Chapter 5: Continuity and Differentiability

CONTENTS — 8 exercises · 137 questions

5.1	Exercise 5.1	34 Q
5.2	Exercise 5.2	10 Q
5.3	Exercise 5.3	15 Q
5.4	Exercise 5.4	10 Q
5.5	Exercise 5.5	18 Q
5.6	Exercise 5.6	11 Q
5.7	Exercise 5.7	17 Q
Misc	Miscellaneous Exercise	22 Q

Exercise 5.1 — Exercise 5.1

Q1 Prove that the function f(x) = 5x - 3 is continuous at x = 0, at x = -3 and at x = 5.

SOLUTION

f(x) = 5x - 3 is a polynomial, so it is continuous at every real number. We verify at the three given points.

At x = 0: f(0) = -3. lim_{x->0} (5x - 3) = -3 = f(0). Continuous at x = 0.

At x = -3: f(-3) = -18. lim_{x->-3} (5x - 3) = -18 = f(-3). Continuous at x = -3.

At x = 5: f(5) = 22. lim_{x->5} (5x - 3) = 22 = f(5). Continuous at x = 5.

Answer: f is continuous at x = 0, x = -3, and x = 5.

Q2 Examine the continuity of the function f(x) = $2x^2$ - 1 at x = 3.

SOLUTION

f(3) = 2(9) - 1 = 17.

lim_{x->3} ($2x^2$ - 1) = 2(9) - 1 = 17 = f(3).

Since the limit equals the function value, f is continuous at x = 3.

Answer: f is continuous at x = 3.

Q3 Examine the following functions for continuity.
(a) f(x) = x - 5
(b) f(x) = 1/(x - 5), x ≠ 5
(c) f(x) = (x² - 25)/(x + 5), x ≠ -5
(d) f(x) = |x - 5|

SOLUTION

(a) f(x) = x - 5 is a polynomial. For any real c, lim_{x->c} (x-5) = c-5 = f(c). Continuous for all x in R.

(b) f(x) = 1/(x-5) is defined for all x ≠ 5. For any c ≠ 5, lim_{x->c} 1/(x-5) = 1/(c-5) = f(c). Continuous for all x ≠ 5.

(c) f(x) = (x²-25)/(x+5) = (x-5)(x+5)/(x+5) = x - 5 for x ≠ -5. For any c ≠ -5, lim_{x->c} f(x) = c - 5 = f(c). Continuous for all x ≠ -5.

(d) f(x) = |x-5|. For c > 5: f(c) = c-5, lim = c-5. For c < 5: f(c) = 5-c, lim = 5-c. At c = 5: LHL = 0, RHL = 0, f(5) = 0. Continuous for all x in R.

Answer: (a) Continuous on R. (b) Continuous for all x ≠ 5. (c) Continuous for all x ≠ -5. (d) Continuous on R.

Q4 Prove that the function f(x) = x^n is continuous at x = n, where n is a positive integer.

SOLUTION

f(x) = x^n is a polynomial function, defined for all real x.
f(n) = n^n.
lim_{x->n} x^n = n^n = f(n).
Since the limit equals the function value, f is continuous at x = n.

Answer: f(x) = x^n is continuous at x = n.

Q5 Is the function f defined by
f(x) = x, if x <= 1
f(x) = 5, if x > 1
continuous at x = 0? At x = 1? At x = 2?

SOLUTION

At x = 0: f(0) = 0. lim_{x->0} f(x) = 0 = f(0). Continuous at x = 0.

At x = 1: f(1) = 1 (since 1 <= 1).
LHL: lim_{x->1⁻} x = 1.
RHL: lim_{x->1⁺} 5 = 5.
LHL ≠ RHL, so f is NOT continuous at x = 1.

At x = 2: f(2) = 5. lim_{x->2} f(x) = 5 = f(2). Continuous at x = 2.

Answer: Continuous at x = 0 and x = 2; NOT continuous at x = 1.

