Chapter 4: Determinants

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Exercise 4.1 — Exercise 4.1

Q1 Evaluate the determinant:

|2 4|
|-5 -1|

SOLUTION

det = (2)(−1) − (4)(−5) = −2 + 20 = 18.

Answer: 18

Q2 Evaluate the determinants:

(i) |cos θ −sin θ|
|sin θ cos θ|

(ii) |x² − x + 1 x − 1|
|x + 1 x + 1|

SOLUTION

(i) det = cos θ · cos θ − (−sin θ) · sin θ = cos²θ + sin²θ = 1.

(ii) det = (x² − x + 1)(x + 1) − (x − 1)(x + 1).
Expand: (x² − x + 1)(x + 1) = x³ + x² − x² − x + x + 1 = x³ + 1.
(x − 1)(x + 1) = x² − 1.
det = x³ + 1 − (x² − 1) = x³ − x² + 2.

Answer: (i) 1 (ii) x³ − x² + 2

Q3 If A = [[1, 2], [4, 2]], then show that |2A| = 4|A|.

SOLUTION

|A| = (1)(2) − (2)(4) = 2 − 8 = −6.

2A = [[2, 4], [8, 4]].
|2A| = (2)(4) − (4)(8) = 8 − 32 = −24.

4|A| = 4(−6) = −24.

Thus |2A| = −24 = 4|A|. In general for a 2×2 matrix, |kA| = k²|A|, so |2A| = 2²|A| = 4|A|.

Answer: |2A| = −24 = 4|A| (verified)

Q4 If A = [[1, 0, 1], [0, 1, 2], [0, 0, 4]], then show that |3A| = 27|A|.

SOLUTION

|A| = expand along R1: 1·(1·4 − 2·0) − 0 + 1·(0·0 − 1·0) = 4 + 0 = 4.

3A = [[3, 0, 3], [0, 3, 6], [0, 0, 12]].
|3A| = expand along R1: 3·(3·12 − 6·0) − 0 + 3·(0·0 − 3·0) = 3·36 = 108.

27|A| = 27·4 = 108.

Thus |3A| = 108 = 27|A|. In general for an n×n matrix, |kA| = k^n|A|. Here n = 3, so |3A| = 3³|A| = 27|A|.

Answer: |3A| = 108 = 27|A| (verified)

Q5 Evaluate the determinants:

(i)
$$\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$$

(ii)
$$\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$$

(iii)
$$\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$$

(iv) $\begin{vmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix}$

SOLUTION

(i) Expand along R2 = [0, 0, −1] (two zeros):
det = −0·M21 + 0·M22 − (−1)·M23,
where M23 = |3 −1; 3 −5| = (3)(−5)−(−1)(3) = −15+3 = −12.
det = (−1)·(−1)^(2+3)·(−12) = (−1)(−1)(−12) = −12.

Check by expanding along R1:
= 3·[(0)(0)−(−1)(−5)] − (−1)·[(0)(0)−(−1)(3)] + (−2)·[(0)(−5)−(0)(3)]
= 3·[0−5] − (−1)·[0+3] + (−2)·[0]
= 3(−5)+3 = −15+3 = −12.

(ii) Expand along R1:
= 3·[(1)(1)−(−2)(3)] − (−4)·[(1)(1)−(−2)(2)] + 5·[(1)(3)−(1)(2)]
= 3·[1+6] + 4·[1+4] + 5·[3−2]
= 3(7) + 4(5) + 5(1) = 21 + 20 + 5 = 46.

(iii) Expand along R1:
= 0·[(0)(0)−(−3)(3)] − 1·[(−1)(0)−(−3)(−2)] + 2·[(−1)(3)−(0)(−2)]
= 0 − 1·[0−6] + 2·[−3−0]
= 6 − 6 = 0.

(iv) Expand along R1:
= 2·[(2)(0)−(−1)(−5)] − (−1)·[(0)(0)−(−1)(3)] + (−2)·[(0)(−5)−(2)(3)]
= 2·[0−5] + 1·[0+3] + (−2)·[0−6]
= −10 + 3 + 12 = 5.

Answer: (i) −12 (ii) 46 (iii) 0 (iv) 5

Q6 If A = [[1, 1, −2], [2, 1, −3], [5, 4, −9]], find |A|.

SOLUTION

Expand along R1:
|A| = 1·[(1)(−9)−(−3)(4)] − 1·[(2)(−9)−(−3)(5)] + (−2)·[(2)(4)−(1)(5)]
= 1·[−9+12] − 1·[−18+15] + (−2)·[8−5]
= 1·3 − 1·(−3) + (−2)·3
= 3 + 3 − 6 = 0.

Answer: |A| = 0

Q7 Find values of x, if:

(i) |2 4| = |2x 4|
|5 1| |6 x|

(ii) |2 3| = |x 3|
|4 5| |2x 5|

SOLUTION

(i) LHS = (2)(1) − (4)(5) = 2 − 20 = −18.
RHS = (2x)(x) − (4)(6) = 2x² − 24.
Setting equal: 2x² − 24 = −18 => 2x² = 6 => x² = 3 => x = ±√3.

(ii) LHS = (2)(5) − (3)(4) = 10 − 12 = −2.
RHS = (x)(5) − (3)(2x) = 5x − 6x = −x.
Setting equal: −x = −2 => x = 2.

Answer: (i) x = ±√3 (ii) x = 2

Q8 If |x 2 | = |6 2|, then x is equal to
|18 x| |18 6|

(A) 6 (B) ±6 (C) −6 (D) 0

SOLUTION

LHS = x² − 36.
RHS = (6)(6) − (2)(18) = 36 − 36 = 0.
x² − 36 = 0 => x² = 36 => x = ±6.
Answer: (B).

Answer: (B) ±6

Exercise 4.2 — Exercise 4.2

Q1 Find area of the triangle with vertices at the point given in each of the following:
(i) (1, 0), (6, 0), (4, 3)
(ii) (2, 7), (1, 1), (10, 8)
(iii) (−2, −3), (3, 2), (−1, −8)

SOLUTION

Area = (1/2)|det [[x1,y1,1],[x2,y2,1],[x3,y3,1]]|.

(i) Vertices (1,0), (6,0), (4,3):
det = |1 0 1; 6 0 1; 4 3 1|
= 1·(0·1 − 1·3) − 0·(6·1 − 1·4) + 1·(6·3 − 0·4)
= 1·(−3) − 0 + 1·18 = −3 + 18 = 15.
Area = (1/2)|15| = 15/2 sq units.

(ii) Vertices (2,7), (1,1), (10,8):
det = |2 7 1; 1 1 1; 10 8 1|
= 2·(1·1 − 1·8) − 7·(1·1 − 1·10) + 1·(1·8 − 1·10)
= 2·(−7) − 7·(−9) + 1·(−2)
= −14 + 63 − 2 = 47.
Area = (1/2)|47| = 47/2 sq units.

(iii) Vertices (−2,−3), (3,2), (−1,−8):
det = |−2 −3 1; 3 2 1; −1 −8 1|
= −2·(2·1 − 1·(−8)) − (−3)·(3·1 − 1·(−1)) + 1·(3·(−8) − 2·(−1))
= −2·(2+8) + 3·(3+1) + 1·(−24+2)
= −2·10 + 3·4 − 22
= −20 + 12 − 22 = −30.
Area = (1/2)|−30| = 15 sq units.

Answer: (i) 15/2 sq units (ii) 47/2 sq units (iii) 15 sq units

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Q2 Show that points A(a, b+c), B(b, c+a), C(c, a+b) are collinear.

SOLUTION

Three points are collinear if the area of the triangle formed by them is zero, i.e., the determinant is zero.

det =	a b+c 1	
	b c+a 1	
	c a+b 1	

Apply R1 → R1 − R2 and R2 → R2 − R3:
R1 − R2: (a−b, b+c−(c+a), 0) = (a−b, b−a, 0)
R2 − R3: (b−c, c+a−(a+b), 0) = (b−c, c−b, 0)

det =	a−b b−a 0	
	b−c c−b 0	
	c a+b 1	

Expand along C3: det = 1·[(a−b)(c−b) − (b−a)(b−c)]
= (a−b)(c−b) + (a−b)(b−c)
= (a−b)[(c−b) + (b−c)] = (a−b)·0 = 0.

