Chapter 3: Matrices

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Exercise 3.1 — Exercise 3.1

Q1 In the matrix A = [[2, 5, 19, -7], [35, -2, 5/2, 12], [√3, 1, -5, 17]], write:
(i) The order of the matrix,
(ii) The number of elements,
(iii) Write the elements a_13, a_21, a_33, a_24, a_23.

SOLUTION

(i) A has 3 rows and 4 columns, so the order is 3 × 4.

(ii) Number of elements = 3 × 4 = 12.

(iii) The element a_ij lies in the i-th row and j-th column.
a_13 = element in row 1, column 3 = 19.
a_21 = element in row 2, column 1 = 35.
a_33 = element in row 3, column 3 = −5.
a_24 = element in row 2, column 4 = 12.
a_23 = element in row 2, column 3 = 5/2.

Answer: (i) 3 × 4. (ii) 12. (iii) a_13 = 19, a_21 = 35, a_33 = −5, a_24 = 12, a_23 = 5/2.

Q2 If a matrix has 24 elements, what are the possible orders it can have? What, if it has 13 elements?

SOLUTION

A matrix of order m × n has mn elements. We list all pairs (m, n) with mn = 24:
1 × 24, 2 × 12, 3 × 8, 4 × 6, 6 × 4, 8 × 3, 12 × 2, 24 × 1.
So there are 8 possible orders.

For 13 elements: 13 is prime, so the only factor pairs are 1 × 13 = 13.
Possible orders: 1 × 13 and 13 × 1.

Answer: 24 elements: 8 possible orders (1×24, 2×12, 3×8, 4×6, 6×4, 8×3, 12×2, 24×1). 13 elements: 2 possible orders (1×13, 13×1).

Q3 If a matrix has 18 elements, what are the possible orders it can have? What, if it has 5 elements?

SOLUTION

For 18 elements: list all (m, n) with mn = 18:
1 × 18, 2 × 9, 3 × 6, 6 × 3, 9 × 2, 18 × 1.
So there are 6 possible orders.

For 5 elements: 5 is prime, so the only factor pairs are 1 × 5 = 5.
Possible orders: 1 × 5 and 5 × 1.

Answer: 18 elements: 6 possible orders (1×18, 2×9, 3×6, 6×3, 9×2, 18×1). 5 elements: 2 possible orders (1×5, 5×1).

Q4 Construct a 2 × 2 matrix, A = [a_ij], whose elements are given by:
(i) $a_{ij} = (i + j)^2 / 2$
(ii) $a_{ij} = i / j$
(iii) $a_{ij} = (i + 2j)^2 / 2$

SOLUTION

(i) $a_{ij} = (i + j)^2 / 2$.
$a_{11} = (1+1)^2/2 = 4/2 = 2$.
$a_{12} = (1+2)^2/2 = 9/2$.
$a_{21} = (2+1)^2/2 = 9/2$.
$a_{22} = (2+2)^2/2 = 16/2 = 8$.
A = [[2, 9/2], [9/2, 8]].

(ii) $a_{ij} = i/j$.
$a_{11} = 1/1 = 1$.
$a_{12} = 1/2$.
$a_{21} = 2/1 = 2$.
$a_{22} = 2/2 = 1$.
$A = [[1, 1/2], [2, 1]]$.

(iii) $a_{ij} = (i + 2j)^2 / 2$.
$a_{11} = (1+2)^2/2 = 9/2$.
$a_{12} = (1+4)^2/2 = 25/2$.
$a_{21} = (2+2)^2/2 = 16/2 = 8$.
$a_{22} = (2+4)^2/2 = 36/2 = 18$.
$A = [[9/2, 25/2], [8, 18]]$.

Answer: (i) [[2, 9/2], [9/2, 8]]. (ii) [[1, 1/2], [2, 1]]. (iii) [[9/2, 25/2], [8, 18]].

Q5 Construct a 3 × 4 matrix, whose elements are given by:
(i) $a_{ij} = (1/2)|-3i + j|$
(ii) $a_{ij} = 2i - j$

SOLUTION

(i) $a_{ij} = (1/2)$	$-3i + j$	.						
Row 1 (i=1): $a_{11}=(1/2)$	$-3+1$	$=1$, $a_{12}=(1/2)$	$-3+2$	$=1/2$, $a_{13}=(1/2)$	$-3+3$	$=0$, $a_{14}=(1/2)$	$-3+4$	$=1/2$.
Row 2 (i=2): $a_{21}=(1/2)$	$-6+1$	$=5/2$, $a_{22}=(1/2)$	$-6+2$	$=2$, $a_{23}=(1/2)$	$-6+3$	$=3/2$, $a_{24}=(1/2)$	$-6+4$	$=1$.
Row 3 (i=3): $a_{31}=(1/2)$	$-9+1$	$=4$, $a_{32}=(1/2)$	$-9+2$	$=7/2$, $a_{33}=(1/2)$	$-9+3$	$=3$, $a_{34}=(1/2)$	$-9+4$	$=5/2$.

$A = [[1, 1/2, 0, 1/2], [5/2, 2, 3/2, 1], [4, 7/2, 3, 5/2]]$.

(ii) $a_{ij} = 2i - j$.
Row 1 (i=1): $2-1=1$, $2-2=0$, $2-3=-1$, $2-4=-2$.
Row 2 (i=2): $4-1=3$, $4-2=2$, $4-3=1$, $4-4=0$.
Row 3 (i=3): $6-1=5$, $6-2=4$, $6-3=3$, $6-4=2$.

A = [[1, 0, -1, -2], [3, 2, 1, 0], [5, 4, 3, 2]].

Answer: (i) [[1, 1/2, 0, 1/2], [5/2, 2, 3/2, 1], [4, 7/2, 3, 5/2]]. (ii) [[1, 0, -1, -2], [3, 2, 1, 0], [5, 4, 3, 2]].

Q6 Find the values of x, y and z from the following equations:
(i) [[4, 3], [x, 5]] = [[y, z], [1, 5]]
(ii) [[x + y, 2], [5 + z, xy]] = [[6, 2], [5, 8]]
(iii) [[x + y + z], [x + z], [y + z]] = [[9], [5], [7]]

SOLUTION

(i) By equality of matrices, corresponding elements must be equal:
4 = y => y = 4.
3 = z => z = 3.
x = 1.
5 = 5. ✓

(ii) By equality of matrices:
x + y = 6 … (1)
xy = 8 … (2)
5 + z = 5 => z = 0.
From (1) and (2): x and y are roots of $t^2 - 6t + 8 = 0$ => (t−2)(t−4) = 0.
So x = 2, y = 4 or x = 4, y = 2.

(iii) By equality:
x + y + z = 9 … (1)
x + z = 5 … (2)
y + z = 7 … (3)
From (1) − (2): y = 4.
Substituting in (3): 4 + z = 7 => z = 3.
From (2): x = 5 − z = 5 − 3 = 2.
So x = 2, y = 4, z = 3.

Answer: (i) x = 1, y = 4, z = 3. (ii) x = 2, y = 4, z = 0 (or x = 4, y = 2, z = 0). (iii) x = 2, y = 4, z = 3.

Q7 Find the value of a, b, c and d from the equation:
[[a - b, 2a + c], [2a - b, 3c + d]] = [[-1, 5], [0, 13]]

SOLUTION

By equality of matrices:
a − b = −1 … (1)
2a + c = 5 … (2)
2a − b = 0 … (3)
3c + d = 13 … (4)

From (1) and (3): Subtract (1) from (3): (2a − b) − (a − b) = 0 − (−1) => a = 1.
From (1): b = a + 1 = 1 + 1 = 2.
From (2): c = 5 − 2a = 5 − 2 = 3.
From (4): d = 13 − 3c = 13 − 9 = 4.