Q6 Find all points of discontinuity of f, where f is defined by
f(x) = 2x + 3, if x <= 2
f(x) = 2x - 3, if x > 2

SOLUTION

For x < 2 or x > 2, f is a polynomial in each piece, so it is continuous on those intervals.
Check at x = 2:
f(2) = 2(2) + 3 = 7.
LHL: lim_{x->2⁻} (2x+3) = 7.
RHL: lim_{x->2⁺} (2x-3) = 1.
LHL ≠ RHL, so f is discontinuous at x = 2.

Answer: f is discontinuous at x = 2. No other points of discontinuity.

Q7 Find all points of discontinuity of f, where f is defined by
f(x) = |x| + 3, if x <= -3
f(x) = -2x, if -3 < x < 3
f(x) = 6x + 2, if x >= 3

SOLUTION

Check at x = -3:

f(-3) =	-3	+3 = 6.
LHL: lim_{x->-3⁻} (	x	+3) = 3+3 = 6.

RHL: lim_{x->-3⁺} (-2x) = 6.
LHL = RHL = f(-3) = 6. Continuous at x = -3.

Check at x = 3:
f(3) = 6(3)+2 = 20.

LHL: $\lim_{x \to 3^-} (-2x) = -6$.

RHL: $\lim_{x \to 3^+} (6x+2) = 20$.

LHL ≠ RHL. Discontinuous at x = 3.

Answer: f is discontinuous at x = 3. No other points of discontinuity.

Q8 Find all points of discontinuity of f, where f is defined by

f(x) = |x|/x, if x ≠ 0

f(x) = 0, if x = 0

SOLUTION

For x > 0: |x|/x = x/x = 1. For x < 0: |x|/x = -x/x = -1.

At x = 0: f(0) = 0.

LHL: $\lim_{x \to 0^-}$	x	/x = -1.
RHL: $\lim_{x \to 0^+}$	x	/x = 1.

LHL ≠ RHL. Discontinuous at x = 0.

Answer: f is discontinuous at x = 0. No other points of discontinuity.

Q9 Find all points of discontinuity of f, where f is defined by

f(x) = x/|x|, if x < 0

f(x) = -1, if x >= 0

SOLUTION

For x < 0: |x| = -x, so x/|x| = x/(-x) = -1.

So f(x) = -1 for x < 0 as well.

Thus f(x) = -1 for all x in R.

Since f is a constant function, it is continuous everywhere.

Answer: f has no points of discontinuity. f is continuous for all x in R.

Q10 Find all points of discontinuity of f, where f is defined by
$f(x) = x + 1$, if $x >= 1$
$f(x) = x^2 + 1$, if $x < 1$

SOLUTION

For $x > 1$: f is polynomial, continuous. For $x < 1$: f is polynomial, continuous.
At $x = 1$: $f(1) = 1+1 = 2$.
LHL: $\lim_{x->1^-} (x^{2+}1) = 2$.
RHL: $\lim_{x->1^+} (x+1) = 2$.
LHL = RHL = f(1) = 2. Continuous at $x = 1$.

Answer: f has no points of discontinuity. f is continuous for all x in R.

Q11 Find all points of discontinuity of f, where f is defined by
$f(x) = x^3 - 3$, if $x <= 2$
$f(x) = x^2 + 1$, if $x > 2$

SOLUTION

At $x = 2$: $f(2) = 8-3 = 5$.
LHL: $\lim_{x->2^-} (x^{3-}3) = 5$.
RHL: $\lim_{x->2^+} (x^{2+}1) = 5$.
LHL = RHL = f(2) = 5. Continuous at $x = 2$.

Answer: f has no points of discontinuity. f is continuous for all x in R.

Q12 Find all points of discontinuity of f, where f is defined by
$f(x) = x^{10} - 1$, if $x <= 1$
$f(x) = x^2$, if $x > 1$

SOLUTION

At $x = 1$: $f(1) = 1-1 = 0$.
LHL: $\lim_{x->1^-} (x^{10-}1) = 0$.
RHL: $\lim_{x->1^+} x^2 = 1$.
LHL ≠ RHL. Discontinuous at $x = 1$.

Answer: f is discontinuous at x = 1. No other points of discontinuity.