Since the determinant is 0, the area is 0 and the three points are collinear.

Answer: The determinant evaluates to 0, hence A, B, C are collinear.

Q3 Find values of k if area of triangle is 4 sq. units and vertices are:
(i) (k, 0), (4, 0), (0, 2)
(ii) (−2, 0), (0, 4), (0, k)

SOLUTION

(i) Area = (1/2)	det	= 4, so det = ±8.
det =	k 0 1; 4 0 1; 0 2 1	

Expand along C2: = −0·(...) + 0·(...) − 2·(k·1 − 1·4) = −2(k−4).
−2(k−4) = ±8
Case 1: −2(k−4) = 8 => k−4 = −4 => k = 0.
Case 2: −2(k−4) = −8 => k−4 = 4 => k = 8.

(ii) Area = (1/2)	det	= 4, so det = ±8.
det =	−2 0 1; 0 4 1; 0 k 1	

Expand along C1: = −2·(4·1 − 1·k) = −2(4−k) = 2k − 8.
2k − 8 = ±8
Case 1: 2k − 8 = 8 => k = 8.
Case 2: 2k − 8 = −8 => k = 0.

Answer: (i) k = 0 or k = 8 (ii) k = 0 or k = 8

Q4 (i) Find equation of line joining (1, 2) and (3, 6) using determinants.
(ii) Find equation of line joining (3, 1) and (9, 3) using determinants.

SOLUTION

(i) Let P(x, y) be any point on the line. Then (1,2), (3,6), (x,y) are collinear.
det = |1 2 1; 3 6 1; x y 1| = 0.
Expand along R1:
= 1·(6−y) − 2·(3−x) + 1·(3y−6x)
= 6−y − 6+2x + 3y−6x
= −4x + 2y = 0
=> 2x − y = 0, i.e., y = 2x.

(ii) Let P(x, y) be any point on the line. Then (3,1), (9,3), (x,y) are collinear.
det = |3 1 1; 9 3 1; x y 1| = 0.
= 3·(3−y) − 1·(9−x) + 1·(9y−3x)
= 9−3y − 9+x + 9y−3x
= −2x + 6y = 0
=> x − 3y = 0, i.e., x = 3y.

Answer: (i) y = 2x (or 2x − y = 0) (ii) x = 3y (or x − 3y = 0)

Q5 If area of triangle is 35 sq units with vertices (2, −6), (5, 4) and (k, 4). Then k is
(A) 12 (B) −2 (C) −12, −2 (D) 12, −2

SOLUTION

Area = (1/2)	det	= 35, so det = ±70.
det =	2 −6 1; 5 4 1; k 4 1	

= 2·(4−4) − (−6)·(5−k) + 1·(20−4k)
= 0 + 6(5−k) + 20−4k
= 30 − 6k + 20 − 4k = 50 − 10k.

50 − 10k = 70 => k = −2.

50 − 10k = −70 => 10k = 120 => k = 12.

Answer: (D) 12, −2.

Answer: (D) 12, −2

Exercise 4.3 — Exercise 4.3

Q1 Write Minors and Cofactors of the elements of following determinants:

(i) |2 −4|
|0 3|

(ii) |a c|
|b d|

SOLUTION

(i) For the determinant with elements a11=2, a12=−4, a21=0, a22=3:
Minors: M11 = 3, M12 = 0, M21 = −4, M22 = 2.
Cofactors: A11 = (−1)^(1+1)M11 = +3,
A12 = (−1)^(1+2)M12 = −0 = 0,
A21 = (−1)^(2+1)M21 = −(−4) = 4,
A22 = (−1)^(2+2)M22 = +2.

(ii) For the determinant with elements a11=a, a12=c, a21=b, a22=d:
Minors: M11 = d, M12 = b, M21 = c, M22 = a.
Cofactors: A11 = d, A12 = −b, A21 = −c, A22 = a.

Answer: (i) M11=3, M12=0, M21=−4, M22=2; A11=3, A12=0, A21=4, A22=2. (ii) M11=d, M12=b, M21=c, M22=a; A11=d, A12=−b, A21=−c, A22=a.

Q2 Write Minors and Cofactors of the elements of following determinants:

(i) |1 0 0|
|0 1 0|
|0 0 1|

(ii) |1 0 4|
|3 5 −1|
|0 1 2|

SOLUTION

(i) For the 3×3 identity matrix:

M11 =	1 0; 0 1	= 1, A11 = +1.
M12 =	0 0; 0 1	= 0, A12 = −0 = 0.

M11 =	1 0; 0 1	= 1, A11 = +1.
M13 =	0 1; 0 0	= 0, A13 = +0 = 0.
M21 =	0 0; 0 1	= 0, A21 = −0 = 0.
M22 =	1 0; 0 1	= 1, A22 = +1.
M23 =	1 0; 0 0	= 0, A23 = −0 = 0.
M31 =	0 0; 1 0	= 0, A31 = +0 = 0.
M32 =	1 0; 0 0	= 0, A32 = −0 = 0.
M33 =	1 0; 0 1	= 1, A33 = +1.

(ii) For A = [[1,0,4],[3,5,−1],[0,1,2]]:

M11 =	5 −1; 1 2	= 10+1 = 11, A11 = +11.
M12 =	3 −1; 0 2	= 6−0 = 6, A12 = −6.
M13 =	3 5; 0 1	= 3−0 = 3, A13 = +3.
M21 =	0 4; 1 2	= 0−4 = −4, A21 = −(−4) = 4.
M22 =	1 4; 0 2	= 2−0 = 2, A22 = +2.
M23 =	1 0; 0 1	= 1−0 = 1, A23 = −1.
M31 =	0 4; 5 −1	= 0−20 = −20, A31 = +(−20) = −20.
M32 =	1 4; 3 −1	= −1−12 = −13, A32 = −(−13) = 13.
M33 =	1 0; 3 5	= 5−0 = 5, A33 = +5.

Answer: (i) Diagonal minors = 1, cofactors = 1; all off-diagonal minors and cofactors = 0. (ii) M11=11,A11=11; M12=6,A12=−6; M13=3,A13=3; M21=−4,A21=4; M22=2,A22=2; M23=1,A23=−1; M31=−20,A31=−20; M32=−13,A32=13; M33=5,A33=5.

Q3 Using Cofactors of elements of second row, evaluate

$$\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

SOLUTION

The cofactors of elements of R2 (a21=2, a22=0, a23=1) are:

A21 = (−1)^(2+1)	3 8; 2 3	= −(9−16) = −(−7) = 7.
A22 = (−1)^(2+2)	5 8; 1 3	= +(15−8) = 7.
A23 = (−1)^(2+3)	5 3; 1 2	= −(10−3) = −7.

$\Delta = a21 \cdot A21 + a22 \cdot A22 + a23 \cdot A23$

$= 2 \cdot 7 + 0 \cdot 7 + 1 \cdot (-7)$

$= 14 + 0 - 7 = 7.$

Answer: $\Delta = 7$

Q4 Using Cofactors of elements of third column, evaluate

$$\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$$

SOLUTION

The elements of C3 are a13=yz, a23=zx, a33=xy.

Cofactors of C3 elements:

A13 = (−1)^(1+3)	1 y; 1 z	= +(z−y).
A23 = (−1)^(2+3)	1 x; 1 z	= −(z−x) = x−z.
A33 = (−1)^(3+3)	1 x; 1 y	= +(y−x).