Answer: a = 1, b = 2, c = 3, d = 4.

Q8 A = [a_ij]_{m × n} is a square matrix, if
(A) m < n
(B) m > n
(C) m = n
(D) None of these

SOLUTION

A matrix is said to be a square matrix if the number of rows equals the number of columns, i.e., m = n.

Answer: (C) m = n.

Q9 Which of the given values of x and y make the following pair of matrices equal:
[[3x + 7, 5], [y + 1, 2 − 3x]] and [[0, y − 2], [8, 4]]
(A) x = −1/3, y = 7
(B) Not possible to find
(C) y = 7, x = −2/3
(D) x = −1/3, y = −2/3

SOLUTION

For the matrices to be equal, all corresponding elements must be equal simultaneously.

From position (1,1): $3x + 7 = 0 \Rightarrow x = -7/3$.

From position (2,2): $2 - 3x = 4 \Rightarrow -3x = 2 \Rightarrow x = -2/3$.

These give two different values of x. No single value of x can satisfy both conditions simultaneously.

Hence it is not possible to find values of x and y that make the matrices equal.

Answer: (B) Not possible to find.

Q10 The number of all possible matrices of order 3×3 with each entry 0 or 1 is:

(A) 27

(B) 18

(C) 81

(D) 512

SOLUTION

A 3×3 matrix has $3 \times 3 = 9$ entries.

Each entry can independently be 0 or 1, giving 2 choices per entry.

Total number of matrices = $2^9 = 512$.

Answer: (D) 512.

Exercise 3.2 — Exercise 3.2

Q1 Let A = [[2, 4], [3, 2]], B = [[1, 3], [-2, 5]], C = [[-2, 5], [3, 4]]. Find each of the following:
(i) A + B
(ii) A – B
(iii) 3A – C
(iv) AB
(v) BA

SOLUTION

(i) A + B = [[2+1, 4+3], [3+(−2), 2+5]] = [[3, 7], [1, 7]].

(ii) A − B = [[2−1, 4−3], [3−(−2), 2−5]] = [[1, 1], [5, −3]].

(iii) 3A = [[6, 12], [9, 6]]. So 3A − C = [[6−(−2), 12−5], [9−3, 6−4]] = [[8, 7], [6, 2]].

(iv) AB = [[2×1+4×(−2), 2×3+4×5], [3×1+2×(−2), 3×3+2×5]] = [[2−8, 6+20], [3−4, 9+10]] = [[−6, 26], [−1, 19]].

(v) BA = [[1×2+3×3, 1×4+3×2], [(−2)×2+5×3, (−2)×4+5×2]] = [[2+9, 4+6], [−4+15, −8+10]] = [[11, 10], [11, 2]].

Answer: (i) [[3, 7], [1, 7]]. (ii) [[1, 1], [5, −3]]. (iii) [[8, 7], [6, 2]]. (iv) [[−6, 26], [−1, 19]]. (v) [[11, 10], [11, 2]].

Q2 Compute the following:
(i) [[a, b], [-b, a]] + [[a, b], [b, a]]
(ii) [[a² + b², b² + c²], [a² + c², a² + b²]] + [[2ab, 2bc], [-2ac, -2ab]]
(iii) [[-1, 4, -6], [8, 5, 16], [2, 8, 5]] + [[12, 7, 6], [8, 0, 5], [3, 2, 4]]
(iv) [[cos² x, sin² x], [sin² x, cos² x]] + [[sin² x, cos² x], [cos² x, sin² x]]

SOLUTION

(i) [[a+a, b+b], [−b+b, a+a]] = [[2a, 2b], [0, 2a]].

(ii) [[a²+b²+2ab, b²+c²+2bc], [a²+c²−2ac, a²+b²−2ab]] = [[(a+b)², (b+c)²], [(a−c)², (a−b)²]].

(iii) [[-1+12, 4+7, -6+6], [8+8, 5+0, 16+5], [2+3, 8+2, 5+4]] = [[11, 11, 0], [16, 5, 21], [5, 10, 9]].

(iv) [[cos² x + sin² x, sin² x + cos² x], [sin² x + cos² x, cos² x + sin² x]] = [[1, 1], [1, 1]].

Answer: (i) [[2a, 2b], [0, 2a]]. (ii) [[(a+b)², (b+c)²], [(a-c)², (a-b)²]]. (iii) [[11, 11, 0], [16, 5, 21], [5, 10, 9]]. (iv) [[1, 1], [1, 1]].

Q3 Compute the indicated products:

(i) [[a, b], [-b, a]] × [[a, -b], [b, a]]

(ii) [[2], [3], [1]] × [2, 3, 4]

(iii) [[-1, 2], [2, 3]] × [[1, 2, 3], [2, 3, 1]]

(iv) [[2, 3, 4], [3, 4, 5], [4, 5, 6]] × [[1, -3], [0, 2], [3, 0]]

(v) [[2, 1], [3, 2], [-1, 1]] × [[1, 0, 1], [-1, 2, 1]]

(vi) [[3, -1, 3], [-1, 0, 2]] × [[2, -3], [1, 0], [3, 1]]

SOLUTION

(i) [[a×a+b×b, a×(−b)+b×a], [(−b)×a+a×b, (−b)×(−b)+a×a]] = [[a²+b², 0], [0, a²+b²]].

(ii) (3×1 matrix) × (1×3 matrix) = 3×3 matrix.
[[2×2, 2×3, 2×4], [3×2, 3×3, 3×4], [1×2, 1×3, 1×4]] = [[4, 6, 8], [6, 9, 12], [2, 3, 4]].

(iii) [[-1×1+2×2, -1×2+2×3, -1×3+2×1], [2×1+3×2, 2×2+3×3, 2×3+3×1]] = [[3, 4, -1], [8, 13, 9]].

(iv) [[2×1+3×0+4×3, 2×(−3)+3×2+4×0], [3×1+4×0+5×3, 3×(−3)+4×2+5×0], [4×1+5×0+6×3, 4×(−3)+5×2+6×0]]
= [[2+0+12, −6+6+0], [3+0+15, −9+8+0], [4+0+18, −12+10+0]] = [[14, 0], [18, -1], [22, -2]].

(v) [[2×1+1×(−1), 2×0+1×2, 2×1+1×1], [3×1+2×(−1), 3×0+2×2, 3×1+2×1], [(−1)×1+1×(−1), (−1)×0+1×2, (−1)×1+1×1]]
= [[1, 2, 3], [1, 4, 5], [-2, 2, 0]].

(vi) [[3×2+(−1)×1+3×3, 3×(−3)+(−1)×0+3×1], [(−1)×2+0×1+2×3, (−1)×(−3)+0×0+2×1]]
= [[6−1+9, −9+0+3], [−2+0+6, 3+0+2]] = [[14, -6], [4, 5]].

Answer: (i) [[a²+b², 0], [0, a²+b²]]. (ii) [[4,6,8],[6,9,12],[2,3,4]]. (iii) [[3,4,-1],[8,13,9]]. (iv) [[14,0],[18,-1],[22,-2]]. (v) [[1,2,3],[1,4,5],[-2,2,0]]. (vi) [[14,-6],[4,5]].

Q4 If A = [[1, 2, -3], [5, 0, 2], [1, -1, 1]], B = [[3, -1, 2], [4, 2, 5], [2, 0, 3]] and C = [[4, 1, 2], [0, 3, 2], [1, -2, 3]], then compute (A + B) and (B – C). Also, verify that A + (B – C) = (A + B) – C.