Q13 Is the function defined by
f(x) = x + 5, if x <= 1
f(x) = x - 5, if x > 1
a continuous function?

SOLUTION

At x = 1: f(1) = 1+5 = 6.
LHL: $\lim_{x \to 1^-} (x+5) = 6$.
RHL: $\lim_{x \to 1^+} (x-5) = -4$.
LHL ≠ RHL. f is NOT continuous at x = 1.
For all other points, f is a polynomial in each piece and hence continuous.

Answer: f is NOT a continuous function; it is discontinuous at x = 1.

Q14 Discuss the continuity of the function f, where f is defined by
f(x) = 3, if 0 <= x <= 1
f(x) = 4, if 1 < x < 3
f(x) = 5, if 3 <= x <= 10

SOLUTION

On (0,1), (1,3), (3,10): f is constant, hence continuous in each open interval.
At x = 1: f(1) = 3. LHL = 3. RHL = 4. LHL ≠ RHL. Discontinuous at x = 1.
At x = 3: f(3) = 5. LHL = 4. RHL = 5. LHL ≠ RHL. Discontinuous at x = 3.

Answer: f is discontinuous at x = 1 and x = 3; continuous at all other points in [0, 10].

Q15 Discuss the continuity of the function f, where f is defined by
f(x) = 2x, if x < 0
f(x) = 0, if 0 <= x <= 1
f(x) = 4x, if x > 1

SOLUTION

At x = 0: f(0) = 0. LHL = $\lim_{x \to 0^-} 2x = 0$. RHL = 0. Continuous at x = 0.
At x = 1: f(1) = 0. LHL = 0. RHL = $\lim_{x \to 1^+} 4x = 4$. Discontinuous at x = 1.

Answer: f is discontinuous at x = 1; continuous at all other points.

Q16 Discuss the continuity of the function f, where f is defined by
f(x) = -2, if x <= -1
f(x) = 2x, if -1 < x <= 1
f(x) = 2, if x > 1

SOLUTION

At x = -1: f(-1) = -2. LHL = -2. RHL = lim_{x->-1⁺} 2x = -2. Continuous at x = -1.
At x = 1: f(1) = 2. LHL = lim_{x->1⁻} 2x = 2. RHL = 2. Continuous at x = 1.

Answer: f is continuous for all x in R. No points of discontinuity.

Q17 Find the relationship between a and b so that the function f defined by
f(x) = ax + 1, if x <= 3
f(x) = bx + 3, if x > 3
is continuous at x = 3.

SOLUTION

For continuity at x = 3: LHL = f(3) = RHL.
f(3) = 3a + 1.
LHL: lim_{x->3⁻} (ax+1) = 3a+1.
RHL: lim_{x->3⁺} (bx+3) = 3b+3.
Setting equal: 3a+1 = 3b+3 => 3a - 3b = 2 => a - b = 2/3.

Answer: a - b = 2/3.

Q18 For what value of λ is the function defined by
f(x) = λ(x² - 2x), if x <= 0
f(x) = 4x + 1, if x > 0
continuous at x = 0? What about continuity at x = 1?

SOLUTION

At x = 0: f(0) = λ(0-0) = 0.
LHL: lim_{x->0⁻} λ(x²⁻2x) = 0.
RHL: lim_{x->0⁺} (4x+1) = 1.
LHL = 0, RHL = 1. These can never be equal regardless of λ. So no value of λ makes f continuous at x = 0.

At x = 1: For x > 0, f(x) = 4x+1, which is a polynomial. f(1) = 5, lim_{x->1} (4x+1) = 5. Continuous at x = 1 for every value of λ.

Answer: No value of λ makes f continuous at x = 0. f is continuous at x = 1 for all values of λ.

Q19 Show that the function defined by g(x) = x - [x] is discontinuous at all integral points. Here [x] denotes the greatest integer less than or equal to x.