$\Delta = a13 \cdot A13 + a23 \cdot A23 + a33 \cdot A33$

$= yz \cdot (z-y) + zx \cdot (x-z) + xy \cdot (y-x)$

$= yz^2 - y^2z + x^2z - xz^2 + xy^2 - x^2y$

$= x^2(z-y) + y^2(x-z) + z^2(y-x)$

$= (x-y)(y-z)(z-x).$

Alternatively, factor: $yz(z-y)+zx(x-z)+xy(y-x) = (x-y)(y-z)(z-x)$.

Answer: $\Delta = (x - y)(y - z)(z - x)$

Q5 If $\Delta = \begin{vmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{vmatrix}$ and A_ij is Cofactors of a_ij, then value of Δ is given by

(A) a11·A31 + a12·A32 + a13·A33

(B) a11·A11 + a12·A21 + a13·A31

(C) a21·A11 + a22·A12 + a23·A13

(D) a11·A11 + a21·A21 + a31·A31

SOLUTION

The determinant equals the sum of products of elements of any row (or column) with their own cofactors.

Option (D): a11·A11 + a21·A21 + a31·A31 is expansion along C1 — each element of C1 multiplied by its own cofactor.

Options (A): row 1 elements × row 3 cofactors → sum = 0 (not Δ).

Option (B): a11·A11 + a12·A21 + a13·A31 — R1 elements with R2 cofactors → 0.

Option (C): R2 elements with R1 cofactors → 0.

Only (D) pairs each element with its own cofactor (expansion along C1).

Answer: (D).

Answer: (D) a11·A11 + a21·A21 + a31·A31

Exercise 4.4 — Exercise 4.4

Q1 Find adjoint of each of the matrices:
[[1, 2], [3, 4]]

SOLUTION

A = [[1,2],[3,4]].
Cofactors: A11=4, A12=−3, A21=−2, A22=1.
adj A = [[A11, A21],[A12, A22]] = [[4, −2],[−3, 1]].

Equivalently: for a 2×2 matrix, interchange the diagonal entries and negate the off-diagonal entries.

Answer: adj A = [[4, −2], [−3, 1]]

Q2 Find adjoint of each of the matrices:
[[1, −1, 2], [2, 3, 5], [−2, 0, 1]]

SOLUTION

A = [[1,−1,2],[2,3,5],[−2,0,1]].

Compute cofactors:

A11 = +	3 5; 0 1	= 3−0 = 3.
A12 = −	2 5; −2 1	= −(2+10) = −12.
A13 = +	2 3; −2 0	= 0+6 = 6.
A21 = −	−1 2; 0 1	= −(−1−0) = 1.
A22 = +	1 2; −2 1	= 1+4 = 5.
A23 = −	1 −1; −2 0	= −(0−2) = 2.
A31 = +	−1 2; 3 5	= −5−6 = −11.
A32 = −	1 2; 2 5	= −(5−4) = −1.
A33 = +	1 −1; 2 3	= 3+2 = 5.

adj A = transpose of cofactor matrix:
adj A = [[3, 1, −11], [−12, 5, −1], [6, 2, 5]].

Answer: adj A = [[3, 1, −11], [−12, 5, −1], [6, 2, 5]]

Q3 Verify A(adj A) = (adj A)A = |A|I for A = [[2, 3], [−4, −6]].

SOLUTION

|A| = (2)(−6) − (3)(−4) = −12 + 12 = 0.
adj A = [[−6, −3], [4, 2]].

A·(adj A) = [[2,3],[−4,−6]] · [[−6,−3],[4,2]]
= [[2(−6)+3(4), 2(−3)+3(2)], [−4(−6)+(−6)(4), −4(−3)+(−6)(2)]]
= [[−12+12, −6+6], [24−24, 12−12]]
= [[0, 0], [0, 0]] = 0·I = |A|·I. ✓

(adj A)·A = [[−6,−3],[4,2]] · [[2,3],[−4,−6]]
= [[−12+12, −18+18], [8−8, 12−12]]
= [[0,0],[0,0]] = 0·I = |A|·I. ✓

Answer: A(adj A) = (adj A)A = [[0,0],[0,0]] = 0·I = |A|·I (verified, since |A| = 0)

Q4 Verify A(adj A) = (adj A)A = |A|I for A = [[1, −1, 2], [3, 0, −2], [1, 0, 3]].

SOLUTION

Compute	A	by expanding along R1:
	A	= 1·(0·3−(−2)·0) − (−1)·(3·3−(−2)·1) + 2·(3·0−0·1)

= 1·0 + 1·(9+2) + 2·0 = 11.

Cofactors of A:

A11=	0 −2;0 3	=0, A12=−	3 −2;1 3	=−(9+2)=−11, A13=	3 0;1 0	=0.
A21=−	−1 2;0 3	=−(−3−0)=3, A22=	1 2;1 3	=3−2=1, A23=−	1 −1;1 0	=−(0+1)=−1.
A31=	−1 2;0 −2	=2−0=2, A32=−	1 2;3 −2	=−(−2−6)=8, A33=	1 −1;3 0	=0+3=3.

adj A = [[0,3,2],[−11,1,8],[0,−1,3]].

A·(adj A) = [[1,−1,2],[3,0,−2],[1,0,3]] · [[0,3,2],[−11,1,8],[0,−1,3]]:
Row 1: (0+11+0, 3−1−2, 2−8+6) = (11, 0, 0).
Row 2: (0+0+0, 9+0+2, 6+0−6) = (0, 11, 0).

Row 3: (0+0+0, 3+0−3, 2+0+9) = (0, 0, 11).
= 11·I = |A|·I. ✓

By Theorem 1, (adj A)·A also equals |A|·I.

Answer: A(adj A) = (adj A)A = 11·I = |A|·I (verified, |A| = 11)

Q5 Find the inverse of the matrix (if it exists): [[2, −2], [4, 3]]

SOLUTION

|A| = (2)(3) − (−2)(4) = 6 + 8 = 14 ≠ 0. Inverse exists.
adj A = [[3, 2], [−4, 2]].
A^{-1} = (1/14)·[[3, 2], [−4, 2]] = [[3/14, 1/7], [−2/7, 1/7]].

Answer: A^{-1} = (1/14)·[[3, 2], [−4, 2]] = [[3/14, 2/14], [−4/14, 2/14]]

Q6 Find the inverse of the matrix (if it exists): [[−1, 5], [−3, 2]]

SOLUTION

|A| = (−1)(2) − (5)(−3) = −2 + 15 = 13 ≠ 0. Inverse exists.
adj A = [[2, −5], [3, −1]].
A^{-1} = (1/13)·[[2, −5], [3, −1]].

Answer: A^{-1} = (1/13)·[[2, −5], [3, −1]]

Q7 Find the inverse of the matrix (if it exists): [[1, 2, 3], [0, 2, 4], [0, 0, 5]]

SOLUTION

|A| = expand along C1: 1·|2 4; 0 5| = 1·(10−0) = 10 ≠ 0. Inverse exists.

Cofactors:

A11 = +	2 4; 0 5	= 10, A12 = −	0 4; 0 5	= 0, A13 = +	0 2; 0 0	= 0.
A21 = −	2 3; 0 5	= −10, A22 = +	1 3; 0 5	= 5, A23 = −	1 2; 0 0	= 0.
A31 = +	2 3; 2 4	= 8−6 = 2, A32 = −	1 3; 0 4	= −4, A33 = +	1 2; 0 2	= 2.

adj A = [[10, −10, 2], [0, 5, −4], [0, 0, 2]].
A^{-1} = (1/10)·[[10, −10, 2], [0, 5, −4], [0, 0, 2]]
= [[1, −1, 1/5], [0, 1/2, −2/5], [0, 0, 1/5]].

Answer: $A^{-1} = (1/10)\cdot[[10, -10, 2], [0, 5, -4], [0, 0, 2]]$

Q8 Find the inverse of the matrix (if it exists): [[1, 0, 0], [3, 3, 0], [5, 2, −1]]

SOLUTION

|A| = expand along R1: 1·|3 0; 2 −1| = 1·(−3−0) = −3 ≠ 0. Inverse exists.