SOLUTION

A + B = [[1+3, 2+(−1), −3+2], [5+4, 0+2, 2+5], [1+2, −1+0, 1+3]] = [[4, 1, -1], [9, 2, 7], [3, -1, 4]].

B − C = [[3−4, −1−1, 2−2], [4−0, 2−3, 5−2], [2−1, 0−(−2), 3−3]] = [[-1, -2, 0], [4, -1, 3], [1, 2, 0]].

A + (B − C) = [[1+(−1), 2+(−2), −3+0], [5+4, 0+(−1), 2+3], [1+1, −1+2, 1+0]] = [[0, 0, -3], [9, -1, 5], [2, 1, 1]].

(A + B) − C = [[4−4, 1−1, −1−2], [9−0, 2−3, 7−2], [3−1, −1−(−2), 4−3]] = [[0, 0, -3], [9, -1, 5], [2, 1, 1]].

Both results are equal, verifying A + (B − C) = (A + B) − C.

Answer: A + B = [[4,1,-1],[9,2,7],[3,-1,4]]; B − C = [[-1,-2,0],[4,-1,3],[1,2,0]]; A+(B−C) = (A+B)−C = [[0,0,-3],[9,-1,5],[2,1,1]]. ✓

Q5 If A = [[2/3, 1/5, 7/5], [3/2, -3/5, 5/2], [7/3, 2/3, 1/5]] and B = [[2/5, 3/5, 1/5], [1/5, 2/5, 4/5], [7/5, 3/5, 6/5]], then compute 3A – 5B.

SOLUTION

3A = [[2, 3/5, 21/5], [9/2, -9/5, 15/2], [7, 2, 3/5]].

5B = [[2, 3, 1], [1, 2, 4], [7, 3, 6]].

3A − 5B:
Row 1: [2−2, 3/5−3, 21/5−1] = [0, 3/5−15/5, 21/5−5/5] = [0, -12/5, 16/5].
Row 2: [9/2−1, -9/5−2, 15/2−4] = [7/2, -9/5−10/5, 15/2−8/2] = [7/2, -19/5, 7/2].
Row 3: [7−7, 2−3, 3/5−6] = [0, -1, 3/5−30/5] = [0, -1, -27/5].

3A − 5B = [[0, -12/5, 16/5], [7/2, -19/5, 7/2], [0, -1, -27/5]].

Answer: 3A − 5B = [[0, -12/5, 16/5], [7/2, -19/5, 7/2], [0, -1, -27/5]].

Q6 Simplify cos θ [[cos θ, sin θ], [-sin θ, cos θ]] + sin θ [[sin θ, -cos θ], [cos θ, sin θ]]

SOLUTION

cos θ × [[cos θ, sin θ], [−sin θ, cos θ]] = [[cos² θ, sin θ cos θ], [−sin θ cos θ, cos² θ]].

sin θ × [[sin θ, −cos θ], [cos θ, sin θ]] = [[sin² θ, −sin θ cos θ], [sin θ cos θ, sin² θ]].

Adding the two matrices:
[[cos² θ + sin² θ, sin θ cos θ − sin θ cos θ], [−sin θ cos θ + sin θ cos θ, cos² θ + sin² θ]]
= [[1, 0], [0, 1]] = I.

Answer: The result is I (the 2 × 2 identity matrix).

Q7 Find X and Y, if
(i) X + Y = [[7, 0], [2, 5]] and X – Y = [[3, 0], [0, 3]]
(ii) 2X + 3Y = [[2, 3], [4, 0]] and 3X + 2Y = [[2, -2], [-1, 5]]

SOLUTION

(i) Adding the two equations:
2X = [[7+3, 0+0], [2+0, 5+3]] = [[10, 0], [2, 8]] => X = [[5, 0], [1, 4]].
Subtracting: 2Y = [[7−3, 0−0], [2−0, 5−3]] = [[4, 0], [2, 2]] => Y = [[2, 0], [1, 1]].

(ii) Multiply first equation by 3: 6X + 9Y = [[6, 9], [12, 0]].
Multiply second equation by 2: 6X + 4Y = [[4, −4], [−2, 10]].
Subtract: 5Y = [[2, 13], [14, −10]] => Y = [[2/5, 13/5], [14/5, −2]].

From first equation: 2X = [[2, 3], [4, 0]] − 3Y = [[2, 3], [4, 0]] − [[6/5, 39/5], [42/5, −6]]
= [[10/5−6/5, 15/5−39/5], [20/5−42/5, 0+6]] = [[4/5, −24/5], [−22/5, 6]].
X = [[2/5, −12/5], [−11/5, 3]].

Answer: (i) X = [[5, 0], [1, 4]], Y = [[2, 0], [1, 1]]. (ii) X = [[2/5, -12/5], [-11/5, 3]], Y = [[2/5, 13/5], [14/5, -2]].

Q8 Find X, if Y = [[3, 2], [1, 4]] and 2X + Y = [[1, 0], [-3, 2]]

SOLUTION

2X = [[1, 0], [−3, 2]] − Y = [[1−3, 0−2], [−3−1, 2−4]] = [[−2, −2], [−4, −2]].
X = [[-1, -1], [-2, -1]].

Answer: X = [[-1, -1], [-2, -1]].

Q9 Find x and y, if 2 [[1, 3], [0, x]] + [[y, 0], [1, 2]] = [[5, 6], [1, 8]]

SOLUTION

2 [[1, 3], [0, x]] = [[2, 6], [0, 2x]].
Adding [[y, 0], [1, 2]]: [[2+y, 6], [1, 2x+2]].
Setting equal to [[5, 6], [1, 8]]:
2 + y = 5 => y = 3.
2x + 2 = 8 => x = 3.
The other entries (6 = 6 and 1 = 1) are automatically satisfied.

Answer: x = 3, y = 3.

Q10 Solve the equation for x, y, z and t, if 2 [[x, z], [y, t]] + 3 [[1, -1], [0, 2]] = 3 [[3, 5], [4, 6]]

SOLUTION

2 [[x, z], [y, t]] = 3 [[3, 5], [4, 6]] − 3 [[1, -1], [0, 2]] = 3 [[3−1, 5−(−1)], [4−0, 6−2]] = 3 [[2, 6], [4, 4]] = [[6, 18], [12, 12]].
[[x, z], [y, t]] = [[3, 9], [6, 6]].
x = 3, y = 6, z = 9, t = 6.

Answer: x = 3, y = 6, z = 9, t = 6.

Q11 If x [[2], [3]] + y [[-1], [1]] = [[10], [5]], find the values of x and y.

SOLUTION

[[2x], [3x]] + [[-y], [y]] = [[10], [5]].
[[2x − y], [3x + y]] = [[10], [5]].
2x − y = 10 … (1)
3x + y = 5 … (2)
Adding: 5x = 15 => x = 3.

From (1): y = 2x − 10 = 6 − 10 = −4.

Answer: x = 3, y = −4.

Q12 Given 3 [[x, y], [z, w]] = [[x, 6], [-1, 2w]] + [[4, x + y], [z + w, 3]], find the values of x, y, z and w.

SOLUTION

3 [[x, y], [z, w]] = [[x + 4, 6 + x + y], [-1 + z + w, 2w + 3]].
Equating elements:
3x = x + 4 => 2x = 4 => x = 2.
3y = 6 + x + y => 2y = 6 + 2 = 8 => y = 4.
3z = −1 + z + w => 2z − w = −1 … (1)
3w = 2w + 3 => w = 3.
From (1): 2z − 3 = −1 => z = 1.