SOLUTION

Let n be any integer. Then g(x) = x - [x].
LHL: $\lim_{x \to n^-} (x - [x]) = n - (n-1) = 1$.
RHL: $\lim_{x \to n^+} (x - [x]) = n - n = 0$.
g(n) = n - [n] = n - n = 0.
Since LHL = 1 ≠ 0 = RHL = g(n), g is discontinuous at every integer n.

Answer: g is discontinuous at all integral points.

Q20 Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at x = π?

SOLUTION

$f(\pi) = \pi^2 - \sin \pi + 5 = \pi^2 - 0 + 5 = \pi^2 + 5$.
$\lim_{x \to \pi} (x^2 - \sin x + 5) = \pi^2 - \sin \pi + 5 = \pi^2 + 5 = f(\pi)$.
f is continuous at x = π.

Answer: Yes, f is continuous at x = π.

Q21 Discuss the continuity of the following functions:
(a) f(x) = sin x + cos x
(b) f(x) = sin x - cos x
(c) f(x) = sin x · cos x

SOLUTION

sin x and cos x are both continuous for all x in R.
(a) f(x) = sin x + cos x: sum of two continuous functions; continuous for all x in R.
(b) f(x) = sin x - cos x: difference of two continuous functions; continuous for all x in R.
(c) f(x) = sin x · cos x: product of two continuous functions; continuous for all x in R.

Answer: All three functions are continuous for all x in R.

Q22 Discuss the continuity of the cosine, cosecant, secant and cotangent functions.

SOLUTION

cos x: continuous for all x in R.

cosec x = 1/sin x: sin x is continuous and sin x = 0 at x = nπ (n integer). By quotient rule, cosec x is continuous for all x except x = nπ.

sec x = 1/cos x: cos x is continuous and cos x = 0 at x = (2n+1)π/2. sec x is continuous for all x except x = (2n+1)π/2.

cot x = cos x/sin x: continuous for all x except x = nπ (where sin x = 0).

Answer: cos x: continuous on R. cosec x: continuous on R - {nπ}. sec x: continuous on R - {(2n+1)π/2}. cot x: continuous on R - {nπ}.

Q23 Find all points of discontinuity of f, where
f(x) = sin x / x, if x < 0
f(x) = x + 1, if x >= 0

SOLUTION

For x < 0 and x ≠ 0: sin x/x is continuous (quotient of continuous functions, x ≠ 0).
For x > 0: x+1 is polynomial, continuous.
At x = 0: f(0) = 0+1 = 1.
LHL: $\lim_{x \to 0^-}$ sin x/x = 1.
RHL: $\lim_{x \to 0^+}$ (x+1) = 1.
LHL = RHL = f(0) = 1. Continuous at x = 0.

Answer: f has no points of discontinuity. f is continuous for all x in R.

Q24 Determine if f defined by
f(x) = x² sin(1/x), if x ≠ 0
f(x) = 0, if x = 0
is a continuous function?

SOLUTION

For x ≠ 0: f(x) = x² sin(1/x). Product of x² (continuous) and sin(1/x) (bounded by 1). f is continuous for x ≠ 0.
At x = 0: f(0) = 0.
|f(x) - f(0)| = |x² sin(1/x)| <= x² -> 0 as x -> 0.

So lim_{x->0} f(x) = 0 = f(0). Continuous at x = 0.

Answer: f is a continuous function for all x in R.

Q25 Examine the continuity of f, where f is defined by
f(x) = sin x - cos x, if x ≠ 0
f(x) = -1, if x = 0

SOLUTION

For x ≠ 0: sin x - cos x is continuous.
At x = 0: f(0) = -1.
lim_{x->0} (sin x - cos x) = sin 0 - cos 0 = 0 - 1 = -1 = f(0).
Continuous at x = 0.

Answer: f is continuous for all x in R.

Q26 Find the value of k so that the function f is continuous at x = π/2:
f(x) = k cos x / (π - 2x), if x ≠ π/2
f(x) = 3, if x = π/2

SOLUTION

For f to be continuous at x = π/2: lim_{x->π/2} k cos x/(π-2x) = 3.
Let h = x - π/2, so x = π/2 + h and h -> 0:
k cos(π/2+h) / (π - 2(π/2+h)) = k(-sin h)/(-2h) = k sin h/(2h) -> k/2 as h -> 0.
Setting k/2 = 3 gives k = 6.