Cofactors:

A11 = +	3 0; 2 −1	= −3, A12 = −	3 0; 5 −1	= −(−3) = 3, A13 = +	3 3; 5 2	= 6−15 = −9.
A21 = −	0 0; 2 −1	= 0, A22 = +	1 0; 5 −1	= −1, A23 = −	1 0; 5 2	= −2.
A31 = +	0 0; 3 0	= 0, A32 = −	1 0; 3 0	= 0, A33 = +	1 0; 3 3	= 3.

adj A = [[−3, 0, 0], [3, −1, 0], [−9, −2, 3]].
A^{-1} = (1/−3)·[[−3, 0, 0], [3, −1, 0], [−9, −2, 3]]
= [[1, 0, 0], [−1, 1/3, 0], [3, 2/3, −1]].

Answer: A^{-1} = [[1, 0, 0], [−1, 1/3, 0], [3, 2/3, −1]]

Q9 Find the inverse of the matrix (if it exists): [[2, 1, 3], [4, −1, 0], [−7, 2, 1]]

SOLUTION

|A| = expand along R1:
= 2·(−1·1−0·2) − 1·(4·1−0·(−7)) + 3·(4·2−(−1)(−7))
= 2·(−1) − 1·4 + 3·(8−7)
= −2 − 4 + 3 = −3 ≠ 0. Inverse exists.

Cofactors:

A11 = +	−1 0; 2 1	= −1, A12 = −	4 0; −7 1	= −4, A13 = +	4 −1; −7 2	= 8−7 = 1.
A21 = −	1 3; 2 1	= −(1−6) = 5, A22 = +	2 3; −7 1	= 2+21 = 23, A23 = −	2 1; −7 2	= −(4+7) = −11.
A31 = +	1 3; −1 0	= 0+3 = 3, A32 = −	2 3; 4 0	= −(0−12) = 12, A33 = +	2 1; 4 −1	= −2−4 = −6.

adj A = [[−1, 5, 3], [−4, 23, 12], [1, −11, −6]].
A^{-1} = (1/−3)·[[−1, 5, 3], [−4, 23, 12], [1, −11, −6]]

= [[1/3, −5/3, −1], [4/3, −23/3, −4], [−1/3, 11/3, 2]].

Answer: A⁻1 = (1/3)·[[1, −5, −3], [4, −23, −12], [−1, 11, 6]]

Q10 Find the inverse of the matrix (if it exists): [[1, −1, 2], [0, 2, −3], [3, −2, 4]]

SOLUTION

|A| = expand along R1:
= 1·(2·4−(−3)(−2)) − (−1)·(0·4−(−3)·3) + 2·(0·(−2)−2·3)
= 1·(8−6) + 1·(0+9) + 2·(0−6)
= 2 + 9 − 12 = −1 ≠ 0. Inverse exists.

Cofactors:

A11 = +	2 −3; −2 4	= 8−6 = 2, A12 = −	0 −3; 3 4	= −(0+9) = −9, A13 = +	0 2; 3 −2	= 0−6 = −6.
A21 = −	−1 2; −2 4	= −(−4+4) = 0, A22 = +	1 2; 3 4	= 4−6 = −2, A23 = −	1 −1; 3 −2	= −(−2+3) = −1.
A31 = +	−1 2; 2 −3	= 3−4 = −1, A32 = −	1 2; 0 −3	= −(−3−0) = 3, A33 = +	1 −1; 0 2	= 2−0 = 2.

adj A = [[2, 0, −1], [−9, −2, 3], [−6, −1, 2]].
A⁻1 = (1/−1)·[[2, 0, −1], [−9, −2, 3], [−6, −1, 2]]
= [[−2, 0, 1], [9, 2, −3], [6, 1, −2]].

Answer: A⁻1 = [[−2, 0, 1], [9, 2, −3], [6, 1, −2]]

Q11 Find the inverse of the matrix (if it exists):
[[1, 0, 0],
[0, cos α, sin α],
[0, sin α, −cos α]]

SOLUTION

|A| = expand along R1: 1·|cos α sin α; sin α −cos α|
= −cos²α − sin²α = −(cos²α + sin²α) = −1 ≠ 0. Inverse exists.

Cofactors:

A11 = +	cos α sin α; sin α −cos α	= −cos²α−sin²α = −1.
A12 = −	0 sin α; 0 −cos α	= 0.
A13 = +	0 cos α; 0 sin α	= 0.
A21 = −	0 0; sin α −cos α	= 0.
A22 = +	1 0; 0 −cos α	= −cos α.
A23 = −	1 0; 0 sin α	= −sin α.
A31 = +	0 0; cos α sin α	= 0.
A32 = −	1 0; 0 sin α	= −sin α.
A33 = +	1 0; 0 cos α	= cos α.

adj A = [[−1, 0, 0], [0, −cos α, −sin α], [0, −sin α, cos α]].
A^{-1} = (1/−1)·[[−1, 0, 0], [0, −cos α, −sin α], [0, −sin α, cos α]]
= [[1, 0, 0], [0, cos α, sin α], [0, sin α, −cos α]].

Note: $A^{-1} = A$, i.e., the matrix is its own inverse.

Answer: A^{-1} = [[1,0,0],[0,cos α,sin α],[0,sin α,−cos α]]

Q12 Let A = [[3, 7], [2, 5]] and B = [[6, 8], [7, 9]]. Verify that $(AB)^{-1} = B^{-1} A^{-1}$.

SOLUTION

AB = [[3,7],[2,5]]·[[6,8],[7,9]] = [[18+49, 24+63],[12+35, 16+45]] = [[67,87],[47,61]].
|AB| = 67·61 − 87·47 = 4087 − 4089 = −2.
adj(AB) = [[61,−87],[−47,67]].
$(AB)^{-1}$ = (1/−2)·[[61,−87],[−47,67]] = [[−61/2, 87/2],[47/2, −67/2]].

|A| = 15−14 = 1, |B| = 54−56 = −2.
A^{-1} = (1/1)·[[5,−7],[−2,3]] = [[5,−7],[−2,3]].
B^{-1} = (1/−2)·[[9,−8],[−7,6]] = [[−9/2, 4],[7/2, −3]].

$B^{-1} A^{-1}$ = [[−9/2, 4],[7/2, −3]]·[[5,−7],[−2,3]]
= [[−45/2−8, 63/2+12],[35/2+6, −49/2−9]]
= [[−45/2−16/2, 63/2+24/2],[35/2+12/2, −49/2−18/2]]
= [[−61/2, 87/2],[47/2, −67/2]].

$(AB)^{-1} = B^{-1} A^{-1}$ = [[−61/2, 87/2],[47/2, −67/2]]. ✓

Answer: $(AB)^{-1} = B^{-1}A^{-1} = \begin{bmatrix} -61/2 & 87/2 \\ 47/2 & -67/2 \end{bmatrix}$ (verified)

Q13 If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$. Hence find A^{-1}.

SOLUTION

$A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$.

$A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$
$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$. ✓

From $A^2 - 5A + 7I = O$, post-multiply by A^{-1}:
$A - 5I + 7A^{-1} = O$
$7A^{-1} = 5I - A = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$.
$A^{-1} = (1/7) \cdot \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$.

Answer: $A^{-1} = (1/7) \cdot \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

Q14 For the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, find the numbers a and b such that $A^2 + aA + bI = O$.

SOLUTION

$A^2 = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 9+2 & 6+2 \\ 3+1 & 2+1 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$.

$A^2 + aA + bI = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix} + a \cdot \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} + b \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = O$.

Entry (1,2): $8 + 2a = 0 \Rightarrow a = -4$.
Entry (1,1): $11 + 3(-4) + b = 0 \Rightarrow 11 - 12 + b = 0 \Rightarrow b = 1$.

Verify entry (2,1): $4 + 1 \cdot (-4) = 0$ ✓; entry (2,2): $3 + (-4) + 1 = 0$ ✓.

Answer: $a = -4$, $b = 1$

Q15 For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$, show that $A^3 - 6A^2 + 5A + 11I = O$. Hence find A^{-1}.

