Chapter 2: Inverse Trigonometric Functions

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Exercise 2.1 — Exercise 2.1

Q1 Find the principal value of $\sin^{-1}(-1/2)$.

SOLUTION

Let $\sin^{-1}(-1/2) = y$. Then $\sin y = -1/2$.

The range of the principal value branch of $\sin^{-1}$ is $[-\pi/2, \pi/2]$.

We know $\sin(-\pi/6) = -1/2$ and $-\pi/6$ lies in $[-\pi/2, \pi/2]$.

Therefore, the principal value of $\sin^{-1}(-1/2)$ is $-\pi/6$.

Answer: $-\pi/6$

Q2 Find the principal value of $\cos^{-1}(\sqrt{3}/2)$.

SOLUTION

Let $\cos^{-1}(\sqrt{3}/2) = y$. Then $\cos y = \sqrt{3}/2$.

The range of the principal value branch of $\cos^{-1}$ is $[0, \pi]$.

We know $\cos(\pi/6) = \sqrt{3}/2$ and $\pi/6$ lies in $[0, \pi]$.

Therefore, the principal value of $\cos^{-1}(\sqrt{3}/2)$ is $\pi/6$.

Answer: $\pi/6$

Q3 Find the principal value of $\text{cosec}^{-1}(2)$.

SOLUTION

Let $\text{cosec}^{-1}(2) = y$. Then $\text{cosec}\ y = 2$, i.e., $\sin y = 1/2$.

The range of the principal value branch of cosec^{-1} is $[-\pi/2, \pi/2] - \{0\}$.

We know $\sin(\pi/6) = 1/2$ and $\pi/6$ lies in $[-\pi/2, \pi/2] - \{0\}$.

Therefore, the principal value of $\text{cosec}^{-1}(2)$ is $\pi/6$.

Answer: $\pi/6$

Q4 Find the principal value of $\tan^{-1}(-\sqrt{3})$.

SOLUTION

Let $\tan^{-1}(-\sqrt{3}) = y$. Then $\tan y = -\sqrt{3}$.

The range of the principal value branch of $\tan^{-1}$ is $(-\pi/2, \pi/2)$.

We know $\tan(-\pi/3) = -\sqrt{3}$ and $-\pi/3$ lies in $(-\pi/2, \pi/2)$.

Therefore, the principal value of $\tan^{-1}(-\sqrt{3})$ is $-\pi/3$.

Answer: $-\pi/3$

Q5 Find the principal value of $\cos^{-1}(-1/2)$.

SOLUTION

Let $\cos^{-1}(-1/2) = y$. Then $\cos y = -1/2$.

The range of the principal value branch of $\cos^{-1}$ is $[0, \pi]$.

We know $\cos(2\pi/3) = -1/2$ and $2\pi/3$ lies in $[0, \pi]$.

Therefore, the principal value of $\cos^{-1}(-1/2)$ is $2\pi/3$.

Answer: $2\pi/3$

Q6 Find the principal value of $\tan^{-1}(-1)$.

SOLUTION

Let $\tan^{-1}(-1) = y$. Then $\tan y = -1$.

The range of the principal value branch of $\tan^{-1}$ is $(-\pi/2, \pi/2)$.

We know $\tan(-\pi/4) = -1$ and $-\pi/4$ lies in $(-\pi/2, \pi/2)$.

Therefore, the principal value of $\tan^{-1}(-1)$ is $-\pi/4$.

Answer: $-\pi/4$

Q7 Find the principal value of $\sec^{-1}(2/\sqrt{3})$.

SOLUTION

Let $\sec^{-1}(2/\sqrt{3}) = y$. Then $\sec y = 2/\sqrt{3}$, i.e., $\cos y = \sqrt{3}/2$.

The range of the principal value branch of $\sec^{-1}$ is $[0, \pi] - \{\pi/2\}$.

We know $\cos(\pi/6) = \sqrt{3}/2$ and $\pi/6$ lies in $[0, \pi] - \{\pi/2\}$.

Therefore, the principal value of $\sec^{-1}(2/\sqrt{3})$ is $\pi/6$.

Answer: $\pi/6$

Q8 Find the principal value of $\cot^{-1}(\sqrt{3})$.

SOLUTION

Let $\cot^{-1}(\sqrt{3}) = y$. Then $\cot y = \sqrt{3}$.

The range of the principal value branch of $\cot^{-1}$ is $(0, \pi)$.

We know $\cot(\pi/6) = \sqrt{3}$ and $\pi/6$ lies in $(0, \pi)$.

Therefore, the principal value of $\cot^{-1}(\sqrt{3})$ is $\pi/6$.

Answer: $\pi/6$

Q9 Find the principal value of $\cos^{-1}(-1/\sqrt{2})$.

SOLUTION

Let $\cos^{-1}(-1/\sqrt{2}) = y$. Then $\cos y = -1/\sqrt{2}$.

The range of the principal value branch of $\cos^{-1}$ is $[0, \pi]$.

We know $\cos(3\pi/4) = -1/\sqrt{2}$ and $3\pi/4$ lies in $[0, \pi]$.