Answer: x = 2, y = 4, z = 1, w = 3.

Q13 If F(x) = [[cos x, -sin x, 0], [sin x, cos x, 0], [0, 0, 1]], show that F(x) F(y) = F(x + y).

SOLUTION

F(x) F(y) = [[cos x, −sin x, 0], [sin x, cos x, 0], [0, 0, 1]] × [[cos y, −sin y, 0], [sin y, cos y, 0], [0, 0, 1]].

Computing the product element by element:
Row 1, Col 1: cos x cos y − sin x sin y = cos(x + y).
Row 1, Col 2: −cos x sin y − sin x cos y = −sin(x + y).
Row 1, Col 3: 0.
Row 2, Col 1: sin x cos y + cos x sin y = sin(x + y).
Row 2, Col 2: −sin x sin y + cos x cos y = cos(x + y).
Row 2, Col 3: 0.
Row 3: [0, 0, 1].

So F(x) F(y) = [[cos(x+y), -sin(x+y), 0], [sin(x+y), cos(x+y), 0], [0, 0, 1]] = F(x + y). ∎

Answer: F(x) F(y) = F(x + y) is proved using the angle-addition formulas cos(x+y) = cos x cos y − sin x sin y and sin(x+y) = sin x cos y + cos x sin y.

Q14 Show that:

(i) [[5, -1], [6, 7]] × [[2, 1], [3, 4]] ≠ [[2, 1], [3, 4]] × [[5, -1], [6, 7]]
(ii) [[1, 2, 3], [0, 1, 0], [1, 1, 0]] × [[-1, 1, 0], [0, -1, 1], [2, 3, 4]] ≠ [[-1, 1, 0], [0, -1, 1], [2, 3, 4]] × [[1, 2, 3], [0, 1, 0], [1, 1, 0]]

SOLUTION

(i) Let A = [[5, -1], [6, 7]] and B = [[2, 1], [3, 4]].
AB = [[5×2+(−1)×3, 5×1+(−1)×4], [6×2+7×3, 6×1+7×4]] = [[10−3, 5−4], [12+21, 6+28]] = [[7, 1], [33, 34]].
BA = [[2×5+1×6, 2×(−1)+1×7], [3×5+4×6, 3×(−1)+4×7]] = [[10+6, −2+7], [15+24, −3+28]] = [[16, 5], [39, 25]].
Since [[7,1],[33,34]] ≠ [[16,5],[39,25]], we have AB ≠ BA. ✓

(ii) Let A = [[1,2,3],[0,1,0],[1,1,0]] and B = [[-1,1,0],[0,-1,1],[2,3,4]].
AB: Row 1: [−1+0+6, 1−2+9, 0+2+12] = [5, 8, 14].
Row 2: [0+0+0, 0−1+0, 0+1+0] = [0, -1, 1].
Row 3: [−1+0+0, 1−1+0, 0+1+0] = [-1, 0, 1].
AB = [[5, 8, 14], [0, -1, 1], [-1, 0, 1]].

BA Row 1: [(−1)×1+1×0+0×1, (−1)×2+1×1+0×1, (−1)×3+1×0+0×0] = [-1, -1, -3].
BA Row 2: [0×1+(−1)×0+1×1, 0×2+(−1)×1+1×1, 0×3+(−1)×0+1×0] = [1, 0, 0].
BA Row 3: [2×1+3×0+4×1, 2×2+3×1+4×1, 2×3+3×0+4×0] = [6, 11, 6].
BA = [[-1, -1, -3], [1, 0, 0], [6, 11, 6]].
Since AB ≠ BA, matrix multiplication is not commutative. ✓

Answer: (i) AB = [[7,1],[33,34]] ≠ [[16,5],[39,25]] = BA. (ii) AB = [[5,8,14],[0,-1,1],[-1,0,1]] ≠ [[-1,-1,-3],[1,0,0],[6,11,6]] = BA. ✓

Q15 Find $A^2 - 5A + 6I$, if A = [[2, 0, 1], [2, 1, 3], [1, -1, 0]]

SOLUTION

$A^2 = A \times A$.
Row 1 of A^2: [2×2+0×2+1×1, 2×0+0×1+1×(−1), 2×1+0×3+1×0] = [5, −1, 2].
Row 2 of A^2: [2×2+1×2+3×1, 2×0+1×1+3×(−1), 2×1+1×3+3×0] = [4+2+3, 0+1−3, 2+3+0] = [9, −2, 5].
Row 3 of A^2: [1×2+(−1)×2+0×1, 1×0+(−1)×1+0×(−1), 1×1+(−1)×3+0×0] = [0, −1, −2].
A^2 = [[5, -1, 2], [9, -2, 5], [0, -1, -2]].

5A = [[10, 0, 5], [10, 5, 15], [5, -5, 0]].

6I = [[6, 0, 0], [0, 6, 0], [0, 0, 6]].

$A^2 - 5A + 6I$ = [[5−10+6, −1−0+0, 2−5+0], [9−10+0, −2−5+6, 5−15+0], [0−5+0, −1+5+0, −2−0+6]]
= [[1, -1, -3], [-1, -1, -10], [-5, 4, 4]].

Answer: $A^2 - 5A + 6I$ = [[1, -1, -3], [-1, -1, -10], [-5, 4, 4]].

Q16 If A = [[1, 0, 2], [0, 2, 1], [2, 0, 3]], prove that $A^3 - 6A^2 + 7A + 2I = 0$.

SOLUTION

Compute $A^2 = A \times A$.
Row 1: [1+0+4, 0+0+0, 2+0+6] = [5, 0, 8].
Row 2: [0+0+2, 0+4+0, 0+2+3] = [2, 4, 5].
Row 3: [2+0+6, 0+0+0, 4+0+9] = [8, 0, 13].
A^2 = [[5, 0, 8], [2, 4, 5], [8, 0, 13]].

Compute $A^3 = A^2 \times A$.
Row 1: [5+0+16, 0+0+0, 10+0+24] = [21, 0, 34].
Row 2: [2+0+10, 0+8+0, 4+4+15] = [12, 8, 23].
Row 3: [8+0+26, 0+0+0, 16+0+39] = [34, 0, 55].
A^3 = [[21, 0, 34], [12, 8, 23], [34, 0, 55]].

$6A^2$ = [[30, 0, 48], [12, 24, 30], [48, 0, 78]].
7A = [[7, 0, 14], [0, 14, 7], [14, 0, 21]].
2I = [[2, 0, 0], [0, 2, 0], [0, 0, 2]].

$A^3 - 6A^2 + 7A + 2I$:
Row 1: [21−30+7+2, 0−0+0+0, 34−48+14+0] = [0, 0, 0].
Row 2: [12−12+0+0, 8−24+14+2, 23−30+7+0] = [0, 0, 0].
Row 3: [34−48+14+0, 0−0+0+0, 55−78+21+2] = [0, 0, 0].
$A^3 - 6A^2 + 7A + 2I = 0$. ∎

Answer: $A^3 - 6A^2 + 7A + 2I = 0$ (verified by direct computation).

Q17 If A = [[3, -2], [4, -2]] and I = [[1, 0], [0, 1]], find k so that $A^2 = kA - 2I$.

SOLUTION

A^2 = [[3, -2], [4, -2]] × [[3, -2], [4, -2]].
= [[3×3+(−2)×4, 3×(−2)+(−2)×(−2)], [4×3+(−2)×4, 4×(−2)+(−2)×(−2)]]
= [[9−8, −6+4], [12−8, −8+4]] = [[1, -2], [4, -4]].

kA − 2I = [[3k, -2k], [4k, -2k]] − [[2, 0], [0, 2]] = [[3k−2, -2k], [4k, -2k−2]].