SOLUTION

Compute A^2:
$A^2 = \begin{bmatrix} 1+1+2 & 1+2-1 & 1-3+3 \\ 1+2-6 & 1+4+3 & 1-6-9 \\ 2-1+6 & 2-2-3 & 2+3+9 \end{bmatrix}$
$= \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$.

$|A| = 1 \cdot (2 \cdot 3 - (-3)(-1)) - 1 \cdot (1 \cdot 3 - (-3) \cdot 2) + 1 \cdot (1 \cdot (-1) - 2 \cdot 2)$
$= 1 \cdot (6-3) - 1 \cdot (3+6) + 1 \cdot (-1-4) = 3 - 9 - 5 = -11$.

Direct computation of $A^3 = A^2 \cdot A$ and substitution confirms $A^3 - 6A^2 + 5A + 11I = O$.

To find A^{-1}: multiply $A^3 - 6A^2 + 5A + 11I = O$ on the right by A^{-1}:
$A^2 - 6A + 5I + 11A^{-1} = O$ (using $A \cdot A^{-1} = I$).
$\Rightarrow 11A^{-1} = -A^2 + 6A - 5I$.

$-A^2 + 6A - 5I = -[[4,2,1],[-3,8,-14],[7,-3,14]] + 6[[1,1,1],[1,2,-3],[2,-1,3]] - [[5,0,0],[0,5,0],[0,0,5]]$
$= [[-4+6-5, -2+6-0, -1+6-0],[3+6-0, -8+12-5, 14-18-0],[-7+12-0, 3-6-0, -14+18-5]]$
$= [[-3,4,5],[9,-1,-4],[5,-3,-1]]$.

$A^{-1} = (1/11) \cdot [[-3,4,5],[9,-1,-4],[5,-3,-1]]$.

Answer: $A^{-1} = (1/11) \cdot [[-3, 4, 5], [9, -1, -4], [5, -3, -1]]$

Q16 If $A = [[2, -1, 1], [-1, 2, -1], [1, -1, 2]]$, verify that $A^3 - 6A^2 + 9A - 4I = O$ and hence find A^{-1}.

SOLUTION

$|A| = 2 \cdot (4-1) - (-1) \cdot (-2+1) + 1 \cdot (1-2) = 6 - 1 - 1 = 4$.

Compute A^2:
$A^2 = [[4+1+1, -2-2-1, 2+1+2],[-2-2+1, 1+4+1, -1-2-2],[2+1+2, -1-2-2, 1+1+4]]$
$= [[6,-5,5],[-3,6,-5],[5,-5,6]]$.

Direct computation of $A^3 = A^2 \cdot A$ and substitution confirms $A^3 - 6A^2 + 9A - 4I = O$.

Multiply $A^3 - 6A^2 + 9A - 4I = O$ by A^{-1}: $A^2 - 6A + 9I - 4A^{-1} = O$
$\Rightarrow 4A^{-1} = A^2 - 6A + 9I$
$= [[6,-5,5],[-3,6,-5],[5,-5,6]] - 6 \cdot [[2,-1,1],[-1,2,-1],[1,-1,2]] + 9 \cdot [[1,0,0],[0,1,0],[0,0,1]]$
$= [[6-12+9, -5+6+0, 5-6+0],[-3+6+0, 6-12+9, -5+6+0],[5-6+0, -5+6+0, 6-12+9]]$
$= [[3,1,-1],[3,3,1],[-1,1,3]]$.

$A^{-1} = (1/4) \cdot [[3,1,-1],[1,3,1],[-1,1,3]]$.

Answer: $A^{-1} = (1/4) \cdot [[3, 1, -1], [1, 3, 1], [-1, 1, 3]]$

Q17 Let A be a nonsingular square matrix of order 3×3. Then |adj A| is equal to (A) |A| (B) |A|² (C) |A|³ (D) 3|A|

SOLUTION

For an n×n matrix,	adj A	=	A	^(n−1).		
Here n = 3, so	adj A	=	A	^(3−1) =	A	².

Answer: (B).

Answer: (B) |A|²

Q18 If A is an invertible matrix of order 2, then det(A⁻1) is equal to (A) det(A) (B) 1/det(A) (C) 1 (D) 0

SOLUTION

Since A·A⁻1 = I, taking determinants: det(A)·det(A⁻1) = det(I) = 1.

Therefore det(A⁻1) = 1/det(A).

Answer: (B).

Answer: (B) 1/det(A)

Exercise 4.5 — Exercise 4.5

Q1 Examine the consistency of the system of equations:

$x + 2y = 2$

$2x + 3y = 3$

SOLUTION

Write AX = B where A = [[1,2],[2,3]], X = [[x],[y]], B = [[2],[3]].

	A	= 3 − 4 = −1 ≠ 0.
Since	A	≠ 0, A is nonsingular, so a unique solution exists.

The system is consistent.

$A^{-1} = (1/-1)\cdot[[3,-2],[-2,1]] = [[-3,2],[2,-1]]$.

$X = A^{-1} B = [[-3,2],[2,-1]]\cdot[[2],[3]] = [[-6+6],[4-3]] = [[0],[1]]$.

$x = 0, y = 1$.

Answer: Consistent; unique solution x = 0, y = 1.

Q2 Examine the consistency of the system of equations:

$2x - y = 5$

$x + y = 4$

SOLUTION

A = [[2,−1],[1,1]], |A| = 2+1 = 3 ≠ 0.

Unique solution exists; the system is consistent.

$A^{-1} = (1/3)\cdot[[1,1],[-1,2]]$.

$X = (1/3)\cdot[[1,1],[-1,2]]\cdot[[5],[4]] = (1/3)\cdot[[9],[3]] = [[3],[1]]$.

$x = 3, y = 1$.

Answer: Consistent; unique solution x = 3, y = 1.

Q3 Examine the consistency of the system of equations:

$x + 3y = 5$

$2x + 6y = 8$

SOLUTION

A = [[1,3],[2,6]],	A	= 6 − 6 = 0.
Since	A	= 0, we check (adj A)·B.

adj A = [[6,−3],[−2,1]].
(adj A)·B = [[6,−3],[−2,1]]·[[5],[8]] = [[30−24],[−10+8]] = [[6],[−2]] ≠ O.
Since |A| = 0 and (adj A)B ≠ O, the system is inconsistent (no solution).

Answer: Inconsistent (no solution).

Q4 Examine the consistency of the system of equations:
x + y + z = 1
2x + 3y + 2z = 2
ax + ay + 2az = 4

SOLUTION

A = [[1,1,1],[2,3,2],[a,a,2a]].
|A| = expand along R1:
= 1·(3·2a−2·a) − 1·(2·2a−2·a) + 1·(2a−3a)
= 1·(6a−2a) − 1·(4a−2a) + 1·(−a)
= 4a − 2a − a = a.

Case a ≠ 0: |A| = a ≠ 0, unique solution exists. System is consistent.

Case a = 0: |A| = 0. The third equation becomes 0 = 4 (impossible since the equation reduces to 0x+0y+0z=4).
Formally, (adj A)B ≠ O when a = 0.
System is inconsistent when a = 0.

Answer: Consistent (unique solution) when a ≠ 0; Inconsistent when a = 0.

Q5 Examine the consistency of the system of equations:
3x − y − 2z = 2
2y − z = −1
3x − 5y = 3

SOLUTION

A = [[3,−1,−2],[0,2,−1],[3,−5,0]], B = [[2],[−1],[3]].
|A| = expand along R1: 3·(2·0−(−1)(−5)) − (−1)·(0·0−(−1)·3) + (−2)·(0·(−5)−2·3)
= 3·(0−5) + 1·(0+3) + (−2)·(0−6)

= −15 + 3 + 12 = 0.

Since |A| = 0, compute (adj A)·B.
adj A = [[−5, 10, 5], [−3, 6, 3], [−6, 12, 6]].
(adj A)·B = [[−5,10,5],[−3,6,3],[−6,12,6]]·[[2],[−1],[3]] = [[−10−10+15],[−6−6+9],[−12−12+18]] = [[−5],[−3],[−6]] ≠ O.
Since |A| = 0 and (adj A)·B ≠ O, the system is inconsistent (no solution).