Therefore, the principal value of $\cos^{-1}(-1/\sqrt{2})$ is $3\pi/4$.

Answer: $3\pi/4$

Q10 Find the principal value of $\text{cosec}^{-1}(-\sqrt{2})$.

SOLUTION

Let $\text{cosec}^{-1}(-\sqrt{2}) = y$. Then $\text{cosec}\ y = -\sqrt{2}$, i.e., $\sin y = -1/\sqrt{2}$.

The range of the principal value branch of cosec^{-1} is $[-\pi/2, \pi/2] - \{0\}$.

We know $\sin(-\pi/4) = -1/\sqrt{2}$ and $-\pi/4$ lies in $[-\pi/2, \pi/2] - \{0\}$.

Therefore, the principal value of $\text{cosec}^{-1}(-\sqrt{2})$ is $-\pi/4$.

Answer: $-\pi/4$

Q11 Find the value of $\tan^{-1}(1) + \cos^{-1}(-1/2) + \sin^{-1}(-1/2)$.

SOLUTION

Evaluate each term using principal values:

$\tan^{-1}(1)$: $\tan(\pi/4) = 1$, so $\tan^{-1}(1) = \pi/4$.

$\cos^{-1}(-1/2)$: $\cos(2\pi/3) = -1/2$, so $\cos^{-1}(-1/2) = 2\pi/3$.

$\sin^{-1}(-1/2)$: $\sin(-\pi/6) = -1/2$, so $\sin^{-1}(-1/2) = -\pi/6$.

Sum $= \pi/4 + 2\pi/3 + (-\pi/6)$
$= \pi/4 + 2\pi/3 - \pi/6$
$= 3\pi/12 + 8\pi/12 - 2\pi/12$
$= 9\pi/12$
$= 3\pi/4$

Answer: $3\pi/4$

Q12 Find the value of $\cos^{-1}(1/2) + 2\sin^{-1}(1/2)$.

SOLUTION

Evaluate each term using principal values:

$\cos^{-1}(1/2)$: $\cos(\pi/3) = 1/2$, so $\cos^{-1}(1/2) = \pi/3$.

$\sin^{-1}(1/2)$: $\sin(\pi/6) = 1/2$, so $\sin^{-1}(1/2) = \pi/6$.

Therefore: $\cos^{-1}(1/2) + 2\sin^{-1}(1/2) = \pi/3 + 2(\pi/6) = \pi/3 + \pi/3 = 2\pi/3$.

Answer: $2\pi/3$

Q13 If $\sin^{-1} x = y$, then
(A) $0 \le y \le \pi$
(B) $-\pi/2 \le y \le \pi/2$
(C) $0 < y < \pi$
(D) $-\pi/2 < y < \pi/2$

SOLUTION

The principal value branch of $\sin^{-1}$ is the function $\sin^{-1} : [-1, 1] \to [-\pi/2, \pi/2]$.

Since $y = \sin^{-1} x$ lies in the range of the principal value branch, y must satisfy $-\pi/2 \le y \le \pi/2$.

The endpoints $-\pi/2$ and $\pi/2$ are included (e.g., $\sin^{-1}(-1) = -\pi/2$ and $\sin^{-1}(1) = \pi/2$).

Therefore the correct option is (B).

Answer: (B) $-\pi/2 \le y \le \pi/2$

Q14 $\tan^{-1}(\sqrt{3}) - \sec^{-1}(-2)$ is equal to

(A) π

(B) $-\pi/3$

(C) $\pi/3$

(D) $2\pi/3$

SOLUTION

Find each principal value:

$\tan^{-1}(\sqrt{3})$: $\tan(\pi/3) = \sqrt{3}$ and $\pi/3$ lies in $(-\pi/2, \pi/2)$, so $\tan^{-1}(\sqrt{3}) = \pi/3$.

$\sec^{-1}(-2)$: $\sec y = -2$ means $\cos y = -1/2$. $\cos(2\pi/3) = -1/2$ and $2\pi/3$ lies in $[0, \pi] - \{\pi/2\}$, so $\sec^{-1}(-2) = 2\pi/3$.

$\tan^{-1}(\sqrt{3}) - \sec^{-1}(-2) = \pi/3 - 2\pi/3 = -\pi/3$.

Therefore the correct option is (B).

Answer: (B) $-\pi/3$

Exercise 2.2 — Exercise 2.2

Q1 Prove that $3 \sin^{-1} x = \sin^{-1}(3x - 4x^3)$, $x \in [-1/2, 1/2]$.

SOLUTION

Let $x = \sin \theta$, so $\theta = \sin^{-1} x$. For $x \in [-1/2, 1/2]$, $\theta \in [-\pi/6, \pi/6]$, which gives $3\theta \in [-\pi/2, \pi/2]$.

Consider $\sin^{-1}(3x - 4x^3) = \sin^{-1}(3 \sin \theta - 4 \sin^3 \theta)$.

Using the triple angle formula: $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$.

So $\sin^{-1}(3x - 4x^3) = \sin^{-1}(\sin 3\theta) = 3\theta$ (since $3\theta \in [-\pi/2, \pi/2]$).

And $3\theta = 3 \sin^{-1} x$.

