 => −x1 = −x2 => x1 = x2.
f is one-one.

Onto: For a given y ∈ B = R − {1}, solve (x − 2)/(x − 3) = y.
x − 2 = y(x − 3) => x − 2 = yx − 3y => x(1 − y) = 2 − 3y => x = (2 − 3y)/(1 − y) = (3y − 2)/(y − 1).
Since y ≠ 1, the denominator is nonzero, giving a real x.
Check x ≠ 3: (3y − 2)/(y − 1) = 3 => 3y − 2 = 3y − 3 => −2 = −3, contradiction. So x ≠ 3, meaning x ∈ A.
Every y ∈ B has a pre-image in A. f is onto.

f is bijective.

Answer: f is one-one and onto (bijective).

Q11 Let f : R → R be defined as f(x) = x⁴. Choose the correct answer.
(A) f is one-one onto
(B) f is many-one onto
(C) f is one-one but not onto
(D) f is neither one-one nor onto.

SOLUTION

One-one: f(1) = 1⁴ = 1 and f(−1) = (−1)⁴ = 1, but 1 ≠ −1. f is NOT one-one.

Onto: x⁴ ≥ 0 for all x ∈ R, so the range of f is [0, ∞). The element −1 ∈ R is not in the range. f is NOT onto.

Answer: (D).

Answer: (D) f is neither one-one nor onto.

Q12 Let f : R → R be defined as f(x) = 3x. Choose the correct answer.
(A) f is one-one onto
(B) f is many-one onto
(C) f is one-one but not onto
(D) f is neither one-one nor onto.

SOLUTION

One-one: If f(x1) = f(x2) then 3x1 = 3x2 => x1 = x2. f is one-one.

Onto: For any y ∈ R, x = y/3 ∈ R satisfies f(x) = 3(y/3) = y. Every element of R is achieved. f is onto.

f is bijective (one-one and onto). Answer: (A).

Answer: (A) f is one-one onto.

Misc — Miscellaneous Exercise

Q1 Show that the function f : R → {x ∈ R : –1 < x < 1} defined by f(x) = x/(1 + |x|), x ∈ R is one-one and onto function.

SOLUTION

Let Y = {x ∈ R : −1 < x < 1} = (−1, 1). We need to show f : R → Y is one-one and onto.

One-one: Suppose f(x1) = f(x2).
Case 1: x1 ≥ 0 and x2 ≥ 0. Then x1/(1+x1) = x2/(1+x2). Cross-multiplying: x1(1+x2) = x2(1+x1) => x1 + x1*x2 = x2 + x1*x2 => x1 = x2.
Case 2: x1 < 0 and x2 < 0. Then x1/(1−x1) = x2/(1−x2). Cross-multiplying: x1(1−x2) = x2(1−x1) => x1 − x1*x2 = x2 − x1*x2 => x1 = x2.
Case 3: x1 ≥ 0 and x2 < 0 (or vice versa). Then f(x1) = x1/(1+x1) ≥ 0 and f(x2) = x2/(1−x2) < 0. They cannot be equal.
In all cases f(x1) = f(x2) => x1 = x2. f is one-one.

Onto: Let y ∈ (−1, 1). We find x ∈ R with f(x) = y.
If y ≥ 0: set x = y/(1−y). Since 0 ≤ y < 1, we have 1−y > 0, so x ≥ 0. Then f(x) = x/(1+x) = [y/(1−y)] / [1 + y/(1−y)] = [y/(1−y)] / [(1−y+y)/(1−y)] = [y/(1−y)] × [(1−y)/1] = y.
If y < 0: set x = y/(1+y). Since −1 < y < 0, we have 1+y > 0, so x < 0. Then |x| = −x = −y/(1+y), and f(x) = x/(1+|x|) = x/(1−x) = [y/(1+y)] / [1 − y/(1+y)] = [y/(1+y)] / [(1+y−y)/(1+y)] = y.
Every y ∈ (−1,1) has a pre-image. f is onto.

f is bijective.

Answer: f is one-one and onto. For y ≥ 0 in (−1,1): pre-image is y/(1−y); for y < 0: pre-image is y/(1+y).

Q2 Show that the function f : R → R given by f(x) = x³ is injective.

SOLUTION

Suppose f(x1) = f(x2), i.e., x1³ = x2³.

Then x1³ − x2³ = 0 => (x1 − x2)(x1² + x1*x2 + x2²) = 0.

The factor x1² + x1*x2 + x2² = (x1 + x2/2)² + 3x2²/4 ≥ 0, and it equals 0 only if x1 = x2 = 0.

So either $x1 - x2 = 0$ (giving $x1 = x2$) or $x1 = x2 = 0$. In both cases $x1 = x2$.

Therefore f is injective.

Answer: $f(x) = x^3$ is injective: $x1^3 = x2^3$ implies $x1 = x2$ for real numbers.

Q3 Given a non empty set X, consider P(X) which is the set of all subsets of X. Define the relation R in P(X) as follows: For subsets A, B in P(X), ARB if and only if $A \subset B$. Is R an equivalence relation on P(X)? Justify your answer.