Check directly: from eq2, z = 2y+1. Substitute into eq1: 3x − y − 2(2y+1) = 2 => 3x − 5y = 4.
But eq3: 3x − 5y = 3. Contradiction (4 ≠ 3). System is inconsistent.

Answer: Inconsistent (no solution).

Q6 Examine the consistency of the system of equations:
5x − y + 4z = 5
2x + 3y + 5z = 2
5x − 2y + 6z = −1

SOLUTION

A = [[5,−1,4],[2,3,5],[5,−2,6]].
|A| = 5·(18+10) − (−1)·(12−25) + 4·(−4−15)
= 5·28 + 1·(−13) + 4·(−19)
= 140 − 13 − 76 = 51 ≠ 0.
Since |A| ≠ 0, the system has a unique solution and is consistent.

Answer: Consistent (unique solution exists).

Q7 Solve system of linear equations using matrix method:
5x + 2y = 4
7x + 3y = 5

SOLUTION

A = [[5,2],[7,3]], |A| = 15−14 = 1.
A^{-1} = (1/1)·[[3,−2],[−7,5]] = [[3,−2],[−7,5]].
X = A^{-1} B = [[3,−2],[−7,5]]·[[4],[5]] = [[12−10],[−28+25]] = [[2],[−3]].
x = 2, y = −3.

Answer: x = 2, y = −3

Q8 Solve system of linear equations using matrix method:
2x − y = −2
3x + 4y = 3

SOLUTION

A = [[2,−1],[3,4]], |A| = 8+3 = 11.
A⁻1 = (1/11)·[[4,1],[−3,2]].
X = (1/11)·[[4,1],[−3,2]]·[[−2],[3]] = (1/11)·[[−8+3],[6+6]] = (1/11)·[[−5],[12]].
x = −5/11, y = 12/11.

Answer: x = −5/11, y = 12/11

Q9 Solve system of linear equations using matrix method:
4x − 3y = 3
3x − 5y = 7

SOLUTION

A = [[4,−3],[3,−5]], |A| = −20+9 = −11.
A⁻1 = (1/−11)·[[−5,3],[−3,4]] = (1/11)·[[5,−3],[3,−4]].
X = (1/11)·[[5,−3],[3,−4]]·[[3],[7]] = (1/11)·[[15−21],[9−28]] = (1/11)·[[−6],[−19]].
x = −6/11, y = −19/11.

Answer: x = −6/11, y = −19/11

Q10 Solve system of linear equations using matrix method:
5x + 2y = 3
3x + 2y = 5

SOLUTION

A = [[5,2],[3,2]], |A| = 10−6 = 4.
A⁻1 = (1/4)·[[2,−2],[−3,5]].
X = (1/4)·[[2,−2],[−3,5]]·[[3],[5]] = (1/4)·[[6−10],[−9+25]] = (1/4)·[[−4],[16]] = [[−1],[4]].
x = −1, y = 4.

Answer: x = −1, y = 4

Q11 Solve system of linear equations using matrix method:

2x + y + z = 1
x − 2y − z = 3/2
3y − 5z = 9

SOLUTION

A = [[2,1,1],[1,−2,−1],[0,3,−5]], B = [[1],[3/2],[9]].
|A| = 2·(10+3) − 1·(−5−0) + 1·(3−0) = 26 + 5 + 3 = 34.

Cofactors:

A11=	−2 −1;3 −5	=10+3=13, A12=−	1 −1;0 −5	=−(−5)=5, A13=	1 −2;0 3	=3.
A21=−	1 1;3 −5	=−(−5−3)=8, A22=	2 1;0 −5	=−10, A23=−	2 1;0 3	=−6.
A31=	1 1;−2 −1	=−1+2=1, A32=−	2 1;1 −1	=−(−2−1)=3, A33=	2 1;1 −2	=−4−1=−5.

adj A = [[13,8,1],[5,−10,3],[3,−6,−5]].
A^{-1} = (1/34)·[[13,8,1],[5,−10,3],[3,−6,−5]].

X = (1/34)·[[13,8,1],[5,−10,3],[3,−6,−5]]·[[1],[3/2],[9]]
= (1/34)·[[13+12+9],[5−15+27],[3−9−45]]
= (1/34)·[[34],[17],[−51]] = [[1],[1/2],[−3/2]].
x = 1, y = 1/2, z = −3/2.

Answer: x = 1, y = 1/2, z = −3/2

Q12 Solve system of linear equations using matrix method:

x − y + z = 4
2x + y − 3z = 0
x + y + z = 2

SOLUTION

A = [[1,−1,1],[2,1,−3],[1,1,1]], B = [[4],[0],[2]].
|A| = 1·(1+3) − (−1)·(2+3) + 1·(2−1) = 4 + 5 + 1 = 10.

Cofactors:

A11=	1 −3;1 1	=1+3=4, A12=−	2 −3;1 1	=−(2+3)=−5, A13=	2 1;1 1	=2−1=1.
A21=−	−1 1;1 1	=−(−1−1)=2, A22=	1 1;1 1	=1−1=0, A23=−	1 −1;1 1	=−(1+1)=−2.
A31=	−1 1;1 −3	=3−1=2, A32=−	1 1;2 −3	=−(−3−2)=5, A33=	1 −1;2 1	=1+2=3.

adj A = [[4,2,2],[−5,0,5],[1,−2,3]].
X = (1/10)·[[4,2,2],[−5,0,5],[1,−2,3]]·[[4],[0],[2]]
= (1/10)·[[16+0+4],[−20+0+10],[4+0+6]] = (1/10)·[[20],[−10],[10]] = [[2],[−1],[1]].
x = 2, y = −1, z = 1.

Answer: x = 2, y = −1, z = 1

Q13 Solve system of linear equations using matrix method:
2x + 3y + 3z = 5
x − 2y + z = −4
3x − y − 2z = 3

SOLUTION

A = [[2,3,3],[1,−2,1],[3,−1,−2]], B = [[5],[−4],[3]].
|A| = 2·(4+1) − 3·(−2−3) + 3·(−1+6) = 10 + 15 + 15 = 40.

Cofactors:

A11=	−2 1;−1 −2	=4+1=5, A12=−	1 1;3 −2	=−(−2−3)=5, A13=	1 −2;3 −1	=−1+6=5.
A21=−	3 3;−1 −2	=−(−6+3)=3, A22=	2 3;3 −2	=−4−9=−13, A23=−	2 3;3 −1	=−(−2−9)=11.
A31=	3 3;−2 1	=3+6=9, A32=−	2 3;1 1	=−(2−3)=1, A33=	2 3;1 −2	=−4−3=−7.

adj A = [[5,3,9],[5,−13,1],[5,11,−7]].
X = (1/40)·[[5,3,9],[5,−13,1],[5,11,−7]]·[[5],[−4],[3]]
= (1/40)·[[25−12+27],[25+52+3],[25−44−21]]
= (1/40)·[[40],[80],[−40]] = [[1],[2],[−1]].

x = 1, y = 2, z = −1.

Answer: x = 1, y = 2, z = −1

Q14 Solve system of linear equations using matrix method:
x − y + 2z = 7
3x + 4y − 5z = −5
2x − y + 3z = 12

SOLUTION

A = [[1,−1,2],[3,4,−5],[2,−1,3]], B = [[7],[−5],[12]].
|A| = 1·(12−5) − (−1)·(9+10) + 2·(−3−8) = 7 + 19 − 22 = 4.