Therefore, $3 \sin^{-1} x = \sin^{-1}(3x - 4x^3)$. (Proved)

Answer: Proved using the substitution $x = \sin \theta$ and the identity $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$.

Q2 Prove that $3 \cos^{-1} x = \cos^{-1}(4x^3 - 3x)$, $x \in [1/2, 1]$.

SOLUTION

Let $x = \cos \theta$, so $\theta = \cos^{-1} x$. For $x \in [1/2, 1]$, $\theta \in [0, \pi/3]$, which gives $3\theta \in [0, \pi]$.

Consider $\cos^{-1}(4x^3 - 3x) = \cos^{-1}(4 \cos^3 \theta - 3 \cos \theta)$.

Using the triple angle formula: $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$.

So $\cos^{-1}(4x^3 - 3x) = \cos^{-1}(\cos 3\theta) = 3\theta$ (since $3\theta \in [0, \pi]$).

And $3\theta = 3 \cos^{-1} x$.

Therefore, $3 \cos^{-1} x = \cos^{-1}(4x^3 - 3x)$. (Proved)

Answer: Proved using the substitution $x = \cos \theta$ and the identity $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$.

Q3 Write $\tan^{-1}((\sqrt{(1+x^2)} - 1)/x)$, $x \neq 0$ in the simplest form.

SOLUTION

Let $x = \tan \theta$, so $\theta = \tan^{-1} x$.

Then $\sqrt{(1 + x^2)} = \sqrt{(1 + \tan^2 \theta)} = \sec \theta$ (taking positive value).

$\tan^{-1}((\sqrt{(1+x^2)} - 1)/x) = \tan^{-1}((\sec \theta - 1)/\tan \theta)$

$= \tan^{-1}((1 - \cos\theta)/\sin\theta)$ [multiplying numerator and denominator by cos θ]

Using half-angle identities: $1 - \cos\theta = 2\sin^2(\theta/2)$ and $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$:

$= \tan^{-1}(2\sin^2(\theta/2) / (2\sin(\theta/2)\cos(\theta/2)))$
$= \tan^{-1}(\sin(\theta/2) / \cos(\theta/2))$
$= \tan^{-1}(\tan(\theta/2))$
$= \theta/2$
$= (1/2)\tan^{-1} x.$

Answer: $(1/2)\tan^{-1} x$

Q4 Write $\tan^{-1}(\sqrt{((1 - \cos x)/(1 + \cos x))})$, $0 < x < \pi$ in the simplest form.

SOLUTION

Use the half-angle identities:
$1 - \cos x = 2\sin^2(x/2)$ and $1 + \cos x = 2\cos^2(x/2)$.

So $(1 - \cos x)/(1 + \cos x) = 2\sin^2(x/2) / (2\cos^2(x/2)) = \tan^2(x/2)$.

Therefore $\sqrt{((1 - \cos x)/(1 + \cos x))} = |\tan(x/2)| = \tan(x/2)$ for $0 < x < \pi$ (since $x/2 \in (0, \pi/2)$ makes $\tan(x/2) > 0$).

$\tan^{-1}(\sqrt{((1 - \cos x)/(1 + \cos x))}) = \tan^{-1}(\tan(x/2)) = x/2.$

Answer: $x/2$

Q5 Write $\tan^{-1}((\cos x - \sin x)/(\cos x + \sin x))$, $-\pi/4 < x < 3\pi/4$ in the simplest form.

SOLUTION

Divide numerator and denominator by cos x:

$(\cos x - \sin x)/(\cos x + \sin x) = (1 - \tan x)/(1 + \tan x).$

Using the tan subtraction formula: $\tan(A - B) = (\tan A - \tan B)/(1 + \tan A \tan B)$, with $A = \pi/4$ (so $\tan A = 1$):

$(1 - \tan x)/(1 + \tan x) = \tan(\pi/4 - x).$

Therefore $\tan^{-1}((\cos x - \sin x)/(\cos x + \sin x)) = \tan^{-1}(\tan(\pi/4 - x)) = \pi/4 - x.$

(For $-\pi/4 < x < 3\pi/4$, $\pi/4 - x$ lies in $(-\pi/2, \pi/2)$, so the principal value applies directly.)

Answer: $\pi/4 - x$

Q6 Write $\tan^{-1}(x / \sqrt{(a^2 - x^2)})$, $|x| < a$ in the simplest form.

SOLUTION

Let $x = a \sin \theta$, so $\theta = \sin^{-1}(x/a)$. For $|x| < a$, $\theta \in (-\pi/2, \pi/2)$.

Then $\sqrt{(a^2 - x^2)} = \sqrt{(a^2 - a^2 \sin^2 \theta)} = a \cos \theta$ ($\cos \theta > 0$ since $\theta \in (-\pi/2, \pi/2)$).

$x / \sqrt{(a^2 - x^2)} = a \sin \theta / (a \cos \theta) = \tan \theta$.

Therefore $\tan^{-1}(x / \sqrt{(a^2 - x^2)}) = \tan^{-1}(\tan \theta) = \theta = \sin^{-1}(x/a)$.