Setting $A^2 = kA - 2I$:
From (1,1): 1 = 3k − 2 => k = 1.
From (1,2): −2 = −2k => k = 1. ✓
From (2,1): 4 = 4k => k = 1. ✓
From (2,2): −4 = −2k − 2 => k = 1. ✓

Answer: k = 1.

Q18 If A = [[0, -tan(α/2)], [tan(α/2), 0]] and I is the identity matrix of order 2, show that I + A = (I − A) [[cos α, -sin α], [sin α, cos α]].

SOLUTION

Let t = tan(α/2). Then I + A = [[1, -t], [t, 1]] and I − A = [[1, t], [-t, 1]].

(I − A) [[cos α, −sin α], [sin α, cos α]]
= [[cos α + t sin α, −sin α + t cos α], [−t cos α + sin α, t sin α + cos α]].

We use the half-angle identities: sin α = 2 sin(α/2) cos(α/2) and cos α = 1 − 2 sin²(α/2).

Element (1,1): cos α + t sin α = (1 − 2 sin²(α/2)) + (sin(α/2)/cos(α/2)) × 2 sin(α/2) cos(α/2)
= 1 − 2 sin²(α/2) + 2 sin²(α/2) = 1. ✓

Element (1,2): −sin α + t cos α = −t(cos α + 1)/1 + t ... using cos α + 1 = 2 cos²(α/2):
= −2 sin(α/2) cos(α/2) + (sin(α/2)/cos(α/2)) × (2 cos²(α/2) − 1 + 1)/...
More directly: −sin α + t cos α = −t(2 cos²(α/2)) + t × ...
= t(cos α − (cos α + 1)) = −t = −tan(α/2). ✓

Element (2,1): sin α − t cos α = t. By symmetry with (1,2): sin α − t cos α = t. ✓

Element (2,2): t sin α + cos α = 1 (same as (1,1) by symmetry). ✓

Therefore (I − A) [[cos α, −sin α], [sin α, cos α]] = [[1, −t], [t, 1]] = I + A. ∎

Answer: I + A = (I − A) [[cos α, −sin α], [sin α, cos α]] is proved using tan(α/2) substitution and half-angle identities.

Q19 A trust fund has Rs 30,000 that must be invested in two different types of bonds. The first bond pays 5% interest per year, and the second bond pays 7% interest per year. Using matrix multiplication, determine how to divide Rs 30,000 among the two types of bonds. If the trust fund must obtain an annual total interest of:
(a) Rs 1800
(b) Rs 2000

SOLUTION

Let Rs x be invested in the 5% bond, so Rs (30,000 − x) is invested in the 7% bond.
Using matrix multiplication: [x, (30000 − x)] × [[5/100], [7/100]] = Total interest.

(a) [x, 30000−x] × [[0.05], [0.07]] = 1800.
0.05x + 0.07(30000 − x) = 1800.
0.05x + 2100 − 0.07x = 1800.
−0.02x = −300 => x = 15000.
Invest Rs 15,000 in the 5% bond and Rs 15,000 in the 7% bond.

(b) [x, 30000−x] × [[0.05], [0.07]] = 2000.
0.05x + 2100 − 0.07x = 2000.
−0.02x = −100 => x = 5000.
Invest Rs 5,000 in the 5% bond and Rs 25,000 in the 7% bond.

Answer: (a) Rs 15,000 in each bond. (b) Rs 5,000 in the 5% bond and Rs 25,000 in the 7% bond.

Q20 The bookshop of a particular school has 10 dozen chemistry books, 8 dozen physics books, 10 dozen economics books. Their selling prices are Rs 80, Rs 60 and Rs 40 each respectively. Find the total amount the bookshop will receive from selling all the books using matrix algebra.

SOLUTION

Quantities (in individual books): Chemistry = 10 × 12 = 120, Physics = 8 × 12 = 96, Economics = 10 × 12 = 120.
Quantity matrix (1 × 3): [120, 96, 120].

Price matrix (3 × 1): [[80], [60], [40]].

Total amount = [120, 96, 120] × [[80], [60], [40]]
= 120 × 80 + 96 × 60 + 120 × 40
= 9600 + 5760 + 4800 = 20160.

The bookshop will receive Rs 20,160.

Answer: Total amount = Rs 20,160.

Q21 Assume X, Y, Z, W and P are matrices of order 2 × n, 3 × k, 2 × p, n × 3 and p × k, respectively. The restriction on n, k and p so that PY + WY will be defined are:
(A) k = 3, p = n
(B) k is arbitrary, p = 2
(C) p is arbitrary, k = 3
(D) k = 2, p = 3

SOLUTION

P is p × k and Y is 3 × k. For PY to be defined, the number of columns of P must equal the number of rows of Y: k = 3.
W is n × 3 and Y is 3 × k (with k = 3 from above, Y is 3 × 3). For WY to be defined, columns of W = rows of Y: 3 = 3. ✓
For PY + WY to be defined, PY and WY must have the same order.
PY has order p × k = p × 3, and WY has order n × 3.
So we need p = n.
Thus: k = 3 and p = n.

Answer: (A) k = 3, p = n.

Q22 If n = p, then the order of the matrix 7X – 5Z is:
(A) p × 2
(B) 2 × n
(C) n × 3
(D) p × n

SOLUTION

X is of order 2 × n and Z is of order 2 × p.
Given n = p, both X and Z are 2 × n matrices.
7X − 5Z is defined (same order) and has order 2 × n.

Answer: (B) 2 × n.

Exercise 3.3 — Exercise 3.3

Q1 Find the transpose of each of the following matrices:
(i) [[5], [1/2], [-1]]
(ii) [[1, -1], [2, 3]]
(iii) [[-1, 5, 6], [√3, 5, 6], [2, 3, -1]]

SOLUTION

(i) The matrix [[5], [1/2], [-1]] is a 3 × 1 column matrix.
Its transpose is the 1 × 3 row matrix [5, 1/2, -1].

(ii) [[1, -1], [2, 3]] is a 2 × 2 matrix.
Interchanging rows and columns: [[1, 2], [-1, 3]].

(iii) [[-1, 5, 6], [√3, 5, 6], [2, 3, -1]] is a 3 × 3 matrix.
Transpose (swap rows and columns): [[-1, √3, 2], [5, 5, 3], [6, 6, -1]].

Answer: (i) [5, 1/2, -1]. (ii) [[1, 2], [-1, 3]]. (iii) [[-1, √3, 2], [5, 5, 3], [6, 6, -1]].

Q2 If A = [[-1, 2, 3], [5, 7, 9], [-2, 1, 1]] and B = [[-4, 1, -5], [1, 2, 0], [1, 3, 1]], then verify that:
(i) (A + B)' = A' + B'
(ii) (A – B)' = A' – B'

SOLUTION

(i) A + B = [[-1+(−4), 2+1, 3+(−5)], [5+1, 7+2, 9+0], [-2+1, 1+3, 1+1]] = [[-5, 3, -2], [6, 9, 9], [-1, 4, 2]].
(A + B)' = [[-5, 6, -1], [3, 9, 4], [-2, 9, 2]].

A' = [[-1, 5, -2], [2, 7, 1], [3, 9, 1]] and B' = [[-4, 1, 1], [1, 2, 3], [-5, 0, 1]].
A' + B' = [[-5, 6, -1], [3, 9, 4], [-2, 9, 2]]. ✓

(ii) A − B = [[-1+4, 2−1, 3+5], [5−1, 7−2, 9−0], [-2−1, 1−3, 1−1]] = [[3, 1, 8], [4, 5, 9], [-3, -2, 0]].
(A − B)' = [[3, 4, -3], [1, 5, -2], [8, 9, 0]].