Answer: k = 6.

Q27 Find the value of k so that the function f is continuous at x = 2:
f(x) = kx², if x <= 2
f(x) = 3, if x > 2

SOLUTION

f(2) = 4k.
LHL: lim_{x->2⁻} kx² = 4k.
RHL: lim_{x->2⁺} 3 = 3.
For continuity: 4k = 3, so k = 3/4.

Answer: k = 3/4.

Q28 Find the value of k so that the function f is continuous at x = π:
f(x) = kx + 1, if x <= π
f(x) = cos x, if x > π

SOLUTION

f(π) = kπ + 1.
LHL: lim_{x->π⁻} (kx+1) = kπ+1.
RHL: lim_{x->π⁺} cos x = cos π = -1.
For continuity: kπ+1 = -1 => kπ = -2 => k = -2/π.

Answer: k = -2/π.

Q29 Find the value of k so that the function f is continuous at x = 5:
f(x) = kx + 1, if x <= 5
f(x) = 3x - 5, if x > 5

SOLUTION

f(5) = 5k+1.
LHL: lim_{x->5⁻} (kx+1) = 5k+1.
RHL: lim_{x->5⁺} (3x-5) = 10.
For continuity: 5k+1 = 10 => 5k = 9 => k = 9/5.

Answer: k = 9/5.

Q30 Find the values of a and b such that the function defined by
f(x) = 5, if x <= 2
f(x) = ax + b, if 2 < x < 10
f(x) = 21, if x >= 10
is a continuous function.

SOLUTION

At x = 2: lim_{x->2⁺} (ax+b) = 2a+b must equal f(2) = 5. So 2a+b = 5. ... (i)
At x = 10: lim_{x->10⁻} (ax+b) = 10a+b must equal f(10) = 21. So 10a+b = 21. ... (ii)
Subtracting (i) from (ii): 8a = 16, a = 2.
From (i): b = 5 - 4 = 1.

Answer: a = 2, b = 1.

Q31 Show that the function defined by $f(x) = \cos(x^2)$ is a continuous function.

SOLUTION

Let $g(x) = x^2$ and $h(x) = \cos x$. Both are continuous functions on R.

$f(x) = \cos(x^2) = h(g(x)) = (h \circ g)(x)$.

By the theorem on continuity of composite functions, f is continuous on R.

Answer: $f(x) = \cos(x^2)$ is a continuous function on R.

Q32 Show that the function defined by $f(x) = |\cos x|$ is a continuous function.

SOLUTION

Let g(x) = cos x and h(x) =	x	. Both are continuous on R.
f(x) =	cos x	= h(g(x)) = (h ∘ g)(x).

By the theorem on composite functions, f is continuous on R.

Answer: $f(x) = |\cos x|$ is a continuous function on R.

Q33 Examine that sin |x| is a continuous function.

SOLUTION

Let g(x) =	x	and h(x) = sin x. Both are continuous on R.
sin	x	= h(g(x)) = (h ∘ g)(x).
By the theorem on composite functions, sin	x	is continuous on R.

Answer: sin|x| is a continuous function on R.

Q34 Find all the points of discontinuity of f defined by $f(x) = |x| - |x + 1|$.

SOLUTION

The possible critical points are x = 0 and x = -1 (where the expressions inside the absolute values change sign).

For x < -1:	x	= -x,	x+1	= -(x+1). f(x) = -x - (-(x+1)) = 1.
For -1 <= x < 0:	x	= -x,	x+1	= x+1. f(x) = -x - (x+1) = -2x-1.
For x >= 0:	x	= x,	x+1	= x+1. f(x) = x - (x+1) = -1.

At x = -1: LHL = 1, f(-1) = 1, RHL = 1. Continuous.
At x = 0: LHL = -1, f(0) = -1, RHL = -1. Continuous.
f is continuous for all x in R.

Answer: f has no points of discontinuity. f is continuous for all x in R.