Answer: $\sin^{-1}(x/a)$

Q7 Write $\tan^{-1}((3a^2 x - x^3)/(a^3 - 3ax^2))$, $a > 0$; $-a/\sqrt{3} < x < a/\sqrt{3}$ in the simplest form.

SOLUTION

Let $x = a \tan \theta$, so $\theta = \tan^{-1}(x/a)$. For $-a/\sqrt{3} < x < a/\sqrt{3}$, $\theta \in (-\pi/6, \pi/6)$, so $3\theta \in (-\pi/2, \pi/2)$.

Substitute into $(3a^2 x - x^3)/(a^3 - 3ax^2)$:

$= (3a^2 \cdot a \tan \theta - a^3 \tan^3 \theta) / (a^3 - 3a \cdot a^2 \tan^2 \theta)$
$= a^3(3 \tan \theta - \tan^3 \theta) / (a^3(1 - 3 \tan^2 \theta))$
$= (3 \tan \theta - \tan^3 \theta) / (1 - 3 \tan^2 \theta)$.

Using the triple angle formula: $\tan 3\theta = (3 \tan \theta - \tan^3 \theta)/(1 - 3 \tan^2 \theta)$.

So $\tan^{-1}((3a^2 x - x^3)/(a^3 - 3ax^2)) = \tan^{-1}(\tan 3\theta) = 3\theta = 3 \tan^{-1}(x/a)$.

Answer: $3 \tan^{-1}(x/a)$

Q8 Find the value of $\tan^{-1}[2 \cos(2 \sin^{-1}(1/2))]$.

SOLUTION

Step 1: Find $\sin^{-1}(1/2)$.
$\sin(\pi/6) = 1/2$, so $\sin^{-1}(1/2) = \pi/6$.

Step 2: Compute $2 \sin^{-1}(1/2) = 2 \times \pi/6 = \pi/3$.

Step 3: Compute $\cos(\pi/3) = 1/2$.

Step 4: Compute $2 \cos(\pi/3) = 2 \times 1/2 = 1$.

Step 5: $\tan^{-1}(1) = \pi/4$ (since $\tan(\pi/4) = 1$).

Therefore, $\tan^{-1}[2 \cos(2 \sin^{-1}(1/2))] = \pi/4$.

Answer: π/4

Q9 Find the value of $\tan\left[\frac{1}{2}\left(\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right)\right]$, $|x| < 1$, $y > 0$ and $xy < 1$.

SOLUTION

Use the standard identities for $|x| < 1$ and $y > 0$:
$\sin^{-1}\frac{2x}{1+x^2} = 2\tan^{-1} x$
$\cos^{-1}\frac{1-y^2}{1+y^2} = 2\tan^{-1} y$

(These follow from the substitutions $x = \tan\alpha$ and $y = \tan\beta$ respectively.)

So $\frac{1}{2}\left(\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right)$
$= \frac{1}{2}(2\tan^{-1} x + 2\tan^{-1} y)$
$= \tan^{-1} x + \tan^{-1} y.$

Since $xy < 1$, the addition formula gives:
$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}$.

Therefore, $\tan\left[\frac{1}{2}\left(\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right)\right] = \frac{x+y}{1-xy}$.

Answer: $\frac{x+y}{1-xy}$

Q10 Find the value of $\sin^{-1}(\sin(2\pi/3))$.

SOLUTION

Note that $2\pi/3$ does not lie in $[-\pi/2, \pi/2]$, the principal value branch of $\sin^{-1}$.

Use the identity $\sin(\pi - \theta) = \sin\theta$:
$\sin(2\pi/3) = \sin(\pi - \pi/3) = \sin(\pi/3)$.

Since $\pi/3 \in [-\pi/2, \pi/2]$:
$\sin^{-1}(\sin(2\pi/3)) = \sin^{-1}(\sin(\pi/3)) = \pi/3$.

Answer: π/3

Q11 Find the value of $\tan^{-1}(\tan(3\pi/4))$.

SOLUTION

Note that $3\pi/4$ does not lie in $(-\pi/2, \pi/2)$, the principal value branch of $\tan^{-1}$.

Use the identity $\tan(\pi + \theta) = \tan\theta$ (periodicity):
$\tan(3\pi/4) = \tan(\pi - \pi/4) = -\tan(\pi/4) = -1$.

So $\tan^{-1}(\tan(3\pi/4)) = \tan^{-1}(-1) = -\pi/4$ (since $-\pi/4 \in (-\pi/2, \pi/2)$).

Answer: $-\pi/4$

Q12 Find the value of $\tan(\sin^{-1}(3/5) + \cot^{-1}(3/2))$.

SOLUTION

Step 1: Let $\alpha = \sin^{-1}(3/5)$, so $\sin\alpha = 3/5$ and $\cos\alpha = 4/5$ (since α is in first quadrant).
Thus $\tan\alpha = \sin\alpha / \cos\alpha = (3/5)/(4/5) = 3/4$.

Step 2: Let $\beta = \cot^{-1}(3/2)$, so $\cot\beta = 3/2$ and $\tan\beta = 2/3$.