A' − B' = [[-1+4, 5−1, -2−1], [2−1, 7−2, 1−3], [3+5, 9−0, 1−1]] = [[3, 4, -3], [1, 5, -2], [8, 9, 0]]. ✓

Answer: Both (A+B)' = A'+B' and (A−B)' = A'−B' are verified.

Q3 If A' = [[3, 4], [-1, 2], [0, 1]] and B = [[-1, 2, 1], [1, 2, 3]], then verify that:
(i) (A + B)' = A' + B'
(ii) (A – B)' = A' – B'

SOLUTION

Since A' = [[3, 4], [-1, 2], [0, 1]], we get A = [[3, -1, 0], [4, 2, 1]] (2 × 3 matrix).

(i) A + B = [[3−1, −1+2, 0+1], [4+1, 2+2, 1+3]] = [[2, 1, 1], [5, 4, 4]].
(A + B)' = [[2, 5], [1, 4], [1, 4]].

B' = [[-1, 1], [2, 2], [1, 3]].
A' + B' = [[3+(−1), 4+1], [−1+2, 2+2], [0+1, 1+3]] = [[2, 5], [1, 4], [1, 4]]. ✓

(ii) A − B = [[3+1, −1−2, 0−1], [4−1, 2−2, 1−3]] = [[4, -3, -1], [3, 0, -2]].
(A − B)' = [[4, 3], [-3, 0], [-1, -2]].

A' − B' = [[3+1, 4−1], [−1−2, 2−2], [0−1, 1−3]] = [[4, 3], [-3, 0], [-1, -2]]. ✓

Answer: Both (A+B)' = A'+B' and (A−B)' = A'−B' are verified.

Q4 If A' = [[-2, 3], [1, 2]] and B = [[-1, 0], [1, 2]], then find (A + 2B)'.

SOLUTION

A' = [[-2, 3], [1, 2]] => A = [[-2, 1], [3, 2]].
2B = [[-2, 0], [2, 4]].
A + 2B = [[-2+(−2), 1+0], [3+2, 2+4]] = [[-4, 1], [5, 6]].
(A + 2B)' = [[-4, 5], [1, 6]].

Answer: (A + 2B)' = [[-4, 5], [1, 6]].

Q5 For the matrices A and B, verify that (AB)' = B'A', where:
(i) A = [[1], [-4], [3]], B = [-1, 2, 1]
(ii) A = [[0], [1], [2]], B = [1, 5, 7]

SOLUTION

(i) A is 3×1, B is 1×3, so AB is 3×3.
AB = [[1×(−1), 1×2, 1×1], [(−4)×(−1), (−4)×2, (−4)×1], [3×(−1), 3×2, 3×1]]
= [[-1, 2, 1], [4, -8, -4], [-3, 6, 3]].
(AB)' = [[-1, 4, -3], [2, -8, 6], [1, -4, 3]].

B' = [[-1], [2], [1]] and A' = [1, -4, 3].
B'A' = [[-1], [2], [1]] × [1, -4, 3] = [[-1, 4, -3], [2, -8, 6], [1, -4, 3]]. ✓

(ii) A is 3×1, B is 1×3.
AB = [[0×1, 0×5, 0×7], [1×1, 1×5, 1×7], [2×1, 2×5, 2×7]] = [[0, 0, 0], [1, 5, 7], [2, 10, 14]].
(AB)' = [[0, 1, 2], [0, 5, 10], [0, 7, 14]].

B' = [[1], [5], [7]] and A' = [0, 1, 2].
B'A' = [[1], [5], [7]] × [0, 1, 2] = [[0, 1, 2], [0, 5, 10], [0, 7, 14]]. ✓

Answer: (AB)' = B'A' is verified for both cases.

Q6 If (i) A = [[cos α, sin α], [-sin α, cos α]], then verify that A'A = I.
(ii) If A = [[sin α, cos α], [-cos α, sin α]], then verify that A'A = I.

SOLUTION

(i) A = [[cos α, sin α], [−sin α, cos α]].
A' = [[cos α, −sin α], [sin α, cos α]].
A'A = [[cos α, −sin α], [sin α, cos α]] × [[cos α, sin α], [−sin α, cos α]].
= [[cos² α + sin² α, cos α sin α − sin α cos α], [sin α cos α − cos α sin α, sin² α + cos² α]]
= [[1, 0], [0, 1]] = I. ✓

(ii) A = [[sin α, cos α], [−cos α, sin α]].
A' = [[sin α, −cos α], [cos α, sin α]].
A'A = [[sin α, −cos α], [cos α, sin α]] × [[sin α, cos α], [−cos α, sin α]].
= [[sin² α + cos² α, sin α cos α − cos α sin α], [cos α sin α − sin α cos α, cos² α + sin² α]]
= [[1, 0], [0, 1]] = I. ✓

Answer: A'A = I is verified for both matrices.

Q7 (i) Show that the matrix A = [[1, -1, 5], [-1, 2, 1], [5, 1, 3]] is a symmetric matrix.
(ii) Show that the matrix A = [[0, 1, -1], [-1, 0, 1], [1, -1, 0]] is a skew symmetric matrix.

SOLUTION

(i) A = [[1, -1, 5], [-1, 2, 1], [5, 1, 3]].
A' = [[1, -1, 5], [-1, 2, 1], [5, 1, 3]].
Since A' = A, the matrix A is symmetric. ∎

(ii) A = [[0, 1, -1], [-1, 0, 1], [1, -1, 0]].

A' = [[0, -1, 1], [1, 0, -1], [-1, 1, 0]].
−A = [[0, -1, 1], [1, 0, -1], [-1, 1, 0]].
Since A' = −A, the matrix A is skew symmetric. ∎

Answer: (i) A' = A, so A is symmetric. (ii) A' = −A, so A is skew symmetric.

Q8 For the matrix A = [[1, 5], [6, 7]], verify that:
(i) (A + A') is a symmetric matrix
(ii) (A – A') is a skew symmetric matrix

SOLUTION

A = [[1, 5], [6, 7]] and A' = [[1, 6], [5, 7]].

(i) A + A' = [[2, 11], [11, 14]].
Let B = A + A'. Then B' = [[2, 11], [11, 14]] = B.
So A + A' is symmetric. ✓

(ii) A − A' = [[1−1, 5−6], [6−5, 7−7]] = [[0, -1], [1, 0]].
Let C = A − A'. Then C' = [[0, 1], [-1, 0]] = −C.
So A − A' is skew symmetric. ✓

Answer: (i) A + A' = [[2, 11], [11, 14]] is symmetric. (ii) A − A' = [[0, -1], [1, 0]] is skew symmetric.

Q9 Find (1/2)(A + A') and (1/2)(A – A'), when A = [[0, a, b], [-a, 0, c], [-b, -c, 0]]

SOLUTION

A' = [[0, -a, -b], [a, 0, -c], [b, c, 0]].

(1/2)(A + A') = (1/2) × [[0+(0), a+(−a), b+(−b)], [(−a)+a, 0+0, c+(−c)], [(−b)+b, (−c)+c, 0+0]]
= (1/2) × [[0, 0, 0], [0, 0, 0], [0, 0, 0]] = [[0, 0, 0], [0, 0, 0], [0, 0, 0]].

(1/2)(A − A') = (1/2) × [[0, a+a, b+b], [(−a)−a, 0, c+c], [(−b)−b, (−c)−c, 0]]
= (1/2) × [[0, 2a, 2b], [-2a, 0, 2c], [-2b, -2c, 0]]
= [[0, a, b], [-a, 0, c], [-b, -c, 0]] = A.