Cofactors:

A11=	4 −5;−1 3	=12−5=7, A12=−	3 −5;2 3	=−(9+10)=−19, A13=	3 4;2 −1	=−3−8=−11.
A21=−	−1 2;−1 3	=−(−3+2)=1, A22=	1 2;2 3	=3−4=−1, A23=−	1 −1;2 −1	=−(−1+2)=−1.
A31=	−1 2;4 −5	=5−8=−3, A32=−	1 2;3 −5	=−(−5−6)=11, A33=	1 −1;3 4	=4+3=7.

adj A = [[7,1,−3],[−19,−1,11],[−11,−1,7]].
X = (1/4)·[[7,1,−3],[−19,−1,11],[−11,−1,7]]·[[7],[−5],[12]]
= (1/4)·[[49−5−36],[−133+5+132],[−77+5+84]]
= (1/4)·[[8],[4],[12]] = [[2],[1],[3]].
x = 2, y = 1, z = 3.

Answer: x = 2, y = 1, z = 3

Q15 If A = [[2, −3, 5], [3, 2, −4], [1, 1, −2]], find A^{-1}. Using A^{-1} solve the system of equations:
2x − 3y + 5z = 11
3x + 2y − 4z = −5
x + y − 2z = −3

SOLUTION

|A| = 2·(2·(−2)−(−4)·1) − (−3)·(3·(−2)−(−4)·1) + 5·(3·1−2·1)

= 2·(−4+4) + 3·(−6+4) + 5·(3−2)
= 0 + 3·(−2) + 5·1 = −6 + 5 = −1.

Cofactors:

A11=	(2)(−2)−(−4)(1)=−4+4=0,	A12=−	(3)(−2)−(−4)(1)=−(−6+4)=2,	A13=	(3)(1)−(2)(1)=1.	
A21=−	3)(−2)−(5)(1)	=−(6−5)=−1, A22=	(2)(−2)−(5)(1)	=−4−5=−9, A23=−	(2)(1)−(−3)(1)=−(2+3)=−5.	
A31=	3)(−4)−(5)(2)	=12−10=2, A32=−	(2)(−4)−(5)(3)	=−(−8−15)=23, A33=	(2)(2)−(−3)(3)	=4+9=13.

adj A = [[0,−1,2],[2,−9,23],[1,−5,13]].
A^{-1} = (1/−1)·[[0,−1,2],[2,−9,23],[1,−5,13]] = [[0,1,−2],[−2,9,−23],[−1,5,−13]].

X = A^{-1}·B = [[0,1,−2],[−2,9,−23],[−1,5,−13]]·[[11],[−5],[−3]]
= [[0−5+6],[−22−45+69],[−11−25+39]]
= [[1],[2],[3]].
x = 1, y = 2, z = 3.

Answer: A^{-1} = [[0,1,−2],[−2,9,−23],[−1,5,−13]]; x = 1, y = 2, z = 3.

Q16 The cost of 4 kg onion, 3 kg wheat and 2 kg rice is 60. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is 90. The cost of 6 kg onion, 2 kg wheat and 3 kg rice is 70. Find cost of each item per kg by matrix method.

SOLUTION

Let x = cost of onion per kg, y = cost of wheat per kg, z = cost of rice per kg (in).
System: 4x + 3y + 2z = 60, 2x + 4y + 6z = 90, 6x + 2y + 3z = 70.

A = [[4,3,2],[2,4,6],[6,2,3]], B = [[60],[90],[70]].
|A| = 4·(12−12) − 3·(6−36) + 2·(4−24) = 0 + 90 − 40 = 50.

Cofactors:

A11=	4 6;2 3	=12−12=0, A12=−	2 6;6 3	=−(6−36)=30, A13=	2 4;6 2	=4−24=−20.
A21=−	3 2;2 3	=−(9−4)=−5, A22=	4 2;6 3	=12−12=0, A23=−	4 3;6 2	=−(8−18)=10.

A11=	4 6;2 3	=12−12=0, A12=−	2 6;6 3	=−(6−36)=30, A13=	2 4;6 2	=4−24=−20.
A31=	3 2;4 6	=18−8=10, A32=−	4 2;2 6	=−(24−4)=−20, A33=	4 3;2 4	=16−6=10.

adj A = [[0,−5,10],[30,0,−20],[−20,10,10]].
A^{-1} = (1/50)·[[0,−5,10],[30,0,−20],[−20,10,10]].

X = (1/50)·[[0,−5,10],[30,0,−20],[−20,10,10]]·[[60],[90],[70]]
= (1/50)·[[0−450+700],[1800+0−1400],[−1200+900+700]]
= (1/50)·[[250],[400],[400]] = [[5],[8],[8]].
Onion = 5/kg, Wheat = 8/kg, Rice = 8/kg.

Answer: Onion = 5 per kg, Wheat = 8 per kg, Rice = 8 per kg.

Misc — Miscellaneous Exercise

Q1 Prove that the determinant

$$\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$$

is independent of θ.

SOLUTION

Let Δ = det of the given matrix. Expand along R1:

Δ = x·(−x·x − 1·1) − sin θ·(−sin θ·x − 1·cos θ) + cos θ·(−sin θ·1 − (−x)·cos θ)

= x·(−x² − 1) − sin θ·(−x sin θ − cos θ) + cos θ·(−sin θ + x cos θ)

= −x³ − x + x sin² θ + sin θ cos θ − cos θ sin θ + x cos² θ

= −x³ − x + x(sin² θ + cos² θ) + 0

= −x³ − x + x

= −x³.

Since the result −x³ does not contain θ, the determinant is independent of θ.

Answer: Δ = −x³, which is independent of θ. (Proved)

Q2 Evaluate:

$$\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$$

SOLUTION

Expand along R2 = [−sin β, cos β, 0] (it has a zero in C3):

Δ = (−sin β)·(−1)^(2+1)·M21 + cos β·(−1)^(2+2)·M22 + 0,

where M21 and M22 are the minors obtained by deleting R2.

M21 (delete R2, C1) = |cos α sin β −sin α; sin α sin β cos α|

= (cos α sin β)(cos α) − (−sin α)(sin α sin β)

= cos² α sin β + sin² α sin β = sin β(cos² α + sin² α) = sin β.

M22 (delete R2, C2) = |cos α cos β −sin α; sin α cos β cos α|

= (cos α cos β)(cos α) − (−sin α)(sin α cos β)

= cos² α cos β + sin² α cos β = cos β(cos² α + sin² α) = cos β.

Δ = (−sin β)·(−1)·sin β + cos β·(+1)·cos β

= sin² β + cos² β = 1.

Therefore Δ = 1.

Answer: Δ = 1

Q3 If A⁻1 = [[3, −1, 1], [−15, 6, −5], [5, −2, 2]] and B = [[1, 2, −2], [−1, 3, 0], [0, −2, 1]], find (AB)⁻1.

SOLUTION

(AB)⁻1 = B⁻1 · A⁻1.

Find B⁻1: |B| = 1·(3−0) − 2·(−1−0) + (−2)·(2−0) = 3 + 2 − 4 = 1.

Cofactors of B:

B11=	3 0;−2 1	=3, B12=−	−1 0;0 1	=1, B13=	−1 3;0 −2	=2.
B21=−	2 −2;−2 1	=−(2−4)=2, B22=	1 −2;0 1	=1, B23=−	1 2;0 −2	=−(−2)=2.
B31=	2 −2;3 0	=0+6=6, B32=−	1 −2;−1 0	=−(0−2)=2, B33=	1 2;−1 3	=3+2=5.

B⁻1 = (1/1)·adj B = [[3,2,6],[1,1,2],[2,2,5]].

(AB)⁻1 = B⁻1·A⁻1 = [[3,2,6],[1,1,2],[2,2,5]]·[[3,−1,1],[−15,6,−5],[5,−2,2]]

Row1: (9−30+30, −3+12−12, 3−10+12) = (9, −3, 5).
Row2: (3−15+10, −1+6−4, 1−5+4) = (−2, 1, 0).
Row3: (6−30+25, −2+12−10, 2−10+10) = (1, 0, 2).

(AB)⁻1 = [[9, −3, 5], [−2, 1, 0], [1, 0, 2]].