Step 3: Use the tan addition formula:
$\tan(\alpha + \beta) = (\tan\alpha + \tan\beta) / (1 - \tan\alpha \cdot \tan\beta)$
$= (3/4 + 2/3) / (1 - (3/4)(2/3))$
$= (9/12 + 8/12) / (1 - 6/12)$
$= (17/12) / (6/12)$
$= 17/6$.

Answer: 17/6

Q13 $\cos^{-1}(\cos(7\pi/6))$ is equal to
(A) $7\pi/6$
(B) $5\pi/6$
(C) $\pi/3$
(D) $\pi/6$

SOLUTION

Note that $7\pi/6$ does not lie in $[0, \pi]$, the principal value branch of $\cos^{-1}$.

$\cos(7\pi/6) = \cos(\pi + \pi/6) = -\cos(\pi/6) = -\sqrt{3}/2$.

Now find the value in $[0, \pi]$ whose cosine is $-\sqrt{3}/2$:
$\cos(5\pi/6) = -\sqrt{3}/2$ and $5\pi/6 \in [0, \pi]$.

Therefore $\cos^{-1}(\cos(7\pi/6)) = 5\pi/6$.

The correct option is (B).

Answer: (B) $5\pi/6$

Q14 $\sin(\pi/3 - \sin^{-1}(-1/2))$ is equal to

(A) 1/2

(B) 1/3

(C) 1/4

(D) 1

SOLUTION

Step 1: Find $\sin^{-1}(-1/2)$.

$\sin(-\pi/6) = -1/2$, so $\sin^{-1}(-1/2) = -\pi/6$.

Step 2: Compute $\pi/3 - \sin^{-1}(-1/2) = \pi/3 - (-\pi/6) = \pi/3 + \pi/6 = 2\pi/6 + \pi/6 = 3\pi/6 = \pi/2$.

Step 3: $\sin(\pi/2) = 1$.

The correct option is (D).

Answer: (D) 1

Q15 $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$ is equal to

(A) π

(B) $-\pi/2$

(C) 0

(D) $2\sqrt{3}$

SOLUTION

Step 1: $\tan^{-1}(\sqrt{3})$: $\tan(\pi/3) = \sqrt{3}$, so $\tan^{-1}(\sqrt{3}) = \pi/3$.

Step 2: $\cot^{-1}(-\sqrt{3})$: $\cot\theta = -\sqrt{3}$ with $\theta \in (0, \pi)$.

$\cot(5\pi/6) = \cos(5\pi/6)/\sin(5\pi/6) = (-\sqrt{3}/2)/(1/2) = -\sqrt{3}$, so $\cot^{-1}(-\sqrt{3}) = 5\pi/6$.

Step 3: $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3}) = \pi/3 - 5\pi/6 = 2\pi/6 - 5\pi/6 = -3\pi/6 = -\pi/2$.

The correct option is (B).

Answer: (B) $-\pi/2$

Misc — Miscellaneous Exercise

Q1 Find the value of $\cos^{-1}(\cos(13\pi/6))$.

SOLUTION

Note that $13\pi/6$ does not lie in $[0, \pi]$, the principal value branch of $\cos^{-1}$.

Reduce $13\pi/6$ modulo 2π: $13\pi/6 - 2\pi = 13\pi/6 - 12\pi/6 = \pi/6$.

So $\cos(13\pi/6) = \cos(\pi/6) = \sqrt{3}/2$.

Since $\pi/6 \in [0, \pi]$: $\cos^{-1}(\cos(13\pi/6)) = \cos^{-1}(\cos(\pi/6)) = \pi/6$.

Answer: $\pi/6$

Q2 Find the value of $\tan^{-1}(\tan(7\pi/6))$.

SOLUTION

Note that $7\pi/6$ does not lie in $(-\pi/2, \pi/2)$, the principal value branch of $\tan^{-1}$.

Since tan has period π: $\tan(7\pi/6) = \tan(7\pi/6 - \pi) = \tan(\pi/6) = 1/\sqrt{3}$.

Since $\pi/6 \in (-\pi/2, \pi/2)$: $\tan^{-1}(\tan(7\pi/6)) = \tan^{-1}(1/\sqrt{3}) = \pi/6$.

Answer: $\pi/6$

Q3 Prove that $2\sin^{-1}(3/5) = \tan^{-1}(24/7)$.

SOLUTION

Let $\theta = \sin^{-1}(3/5)$, so $\sin\theta = 3/5$.

Since $\sin\theta = 3/5$, using the Pythagorean identity: $\cos\theta = 4/5$ (θ in first quadrant).
Therefore $\tan\theta = \sin\theta/\cos\theta = (3/5)/(4/5) = 3/4$.

Now find $\tan(2\theta)$ using the double angle formula:
$\tan(2\theta) = 2\tan\theta / (1 - \tan^2\theta) = 2(3/4) / (1 - (3/4)^2) = (3/2) / (1 - 9/16) = (3/2) / (7/16) = (3/2)(16/7) = 24/7$.

Since $\theta \in (0, \pi/2)$, we have $2\theta \in (0, \pi)$, and since $\tan(2\theta) = 24/7 > 0$, $2\theta \in (0, \pi/2)$.
So $2\theta = \tan^{-1}(24/7)$.