This is consistent: since A is already skew symmetric (A' = −A), we have (A + A')/2 = 0 and (A − A')/2 = A.

Answer: (1/2)(A + A') = [[0, 0, 0], [0, 0, 0], [0, 0, 0]] (zero matrix). (1/2)(A − A') = [[0, a, b], [-a, 0, c], [-b, -c, 0]] = A.

Q10 Express the following matrices as the sum of a symmetric and a skew symmetric matrix:
(i) [[3, 5], [1, -1]]
(ii) [[6, -2, 2], [-2, 3, -1], [2, -1, 3]]
(iii) [[3, 3, -1], [-2, -2, 1], [-4, -5, 2]]
(iv) [[1, 5], [-1, 2]]

SOLUTION

(i) Let A = [[3, 5], [1, -1]]. Then A' = [[3, 1], [5, -1]].
Symmetric part: P = (1/2)(A + A') = (1/2) [[6, 6], [6, -2]] = [[3, 3], [3, -1]].
Skew symmetric part: Q = (1/2)(A − A') = (1/2) [[0, 4], [-4, 0]] = [[0, 2], [-2, 0]].
Verify: P + Q = [[3, 5], [1, -1]] = A. ✓

(ii) Let A = [[6, -2, 2], [-2, 3, -1], [2, -1, 3]]. Then A' = [[6, -2, 2], [-2, 3, -1], [2, -1, 3]] = A.
P = (1/2)(A + A') = A = [[6, -2, 2], [-2, 3, -1], [2, -1, 3]] (already symmetric).
Q = (1/2)(A − A') = 0 (zero matrix, since A is symmetric).
A = P + Q = A + 0 = A. ✓

(iii) Let A = [[3, 3, -1], [-2, -2, 1], [-4, -5, 2]]. Then A' = [[3, -2, -4], [3, -2, -5], [-1, 1, 2]].
P = (1/2)(A + A') = (1/2) [[6, 1, -5], [1, -4, -4], [-5, -4, 4]] = [[3, 1/2, -5/2], [1/2, -2, -2], [-5/2, -2, 2]].
Q = (1/2)(A − A') = (1/2) [[0, 5, 3], [-5, 0, 6], [-3, -6, 0]] = [[0, 5/2, 3/2], [-5/2, 0, 3], [-3/2, -3, 0]].
Verify: P + Q = [[3, 3, -1], [-2, -2, 1], [-4, -5, 2]] = A. ✓

(iv) Let A = [[1, 5], [-1, 2]]. Then A' = [[1, -1], [5, 2]].
P = (1/2)(A + A') = (1/2) [[2, 4], [4, 4]] = [[1, 2], [2, 2]].
Q = (1/2)(A − A') = (1/2) [[0, 6], [-6, 0]] = [[0, 3], [-3, 0]].
Verify: P + Q = [[1, 5], [-1, 2]] = A. ✓

Answer: (i) P = [[3,3],[3,-1]], Q = [[0,2],[-2,0]]. (ii) P = A (already symmetric), Q = 0. (iii) P = [[3,1/2,-5/2],[1/2,-2,-2],[-5/2,-2,2]], Q = [[0,5/2,3/2],[-5/2,0,3],[-3/2,-3,0]]. (iv) P = [[1,2],[2,2]], Q = [[0,3],[-3,0]].

Q11 If A, B are symmetric matrices of same order, then AB – BA is a:
(A) Skew symmetric matrix
(B) Symmetric matrix
(C) Zero matrix
(D) Identity matrix

SOLUTION

Since A and B are symmetric, A' = A and B' = B.
Let C = AB − BA. Then:
C' = (AB − BA)' = (AB)' − (BA)' = B'A' − A'B' = BA − AB = −(AB − BA) = −C.
Since C' = −C, the matrix AB − BA is skew symmetric.

Answer: (A) Skew symmetric matrix.

Q12 If A = [[cos α, -sin α], [sin α, cos α]], and A + A' = I, then the value of α is:
(A) π/6
(B) π/3
(C) π
(D) 3π/2

SOLUTION

A + A' = [[cos α, −sin α], [sin α, cos α]] + [[cos α, sin α], [−sin α, cos α]] = [[2 cos α, 0], [0, 2 cos α]].
Setting A + A' = I = [[1, 0], [0, 1]]:
2 cos α = 1 => cos α = 1/2 => α = π/3.

Answer: (B) π/3.

Exercise 3.4 — Exercise 3.4

Q1 Matrices A and B will be inverse of each other only if:

(A) AB = BA

(B) AB = BA = 0

(C) AB = 0, BA = I

(D) AB = BA = I

SOLUTION

By definition, B is the inverse of A (and A is the inverse of B) if and only if AB = BA = I, where I is the identity matrix of the appropriate order.

This is Definition 6 in the chapter.

Answer: (D) AB = BA = I.

Misc — Miscellaneous Exercise

Q1 If A and B are symmetric matrices, prove that AB – BA is a skew symmetric matrix.

SOLUTION

Since A and B are symmetric, A' = A and B' = B.
Let C = AB − BA.
C' = (AB − BA)' = (AB)' − (BA)'
= B'A' − A'B' (using the property (PQ)' = Q'P')
= BA − AB (since B' = B and A' = A)
= −(AB − BA) = −C.
Since C' = −C, the matrix AB − BA is skew symmetric. ∎

Answer: (AB − BA)' = BA − AB = −(AB − BA), so AB − BA is skew symmetric.

Q2 Show that the matrix B'AB is symmetric or skew symmetric according as A is symmetric or skew symmetric.

SOLUTION

Case 1: A is symmetric, i.e., A' = A.
Let C = B'AB. Then:
C' = (B'AB)' = B'A'(B')' = B'A'B = B'AB = C (since A' = A and (B')' = B).
So C = C', meaning B'AB is symmetric. ✓

Case 2: A is skew symmetric, i.e., A' = −A.
C' = (B'AB)' = B'A'(B')' = B'A'B = B'(−A)B = −B'AB = −C.
So C' = −C, meaning B'AB is skew symmetric. ✓

Answer: B'AB is symmetric when A is symmetric, and skew symmetric when A is skew symmetric.

Q3 Find the values of x, y, z if the matrix A = [[0, 2y, z], [x, y, -z], [x, -y, z]] satisfies the equation A'A = I.

SOLUTION

A' = [[0, x, x], [2y, y, -y], [z, -z, z]].

Computing A'A:
$(A'A)_{11} = 0^2 + x^2 + x^2 = 2x^2$.

(A'A)_22 = (2y)² + y² + (-y)² = 4y² + y² + y² = 6y².
(A'A)_33 = z² + (-z)² + z² = 3z².
(A'A)_12 = 0×2y + x×y + x×(−y) = 0 + xy − xy = 0.
(A'A)_13 = 0×z + x×(−z) + x×z = 0 − xz + xz = 0.
(A'A)_23 = 2y×z + y×(−z) + (−y)×z = 2yz − yz − yz = 0.

So A'A = [[2x², 0, 0], [0, 6y², 0], [0, 0, 3z²]].

Setting A'A = I:
2x² = 1 => x² = 1/2 => x = ±1/√2.
6y² = 1 => y² = 1/6 => y = ±1/√6.
3z² = 1 => z² = 1/3 => z = ±1/√3.

Answer: x = ±1/√2, y = ±1/√6, z = ±1/√3.

Q4 For what values of x: [1, 2, 1] [[1, 2, 0], [2, 0, 1], [1, 0, 2]] [[0], [2], [x]] = O?