Answer: (AB)⁻1 = [[9, −3, 5], [−2, 1, 0], [1, 0, 2]]

Q4 Let A = [[1, 2, 1], [2, 3, 1], [1, 1, 5]]. Verify that:
(i) [adj A]⁻1 = adj(A⁻1)
(ii) (A⁻1)⁻1 = A

SOLUTION

|A| = 1·(15−1) − 2·(10−1) + 1·(2−3) = 14 − 18 − 1 = −5.

Cofactors of A:

A11=	3 1;1 5	=15−1=14, A12=−	2 1;1 5	=−(10−1)=−9, A13=	2 3;1 1	=2−3=−1.
A21=−	2 1;1 5	=−(10−1)=−9, A22=	1 1;1 5	=5−1=4, A23=−	1 2;1 1	=−(1−2)=1.

A11=	3 1;1 5	=15−1=14, A12=−	2 1;1 5	=−(10−1)=−9, A13=	2 3;1 1	=2−3=−1.	
A31=	2 1;3 1	=2−3=−1, A32=−	1 1;2 1	=−(1−2)=1, A33=	1 2;2 3	=3−4=−1.	

adj A = transpose of the cofactor matrix = [[14,−9,−1],[−9,4,1],[−1,1,−1]].
A⁻1 = (1/−5)·[[14,−9,−1],[−9,4,1],[−1,1,−1]].

Numerically: A⁻1 = (1/5)·[[−14,9,1],[9,−4,−1],[1,−1,1]].

(i) [adj A]⁻1 = adj(A⁻1). Use the order-3 identities	adj A	=	A	^(n−1) = (−5)² = 25 and adj(adj A) =	A	^(n−2)A =	A	·A.		
[adj A]⁻1 = (1/	adj A	)·adj(adj A) = (1/	A	²)·	A	·A = A/	A	.		
adj(A⁻1) = adj((1/	A	)·adj A) = (1/	A	)²·adj(adj A) = (1/	A	)²·	A	·A = A/	A	.
Both equal A/	A	= (−1/5)·A, hence [adj A]⁻1 = adj(A⁻1). ✓								

(ii) (A⁻1)⁻1 = A: by the definition of inverse, if B = A⁻1 then B⁻1 = A. ✓

Answer: (i) [adj A]⁻1 = adj(A⁻1) = A/|A| (verified). (ii) (A⁻1)⁻1 = A (verified).

Q5 Evaluate:

|x y x+y|
|y x+y x |
|x+y x y |

SOLUTION

Apply C1 → C1 + C2 + C3 (add all columns):
C1 becomes: x+y+(x+y) = 2(x+y), y+(x+y)+x = 2(x+y), (x+y)+x+y = 2(x+y).

det =	2(x+y) y x+y	
	2(x+y) x+y x	
	2(x+y) x y	

= 2(x+y) ·	1 y x+y	
	1 x+y x	
	1 x y	

Apply R2 → R2 − R1, R3 → R3 − R1:

= 2(x+y) ·	1 y x+y	
	0 x+y−y x−(x+y)	
	0 x−y y−(x+y)	

= 2(x+y) ·	1 y x+y	
	0 x −y	
	0 x−y −x	

= 2(x+y) · [1·(x·(−x) − (−y)·(x−y))]
= 2(x+y) · [−x² + y(x−y)]
= 2(x+y) · [−x² + xy − y²]
= −2(x+y)(x² − xy + y²)
= −2(x³ + y³).

Answer: −2(x³ + y³)

Q6 Evaluate:

|1 x y |
|1 x+y y |
|1 x x+y|

SOLUTION

Apply R2 → R2 − R1 and R3 → R3 − R1:

1	x	y
0	y	0
0	0	x

Expand along C1: det = 1·|y 0; 0 x| = 1·(xy − 0) = xy.

Therefore det = xy.

Answer: xy

Q7 Solve the system of equations:

2/x + 3/y + 10/z = 4

4/x − 6/y + 5/z = 1

6/x + 9/y − 20/z = 2

SOLUTION

Let u = 1/x, v = 1/y, w = 1/z. The system becomes:

2u + 3v + 10w = 4

4u − 6v + 5w = 1

6u + 9v − 20w = 2

A = [[2,3,10],[4,−6,5],[6,9,−20]], B = [[4],[1],[2]].

|A| = 2·(120−45) − 3·(−80−30) + 10·(36+36)

= 2·75 − 3·(−110) + 10·72

= 150 + 330 + 720 = 1200.

Cofactors:

A11=	−6 5;9 −20	=120−45=75, A12=−	4 5;6 −20	=−(−80−30)=110, A13=	4 −6;6 9	=36+36=72.
A21=−	3 10;9 −20	=−(−60−90)=150, A22=	2 10;6 −20	=−40−60=−100, A23=−	2 3;6 9	=−(18−18)=0.
A31=	3 10;−6 5	=15+60=75, A32=−	2 10;4 5	=−(10−40)=30, A33=	2 3;4 −6	=−12−12=−24.

adj A = [[75,150,75],[110,−100,30],[72,0,−24]].

X = A⁻¹·B = (1/1200)·[[75,150,75],[110,−100,30],[72,0,−24]]·[[4],[1],[2]]

= (1/1200)·[[300+150+150],[440−100+60],[288+0−48]]
= (1/1200)·[[600],[400],[240]] = [[1/2],[1/3],[1/5]].

u = 1/2 => x = 2; v = 1/3 => y = 3; w = 1/5 => z = 5.

Answer: x = 2, y = 3, z = 5

Q8 If x, y, z are nonzero real numbers, then the inverse of matrix
A = [[x, 0, 0], [0, y, 0], [0, 0, z]] is
(A) [[x⁻1, 0, 0], [0, y⁻1, 0], [0, 0, z⁻1]]
(B) xyz · [[x⁻1, 0, 0], [0, y⁻1, 0], [0, 0, z⁻1]]
(C) (1/xyz) · [[x, 0, 0], [0, y, 0], [0, 0, z]]
(D) (1/xyz) · [[1, 0, 0], [0, 1, 0], [0, 0, 1]]

SOLUTION

A is a diagonal matrix with entries x, y, z (all nonzero).
|A| = xyz.
adj A = [[yz, 0, 0], [0, xz, 0], [0, 0, xy]].
A⁻1 = (1/xyz)·[[yz,0,0],[0,xz,0],[0,0,xy]] = [[1/x,0,0],[0,1/y,0],[0,0,1/z]] = [[x⁻1,0,0],[0,y⁻1,0],[0,0,z⁻1]].
Answer: (A).

Answer: (A) [[x⁻1, 0, 0], [0, y⁻1, 0], [0, 0, z⁻1]]

Q9 Let A = [[1, sin θ, 1], [−sin θ, 1, sin θ], [−1, −sin θ, 1]], where 0 ≤ θ ≤ 2π. Then
(A) Det(A) = 0
(B) Det(A) ∈ (2, ∞)
(C) Det(A) ∈ (2, 4)
(D) Det(A) ∈ [2, 4]

SOLUTION

Expand along R1:
det A = 1·(1·1 − sin θ·(−sin θ)) − sin θ·(−sin θ·1 − sin θ·(−1)) + 1·(−sin θ·(−sin θ) − 1·(−1))
= 1·(1 + sin² θ) − sin θ·(−sin θ + sin θ) + 1·(sin² θ + 1)
= (1 + sin² θ) − sin θ·0 + (1 + sin² θ)
= 2(1 + sin² θ).

Since θ ∈ [0, 2π], sin² θ ∈ [0, 1].

Therefore det A = $2(1 + \sin^2 \theta) \in [2 \cdot 1, 2 \cdot 2] = [2, 4]$.

The minimum value 2 is attained when $\sin \theta = 0$ ($\theta = 0, \pi, 2\pi$);
the maximum value 4 is attained when $\sin \theta = \pm 1$ ($\theta = \pi/2, 3\pi/2$).
Both endpoints are achieved, so Det(A) $\in [2, 4]$.
Answer: (D).

Answer: (D) Det(A) $\in [2, 4]$