Therefore $2\sin^{-1}(3/5) = \tan^{-1}(24/7)$. (Proved)

Answer: Proved using the double angle formula for tan, with $\tan\theta = 3/4$ when $\theta = \sin^{-1}(3/5)$.

Q4 Prove that $\sin^{-1}(8/17) + \sin^{-1}(3/5) = \tan^{-1}(77/36)$.

SOLUTION

Let $\alpha = \sin^{-1}(8/17)$ and $\beta = \sin^{-1}(3/5)$. Both are in $(0, \pi/2)$.

From α: $\sin\alpha = 8/17$, $\cos\alpha = \sqrt{(1 - 64/289)} = \sqrt{(225/289)} = 15/17$. So $\tan\alpha = 8/15$.

From β: $\sin\beta = 3/5$, $\cos\beta = 4/5$. So $\tan\beta = 3/4$.

Using the tan addition formula:
$\tan(\alpha + \beta) = (\tan\alpha + \tan\beta) / (1 - \tan\alpha \cdot \tan\beta)$
$= (8/15 + 3/4) / (1 - (8/15)(3/4))$
$= (32/60 + 45/60) / (1 - 24/60)$
$= (77/60) / (36/60)$
$= 77/36$.

Since $\alpha, \beta \in (0, \pi/2)$, their sum $\alpha + \beta \in (0, \pi)$. Also $\tan(\alpha + \beta) = 77/36 > 0$, so $\alpha + \beta \in (0, \pi/2)$.
Therefore $\alpha + \beta = \tan^{-1}(77/36)$, i.e., $\sin^{-1}(8/17) + \sin^{-1}(3/5) = \tan^{-1}(77/36)$. (Proved)

Answer: Proved using $\tan(\alpha + \beta) = (8/15 + 3/4)/(1 - (8/15)(3/4)) = (77/60)/(36/60) = 77/36$.

Q5 Prove that $\cos^{-1}(4/5) + \cos^{-1}(12/13) = \cos^{-1}(33/65)$.

SOLUTION

Let $\alpha = \cos^{-1}(4/5)$ and $\beta = \cos^{-1}(12/13)$. Both lie in $(0, \pi/2)$.

From α: $\cos\alpha = 4/5$, $\sin\alpha = 3/5$.
From β: $\cos\beta = 12/13$, $\sin\beta = 5/13$.

Using the cosine addition formula:
$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$= (4/5)(12/13) - (3/5)(5/13)$
$= 48/65 - 15/65$
$= 33/65$.

Since $\alpha, \beta \in (0, \pi/2)$, we have $\alpha + \beta \in (0, \pi)$, and $\cos(\alpha + \beta) = 33/65 > 0$ so $\alpha + \beta \in (0, \pi/2)$.
Therefore $\alpha + \beta = \cos^{-1}(33/65)$, i.e., $\cos^{-1}(4/5) + \cos^{-1}(12/13) = \cos^{-1}(33/65)$. (Proved)

Answer: Proved using $\cos(\alpha + \beta) = (4/5)(12/13) - (3/5)(5/13) = 33/65$.

Q6 Prove that $\cos^{-1}(12/13) + \sin^{-1}(3/5) = \sin^{-1}(56/65)$.

SOLUTION

Let $\alpha = \cos^{-1}(12/13)$ and $\beta = \sin^{-1}(3/5)$. Both lie in $(0, \pi/2)$.

From α: $\cos \alpha = 12/13$, $\sin \alpha = 5/13$.
From β: $\sin \beta = 3/5$, $\cos \beta = 4/5$.

Using the sine addition formula:
$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$
$= (5/13)(4/5) + (12/13)(3/5)$
$= 4/13 + 36/65$
$= 20/65 + 36/65$
$= 56/65$.

Since $\alpha, \beta \in (0, \pi/2)$, $\alpha + \beta \in (0, \pi)$, and $\sin(\alpha + \beta) = 56/65 > 0$ confirms $\alpha + \beta \in (0, \pi)$.
Moreover $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta = (12/13)(4/5) - (5/13)(3/5) = 48/65 - 15/65 = 33/65 > 0$,
so $\alpha + \beta \in (0, \pi/2)$ and $\alpha + \beta = \sin^{-1}(56/65)$.

Therefore $\cos^{-1}(12/13) + \sin^{-1}(3/5) = \sin^{-1}(56/65)$. (Proved)

Answer: Proved using $\sin(\alpha + \beta) = (5/13)(4/5) + (12/13)(3/5) = 56/65$.

Q7 Prove that $\tan^{-1}(63/16) = \sin^{-1}(5/13) + \cos^{-1}(3/5)$.

SOLUTION

Let $\alpha = \sin^{-1}(5/13)$ and $\beta = \cos^{-1}(3/5)$. Both lie in $(0, \pi/2)$.

From α: $\sin \alpha = 5/13$, $\cos \alpha = 12/13$. So $\tan \alpha = 5/12$.
From β: $\cos \beta = 3/5$, $\sin \beta = 4/5$. So $\tan \beta = 4/3$.