SOLUTION

Reflexive: Every set A is a subset of itself, so $A \subset A$. $(A, A) \in R$ for all $A \in P(X)$. R is reflexive.

Symmetric: If $A \subset B$, does it follow that $B \subset A$? Not necessarily. For example, if $X = \{1, 2\}$, let $A = \{1\}$ and $B = \{1, 2\}$. Then $A \subset B$ but $B \not\subset A$. So R is NOT symmetric.

Since R is not symmetric, R is NOT an equivalence relation.

(R is also transitive: if $A \subset B$ and $B \subset C$ then $A \subset C$, but this is not enough for equivalence.)

Answer: R is not an equivalence relation. It is reflexive and transitive but not symmetric ($A \subset B$ does not imply $B \subset A$ in general).

Q4 Find the number of all onto functions from the set {1, 2, 3, ..., n} to itself.

SOLUTION

An onto function from a finite set to itself is the same as a bijection (one-one and onto), which is a permutation of the set.

The number of permutations of {1, 2, 3, ..., n} is n! (n factorial).

Therefore, the number of onto functions from {1, 2, ..., n} to itself is n!.

Answer: n! (n factorial).

Q5 Let A = {–1, 0, 1, 2}, B = {–4, –2, 0, 2} and f, g : A → B be functions defined by f(x) = x² – x, x ∈ A and g(x) = 2|x – 1/2| – 1, x ∈ A. Are f and g equal? Justify your answer.

SOLUTION

Two functions f : A → B and g : A → B are equal if f(a) = g(a) for all a ∈ A.

Compute f and g at each element of A = {−1, 0, 1, 2}:

x = −1: f(−1) = (−1)² − (−1) = 1 + 1 = 2. g(−1) = 2	−1 − 1/2	− 1 = 2 × (3/2) − 1 = 3 − 1 = 2.
x = 0: f(0) = 0 − 0 = 0. g(0) = 2	0 − 1/2	− 1 = 2 × (1/2) − 1 = 1 − 1 = 0.
x = 1: f(1) = 1 − 1 = 0. g(1) = 2	1 − 1/2	− 1 = 2 × (1/2) − 1 = 1 − 1 = 0.
x = 2: f(2) = 4 − 2 = 2. g(2) = 2	2 − 1/2	− 1 = 2 × (3/2) − 1 = 3 − 1 = 2.

f(a) = g(a) for all a ∈ A. Therefore f and g are equal functions.

Answer: Yes, f and g are equal: f(−1)=g(−1)=2, f(0)=g(0)=0, f(1)=g(1)=0, f(2)=g(2)=2.

Q6 Let A = {1, 2, 3}. Then number of relations containing (1, 2) and (1, 3) which are reflexive and symmetric but not transitive is

(A) 1
(B) 2
(C) 3
(D) 4

SOLUTION

We need R ⊂ A × A, R reflexive and symmetric, R contains (1,2) and (1,3), R is not transitive.

Since R must be reflexive: (1,1),(2,2),(3,3) ∈ R.
Since (1,2) ∈ R and R symmetric: (2,1) ∈ R.
Since (1,3) ∈ R and R symmetric: (3,1) ∈ R.

So R must contain at least {(1,1),(2,2),(3,3),(1,2),(2,1),(1,3),(3,1)}. Call this R0.

Check transitivity of R0: (2,1) ∈ R0 and (1,3) ∈ R0, so need (2,3) ∈ R0 for transitivity — but (2,3) ∉ R0. So R0 is NOT transitive. R0 qualifies.

Can we add more pairs and keep it reflexive, symmetric but not transitive?

Adding (2,3) requires also adding (3,2) for symmetry. Now check transitivity of R0 ∪ {(2,3),(3,2)}:
(2,3) and (3,1) => (2,1) ∈ R. (3,2) and (2,1) => (3,1) ∈ R. (2,1) and (1,3) => (2,3) ∈ R. (3,1) and (1,2) => (3,2) ∈ R. All chains are covered. This relation IS transitive — it no longer qualifies.

Therefore only R0 itself is a relation containing (1,2),(1,3) that is reflexive, symmetric but not transitive.

Answer: (A) 1.

Answer: (A) 1.

Q7 Let A = {1, 2, 3}. Then number of equivalence relations containing (1, 2) is
(A) 1
(B) 2
(C) 3
(D) 4

SOLUTION

An equivalence relation containing (1,2) must also contain (2,1) (symmetry), (1,1),(2,2),(3,3) (reflexivity), and (1,2),(2,1) already give transitivity for this pair.

Smallest equivalence relation containing (1,2): R1 = {(1,1),(2,2),(3,3),(1,2),(2,1)}.

Can we extend R1 to a larger equivalence relation?
If we add (2,3), then by symmetry we need (3,2), and by transitivity (with (1,2)) we need (1,3) and (3,1). This gives the universal relation R2 = A × A.
If we add (1,3), similar reasoning forces the universal relation.
If we add (2,3) and (3,2) without the others: transitivity of (1,2) and (2,3) forces (1,3); then (3,1) is needed for symmetry, giving the universal relation again.

The only equivalence relations containing (1,2) are R1 and A × A. There are exactly 2.

Answer: (B) 2.

Answer: (B) 2.