Exercise 5.2 — Exercise 5.2

Q1 Differentiate $\sin(x^2 + 5)$ with respect to x.

SOLUTION

Let $u = x^2 + 5$. Then $d/dx[\sin(x^2+5)] = \cos(x^2+5) \cdot du/dx = \cos(x^2+5) \cdot 2x$.

Answer: $2x \cos(x^2 + 5)$.

Q2 Differentiate $\cos(\sin x)$ with respect to x.

SOLUTION

$d/dx[\cos(\sin x)] = -\sin(\sin x) \cdot d/dx(\sin x) = -\sin(\sin x) \cdot \cos x$.

Answer: $-\cos x \cdot \sin(\sin x)$.

Q3 Differentiate $\sin(ax + b)$ with respect to x.

SOLUTION

$d/dx[\sin(ax+b)] = \cos(ax+b) \cdot a = a \cos(ax+b)$.

Answer: $a \cos(ax + b)$.

Q4 Differentiate $\sec(\tan(\sqrt{x}))$ with respect to x.

SOLUTION

Let $u = \sqrt{x}$, $v = \tan u$, $f = \sec v$.

$d/dx[\sec(\tan(\sqrt{x}))] = \sec(\tan(\sqrt{x})) \cdot \tan(\tan(\sqrt{x})) \cdot d/dx[\tan(\sqrt{x})]$

$= \sec(\tan(\sqrt{x})) \cdot \tan(\tan(\sqrt{x})) \cdot \sec^2(\sqrt{x}) \cdot d/dx(\sqrt{x})$

$= \sec(\tan(\sqrt{x})) \cdot \tan(\tan(\sqrt{x})) \cdot \sec^2(\sqrt{x}) \cdot 1/(2\sqrt{x})$.

Answer: $\sec(\tan \sqrt{x}) \cdot \tan(\tan \sqrt{x}) \cdot \sec^2(\sqrt{x}) / (2\sqrt{x})$.

Q5 Differentiate $\sin(ax + b) / \cos(cx + d)$ with respect to x.

SOLUTION

Let $u = \sin(ax+b)$, $v = \cos(cx+d)$. By the quotient rule:

$d/dx[u/v] = (u'v - uv') / v^2$

$u' = a \cos(ax+b)$, $v' = -c \sin(cx+d)$.

$= [a \cos(ax+b)\cdot\cos(cx+d) + c \sin(ax+b)\cdot\sin(cx+d)] / \cos^2(cx+d)$

= a cos(ax+b)/cos(cx+d) + c sin(ax+b) tan(cx+d)/cos(cx+d).

Answer: [a cos(ax+b)·cos(cx+d) + c sin(ax+b)·sin(cx+d)] / cos²(cx+d).

Q6 Differentiate cos(x³) · sin²(x⁵) with respect to x.

SOLUTION

By the product rule: d/dx[cos(x³) · sin²(x⁵)]
= [-sin(x³) · 3x²] · sin²(x⁵) + cos(x³) · 2 sin(x⁵) · cos(x⁵) · 5x⁴
= -3x² sin(x³) sin²(x⁵) + 10x⁴ cos(x³) sin(x⁵) cos(x⁵).

Answer: -3x² sin(x³) sin²(x⁵) + 10x⁴ cos(x³) sin(x⁵) cos(x⁵).

Q7 Differentiate 2 cot(x²) with respect to x.

SOLUTION

d/dx[2 cot(x²)] = 2 · (-cosec²(x²)) · 2x = -4x cosec²(x²).

Answer: -4x cosec²(x²).

Q8 Differentiate cos(√x) with respect to x.

SOLUTION

d/dx[cos(√x)] = -sin(√x) · d/dx(√x) = -sin(√x) · 1/(2√x) = -sin(√x)/(2√x).

Answer: -sin(√x) / (2√x).

Q9 Prove that the function f given by f(x) = |x - 1|, x in R, is not differentiable at x = 1.