Using the tan addition formula:
$\tan(\alpha + \beta) = (\tan \alpha + \tan \beta) / (1 - \tan \alpha \cdot \tan \beta)$
$= (5/12 + 4/3) / (1 - (5/12)(4/3))$
$= (5/12 + 16/12) / (1 - 20/36)$
$= (21/12) / (16/36)$
$= (7/4) / (4/9)$
$= (7/4)(9/4)$
$= 63/16$.

Since $\alpha, \beta \in (0, \pi/2)$, $\alpha + \beta \in (0, \pi)$. And $\tan(\alpha + \beta) = 63/16 > 0$, so $\alpha + \beta \in (0,$

π/2).

Therefore α + β = $\tan^{-1}(63/16)$, i.e., $\sin^{-1}(5/13) + \cos^{-1}(3/5) = \tan^{-1}(63/16)$. (Proved)

Answer: Proved using $\tan(\alpha + \beta) = (5/12 + 4/3)/(1 - (5/12)(4/3)) = (21/12)/(16/36) = 63/16$.

Q8 Prove that $\tan^{-1}(\sqrt{x}) = (1/2)\cos^{-1}((1-x)/(1+x))$, $x \in [0, 1]$.

SOLUTION

Let $\tan^{-1}(\sqrt{x}) = \theta$, so $\sqrt{x} = \tan\theta$, i.e., $x = \tan^2\theta$, where $\theta \in [0, \pi/4]$ for $x \in [0, 1]$.

Compute $(1 - x)/(1 + x) = (1 - \tan^2\theta)/(1 + \tan^2\theta)$.

Using the double angle formula for cosine:
$\cos 2\theta = (1 - \tan^2\theta)/(1 + \tan^2\theta)$.

So $(1 - x)/(1 + x) = \cos 2\theta$.

Since $\theta \in [0, \pi/4]$, we have $2\theta \in [0, \pi/2] \subseteq [0, \pi]$, so:
$\cos^{-1}(\cos 2\theta) = 2\theta$.

Therefore $(1/2)\cos^{-1}((1-x)/(1+x)) = (1/2) \cdot 2\theta = \theta = \tan^{-1}(\sqrt{x})$. (Proved)

Answer: Proved using $x = \tan^2\theta$ and the identity $\cos 2\theta = (1 - \tan^2\theta)/(1 + \tan^2\theta)$.

Q9 Prove that $\cot^{-1}((\sqrt{(1+\sin x)} + \sqrt{(1-\sin x)})/(\sqrt{(1+\sin x)} - \sqrt{(1-\sin x)})) = x/2$, $x \in (0, \pi/4)$.

SOLUTION

For $x \in (0, \pi/4)$, $x/2 \in (0, \pi/8)$, so $\cos(x/2) > \sin(x/2) > 0$.

Use the identities:
$1 + \sin x = (\cos(x/2) + \sin(x/2))^2$
$1 - \sin x = (\cos(x/2) - \sin(x/2))^2$

Taking positive square roots (both factors are positive for $x \in (0, \pi/4)$):
$\sqrt{(1 + \sin x)} = \cos(x/2) + \sin(x/2)$
$\sqrt{(1 - \sin x)} = \cos(x/2) - \sin(x/2)$

Numerator: $\sqrt{(1+\sin x)} + \sqrt{(1-\sin x)} = 2\cos(x/2)$
Denominator: $\sqrt{(1+\sin x)} - \sqrt{(1-\sin x)} = 2\sin(x/2)$

So the ratio = $2\cos(x/2) / (2\sin(x/2)) = \cot(x/2)$.

Therefore $\cot^{-1}(\cot(x/2)) = x/2$ (since $x/2 \in (0, \pi/4) \subset (0, \pi)$). (Proved)

Answer: Proved using $1 \pm \sin x = (\cos(x/2) \pm \sin(x/2))^2$ to get the ratio = $\cot(x/2)$.

Q10 Prove that $\tan^{-1}((\sqrt{(1+x)} - \sqrt{(1-x)})/(\sqrt{(1+x)} + \sqrt{(1-x)})) = \pi/4 - (1/2)\cos^{-1} x$, $-1/\sqrt{2} \le x \le 1$. [Hint: Put $x = \cos 2\theta$]

SOLUTION

Put $x = \cos 2\theta$, so $\theta = (1/2)\cos^{-1} x$. For $x \in [-1/\sqrt{2}, 1]$, $\theta \in [0, \pi/4]$.

Then:
$1 + x = 1 + \cos 2\theta = 2\cos^2\theta$, so $\sqrt{(1+x)} = \sqrt{2}\cos\theta$.
$1 - x = 1 - \cos 2\theta = 2\sin^2\theta$, so $\sqrt{(1-x)} = \sqrt{2}\sin\theta$.

(Both $\sqrt{2}\cos\theta$ and $\sqrt{2}\sin\theta$ are non-negative for $\theta \in [0, \pi/4]$.)