SOLUTION

First compute [1, 2, 1] × [[1, 2, 0], [2, 0, 1], [1, 0, 2]]:
= [1×1 + 2×2 + 1×1, 1×2 + 2×0 + 1×0, 1×0 + 2×1 + 1×2]
= [1 + 4 + 1, 2 + 0 + 0, 0 + 2 + 2] = [6, 2, 4].

Now [6, 2, 4] × [[0], [2], [x]] = 6×0 + 2×2 + 4×x = 4 + 4x.

Setting this equal to O (the zero matrix of order 1×1): 4 + 4x = 0 => x = −1.

Answer: x = −1.

Q5 If A = [[3, 1], [-1, 2]], show that A² – 5A + 7I = 0.

SOLUTION

A² = [[3, 1], [-1, 2]] × [[3, 1], [-1, 2]]
= [[3×3+1×(−1), 3×1+1×2], [(−1)×3+2×(−1), (−1)×1+2×2]]
= [[9−1, 3+2], [−3−2, −1+4]] = [[8, 5], [-5, 3]].

5A = [[15, 5], [-5, 10]].

7I = [[7, 0], [0, 7]].

A² − 5A + 7I = [[8−15+7, 5−5+0], [−5+5+0, 3−10+7]] = [[0, 0], [0, 0]] = 0. ∎

Answer: A² − 5A + 7I = 0 is verified by direct computation.

Q6 Find x, if [x, -5, -1] [[1, 0, 2], [0, 2, 1], [2, 0, 3]] [[x], [4], [1]] = O.

SOLUTION

First compute [x, −5, −1] × [[1, 0, 2], [0, 2, 1], [2, 0, 3]]:
= [x×1 + (−5)×0 + (−1)×2, x×0 + (−5)×2 + (−1)×0, x×2 + (−5)×1 + (−1)×3]
= [x − 2, −10, 2x − 8].

Now [x−2, −10, 2x−8] × [[x], [4], [1]]
= (x−2)×x + (−10)×4 + (2x−8)×1
= x^2 − 2x − 40 + 2x − 8 = x^2 − 48.

Setting x^2 − 48 = 0: x^2 = 48 => x = ±4√3.

Answer: x = ±4√3.

Q7 A manufacturer produces three products x, y, z which he sells in two markets. Annual sales are indicated below:
Market I: 10,000 (x), 2,000 (y), 18,000 (z)
Market II: 6,000 (x), 20,000 (y), 8,000 (z)
(a) If unit sale prices of x, y and z are Rs 2.50, Rs 1.50 and Rs 1.00, respectively, find the total revenue in each market with the help of matrix algebra.
(b) If the unit costs of the above three commodities are Rs 2.00, Rs 1.00 and 50 paise respectively. Find the gross profit.

SOLUTION

(a) Sales matrix (2×3): S = [[10000, 2000, 18000], [6000, 20000, 8000]].
Price matrix (3×1): P = [[2.50], [1.50], [1.00]].

Revenue = S × P:
Market I: 10000×2.50 + 2000×1.50 + 18000×1.00 = 25000 + 3000 + 18000 = 46000.
Market II: 6000×2.50 + 20000×1.50 + 8000×1.00 = 15000 + 30000 + 8000 = 53000.
Total revenue: Market I = Rs 46,000, Market II = Rs 53,000.

(b) Unit cost matrix (3×1): C = [[2.00], [1.00], [0.50]].
Profit per unit = P − C = [[0.50], [0.50], [0.50]].

Gross profit = S × (P − C):
Market I: 10000×0.50 + 2000×0.50 + 18000×0.50 = (10000+2000+18000)×0.50 = 30000×0.50 = 15000.

Market II: 6000×0.50 + 20000×0.50 + 8000×0.50 = (6000+20000+8000)×0.50 = 34000×0.50 = 17000.

Gross profit: Market I = Rs 15,000, Market II = Rs 17,000. Total = Rs 32,000.

Answer: (a) Revenue: Market I = Rs 46,000; Market II = Rs 53,000. (b) Gross profit: Market I = Rs 15,000; Market II = Rs 17,000.

Q8 Find the matrix X so that X [[1, 2, 3], [4, 5, 6]] = [[-7, -8, -9], [2, 4, 6]].

SOLUTION

X must be a 2×2 matrix (since X × (2×3) = (2×3), we need X of order 2×2).

Let X = [[a, b], [c, d]].

X [[1, 2, 3], [4, 5, 6]] = [[a+4b, 2a+5b, 3a+6b], [c+4d, 2c+5d, 3c+6d]].

Equating with [[-7, -8, -9], [2, 4, 6]]:

a + 4b = −7 … (1)

2a + 5b = −8 … (2)

c + 4d = 2 … (3)

2c + 5d = 4 … (4)

From (1) and (2): 2×(1) − (2): 2a+8b − 2a−5b = −14+8 => 3b = −6 => b = −2.

From (1): a = −7 − 4×(−2) = −7 + 8 = 1.

From (3) and (4): 2×(3) − (4): 2c+8d − 2c−5d = 4−4 => 3d = 0 => d = 0.

From (3): c = 2 − 4×0 = 2.

X = [[1, -2], [2, 0]].

Answer: X = [[1, -2], [2, 0]].

Q9 If A = [[α, β], [γ, -α]] is such that $A^2 = I$, then:

(A) $1 + \alpha^2 + \beta\gamma = 0$

(B) $1 - \alpha^2 + \beta\gamma = 0$

(C) $1 - \alpha^2 - \beta\gamma = 0$

(D) $1 + \alpha^2 - \beta\gamma = 0$

SOLUTION

$A^2 = [[\alpha, \beta], [\gamma, -\alpha]] \times [[\alpha, \beta], [\gamma, -\alpha]]$

$= [[\alpha^2 + \beta\gamma, \alpha\beta - \alpha\beta], [\alpha\gamma - \alpha\gamma, \beta\gamma + \alpha^2]]$

$= [[\alpha^2 + \beta\gamma, 0], [0, \alpha^2 + \beta\gamma]]$.

For $A^2 = I = [[1, 0], [0, 1]]$:
$\alpha^2 + \beta\gamma = 1 \Rightarrow 1 - \alpha^2 - \beta\gamma = 0$.

Answer: (C) $1 - \alpha^2 - \beta\gamma = 0$.

Q10 If the matrix A is both symmetric and skew symmetric, then:
(A) A is a diagonal matrix
(B) A is a zero matrix
(C) A is a square matrix
(D) None of these

SOLUTION

If A is symmetric: $A' = A$.
If A is skew symmetric: $A' = -A$.
Both hold simultaneously $\Rightarrow A = -A \Rightarrow 2A = 0 \Rightarrow A = 0$.
So A must be the zero matrix.

Answer: (B) A is a zero matrix.

Q11 If A is a square matrix such that $A^2 = A$, then $(I + A)^3 - 7A$ is equal to:
(A) A
(B) I – A
(C) I
(D) 3A

SOLUTION

Given $A^2 = A$ (A is idempotent).
Note: $A^3 = A^2 \times A = A \times A = A^2 = A$.

Expand $(I + A)^3$ using the binomial theorem (I and A commute since $IA = AI = A$):
$(I + A)^3 = I^3 + 3I^2 A + 3I A^2 + A^3$
$= I + 3A + 3A^2 + A^3$
$= I + 3A + 3A + A$ (since $A^2 = A$ and $A^3 = A$)
$= I + 7A$.

Therefore $(I + A)^3 - 7A = I + 7A - 7A = I$.

Answer: (C) I.