SOLUTION

The left-hand derivative at x = 1:
lim_{h->0⁻} [f(1+h) - f(1)] / h = lim_{h->0⁻} [|h| / h] = lim_{h->0⁻} (-h/h) = -1.

The right-hand derivative at x = 1:
lim_{h->0⁺} [f(1+h) - f(1)] / h = lim_{h->0⁺} [h / h] = 1.

Since LHD = -1 ≠ 1 = RHD, f is not differentiable at x = 1.

Answer: f(x) = |x - 1| is not differentiable at x = 1 (LHD = -1, RHD = 1).

Q10 Prove that the greatest integer function defined by f(x) = [x], 0 < x < 3, is not differentiable at x = 1 and x = 2.

SOLUTION

At x = 1: LHD = $\lim_{h\to 0^-}$ ([1+h] - [1])/h = (0-1)/h ... For h in (-1,0), [1+h]=0, [1]=1.

LHD = $\lim_{h\to 0^-}$ (0-1)/h = $\lim_{h\to 0^-}$ (-1/h) = +∞ (does not exist as a finite value).

RHD = $\lim_{h\to 0^+}$ ([1+h]-1)/h = 0/h = 0.

LHD ≠ RHD, so f is not differentiable at x = 1.

At x = 2: LHD = $\lim_{h\to 0^-}$ ([2+h]-2)/h. For h in (-1,0), [2+h]=1, so = (1-2)/h = -1/h -> +∞.

RHD = $\lim_{h\to 0^+}$ ([2+h]-2)/h = 0/h = 0.

LHD ≠ RHD, so f is not differentiable at x = 2.

Answer: f(x) = [x] is not differentiable at x = 1 and x = 2.

Exercise 5.3 — Exercise 5.3

Q1 Find dy/dx if 2x + 3y = sin x.

SOLUTION

Differentiating both sides w.r.t. x:
2 + 3 dy/dx = cos x.
dy/dx = (cos x - 2) / 3.

Answer: dy/dx = (cos x - 2) / 3.

Q2 Find dy/dx if 2x + 3y = sin y.

SOLUTION

Differentiating both sides w.r.t. x:
2 + 3 dy/dx = cos y · dy/dx.
dy/dx (cos y - 3) = 2.
dy/dx = 2 / (cos y - 3).

Answer: dy/dx = 2 / (cos y - 3).

Q3 Find dy/dx if $ax + by^2 = \cos y$.

SOLUTION

Differentiating both sides w.r.t. x:
a + 2by dy/dx = -sin y · dy/dx.
dy/dx (2by + sin y) = -a.
dy/dx = -a / (2by + sin y).

Answer: dy/dx = -a / (2by + sin y).

Q4 Find dy/dx if $xy + y^2 = \tan x + y$.

SOLUTION

Differentiating w.r.t. x:
$y + x\,dy/dx + 2y\,dy/dx = \sec^2 x + dy/dx$.
dy/dx (x + 2y - 1) = $\sec^2 x$ - y.
dy/dx = ($\sec^2 x$ - y) / (x + 2y - 1).

Answer: dy/dx = ($\sec^2 x$ - y) / (x + 2y - 1).

Q5 Find dy/dx if $x^2 + xy + y^2 = 100$.

SOLUTION

Differentiating w.r.t. x:

$2x + y + x\,dy/dx + 2y\,dy/dx = 0$.

$dy/dx\,(x + 2y) = -(2x + y)$.

$dy/dx = -(2x + y) / (x + 2y)$.

Answer: $dy/dx = -(2x + y) / (x + 2y)$.

Q6 Find dy/dx if $x^3 + x^2y + xy^2 + y^3 = 81$.

SOLUTION

Differentiating w.r.t. x:

$3x^2 + 2xy + x^2\,dy/dx + y^2 + 2xy\,dy/dx + 3y^2\,dy/dx = 0$.

$dy/dx\,(x^2 + 2xy + 3y^2) = -(3x^2 + 2xy + y^2)$.

$dy/dx = -(3x^2 + 2xy + y^2) / (x^2 + 2xy + 3y^2)$.

Answer: $dy/dx = -(3x^2 + 2xy + y^2) / (x^2 + 2xy + 3y^2)$.