Numerator: $\sqrt{(1+x)} - \sqrt{(1-x)} = \sqrt{2}\cos\theta - \sqrt{2}\sin\theta = \sqrt{2}(\cos\theta - \sin\theta)$.
Denominator: $\sqrt{(1+x)} + \sqrt{(1-x)} = \sqrt{2}\cos\theta + \sqrt{2}\sin\theta = \sqrt{2}(\cos\theta + \sin\theta)$.

Ratio = $(\cos\theta - \sin\theta)/(\cos\theta + \sin\theta) = (1 - \tan\theta)/(1 + \tan\theta) = \tan(\pi/4 - \theta)$.

For $\theta \in [0, \pi/4]$, $\pi/4 - \theta \in [0, \pi/4] \subset (-\pi/2, \pi/2)$, so:
$\tan^{-1}(\tan(\pi/4 - \theta)) = \pi/4 - \theta = \pi/4 - (1/2)\cos^{-1} x$. (Proved)

Answer: Proved using $x = \cos 2\theta$ so that the ratio reduces to $\tan(\pi/4 - \theta)$, giving $\pi/4 - (1/2)\cos^{-1} x$.

Q11 Solve $2\tan^{-1}(\cos x) = \tan^{-1}(2\,\mathrm{cosec}\, x)$.

SOLUTION

Use the double angle formula for $\tan^{-1}$:
$2\tan^{-1}(\cos x) = \tan^{-1}(2\cos x / (1 - \cos^2 x)) = \tan^{-1}(2\cos x / \sin^2 x)$.

(This uses $2\tan^{-1} t = \tan^{-1}(2t/(1-t^2))$ for the appropriate range of t.)

The equation becomes:
$\tan^{-1}(2\cos x / \sin^2 x) = \tan^{-1}(2 / \sin x)$.

Since $\tan^{-1}$ is one-one, equate arguments:
$2\cos x / \sin^2 x = 2 / \sin x$
$\cos x / \sin x = 1$
$\tan x = 1$
$x = \pi/4 + n\pi$, $n \in Z$.

The principal solution is $x = \pi/4$.

Answer: $x = \pi/4$ (principal solution); general solution $x = n\pi + \pi/4$, $n \in Z$.

Q12 Solve $\tan^{-1}((1-x)/(1+x)) = (1/2)\tan^{-1} x$, $(x > 0)$.

SOLUTION

Use the identity: $\tan^{-1} a - \tan^{-1} b = \tan^{-1}((a-b)/(1+ab))$ (for appropriate values).

Note that $\tan^{-1}(1) - \tan^{-1}(x) = \tan^{-1}((1-x)/(1+x))$ for $x > -1$.

So the equation becomes:
$\pi/4 - \tan^{-1} x = (1/2)\tan^{-1} x$
$\pi/4 = \tan^{-1} x + (1/2)\tan^{-1} x$
$\pi/4 = (3/2)\tan^{-1} x$
$\tan^{-1} x = \pi/6$
$x = \tan(\pi/6) = 1/\sqrt{3}$.

Answer: $x = 1/\sqrt{3}$

Q13 $\sin(\tan^{-1} x)$, $|x| < 1$ is equal to
(A) $x/\sqrt{(1-x^2)}$
(B) $1/\sqrt{(1-x^2)}$
(C) $1/\sqrt{(1+x^2)}$
(D) $x/\sqrt{(1+x^2)}$

SOLUTION

Let $\tan^{-1} x = \theta$, so $\tan\theta = x$, where $\theta \in (-\pi/2, \pi/2)$.

In a right triangle with opposite side x and adjacent side 1, the hypotenuse is $\sqrt{(1 + x^2)}$.

Therefore $\sin\theta$ = opposite/hypotenuse = $x/\sqrt{(1 + x^2)}$.

So $\sin(\tan^{-1} x) = x/\sqrt{(1 + x^2)}$.

The correct option is (D).

Answer: (D) $x/\sqrt{(1+x^2)}$

Q14 $\sin^{-1}(1-x) - 2\sin^{-1} x = \pi/2$, then x is equal to

(A) 0, 1/2

(B) 1, 1/2

(C) 0

(D) 1/2

SOLUTION

Let $\alpha = \sin^{-1}(1-x)$ and $\beta = \sin^{-1} x$. The equation is $\alpha - 2\beta = \pi/2$, so $\alpha = \pi/2 + 2\beta$.

Taking sin of both sides:
$\sin \alpha = \sin(\pi/2 + 2\beta) = \cos(2\beta) = 1 - 2\sin^2 \beta$.

But $\sin \alpha = 1 - x$ and $\sin \beta = x$, so:
$1 - x = 1 - 2x^2$
$2x^2 - x = 0$
$x(2x - 1) = 0$
$x = 0$ or $x = 1/2$.

Check $x = 1/2$:
$\sin^{-1}(1 - 1/2) - 2\sin^{-1}(1/2) = \sin^{-1}(1/2) - 2\sin^{-1}(1/2) = \pi/6 - \pi/3 = -\pi/6 \neq \pi/2$. (Rejected)

Check $x = 0$:
$\sin^{-1}(1) - 2\sin^{-1}(0) = \pi/2 - 0 = \pi/2$. (Accepted)

Therefore $x = 0$.

The correct option is (C).

Answer: (C) 0