Q7 Find dy/dx if $\sin^2 y + \cos(xy) = \kappa$.

SOLUTION

Differentiating w.r.t. x:

$2 \sin y \cos y \cdot dy/dx - \sin(xy) \cdot (y + x\,dy/dx) = 0$.

$\sin(2y)\,dy/dx - y \sin(xy) - x \sin(xy)\,dy/dx = 0$.

$dy/dx\,[\sin(2y) - x \sin(xy)] = y \sin(xy)$.

$dy/dx = y \sin(xy) / (\sin(2y) - x \sin(xy))$.

Answer: $dy/dx = y \sin(xy) / (\sin 2y - x \sin xy)$.

Q8 Find dy/dx if $\sin^2 x + \cos^2 y = 1$.

SOLUTION

Differentiating w.r.t. x:

$2 \sin x \cos x - 2 \cos y \sin y \cdot dy/dx = 0$.

$\sin(2x) = \sin(2y)\,dy/dx$.

$dy/dx = \sin(2x) / \sin(2y)$.

Answer: dy/dx = sin 2x / sin 2y.

Q9 Find dy/dx if y = sin^{-1}(2x / (1 + x²)).

SOLUTION

Put x = tan θ, so θ = tan^{-1} x.
2x/(1+x²) = 2 tan θ/(1+tan² θ) = sin 2θ.
y = sin^{-1}(sin 2θ) = 2θ = 2 tan^{-1} x.
dy/dx = 2 · 1/(1+x²) = 2/(1+x²).

Answer: dy/dx = 2 / (1 + x²).

Q10 Find dy/dx if y = tan^{-1}((3x - x³) / (1 - 3x²)), -1/√3 < x < 1/√3.

SOLUTION

Put x = tan θ, so θ = tan^{-1} x.
(3 tan θ - tan³ θ)/(1 - 3 tan² θ) = tan 3θ.
y = tan^{-1}(tan 3θ) = 3θ = 3 tan^{-1} x.
dy/dx = 3/(1+x²).

Answer: dy/dx = 3 / (1 + x²).

Q11 Find dy/dx if y = cos^{-1}((1 - x²) / (1 + x²)), 0 < x < 1.

SOLUTION

Put x = tan θ, so θ = tan^{-1} x.
(1 - tan² θ)/(1 + tan² θ) = cos 2θ.
y = cos^{-1}(cos 2θ) = 2θ = 2 tan^{-1} x.
dy/dx = 2/(1+x²).

Answer: dy/dx = 2 / (1 + x²).

Q12 Find dy/dx if y = sin^{-1}((1 - x²) / (1 + x²)), 0 < x < 1.

SOLUTION

Put x = tan θ, so θ = tan^{-1} x.
(1 - tan² θ)/(1 + tan² θ) = cos 2θ.
y = sin^{-1}(cos 2θ) = sin^{-1}(sin(π/2 - 2θ)) = π/2 - 2θ = π/2 - 2 tan^{-1} x.
dy/dx = -2/(1+x²).

Answer: dy/dx = -2 / (1 + x²).

Q13 Find dy/dx if y = cos^{-1}(2x / (1 + x²)), -1 < x < 1.

SOLUTION

Put x = tan θ, so θ = tan^{-1} x.
2 tan θ/(1 + tan² θ) = sin 2θ.
y = cos^{-1}(sin 2θ) = cos^{-1}(cos(π/2 - 2θ)) = π/2 - 2θ = π/2 - 2 tan^{-1} x.
dy/dx = -2/(1+x²).

Answer: dy/dx = -2 / (1 + x²).

Q14 Find dy/dx if y = sin^{-1}(2x√(1 - x²)), -1/√2 < x < 1/√2.

SOLUTION

Put x = sin θ, so θ = sin^{-1} x.
2x√(1-x²) = 2 sin θ cos θ = sin 2θ.
y = sin^{-1}(sin 2θ) = 2θ = 2 sin^{-1} x.
dy/dx = 2/√(1-x²).

Answer: dy/dx = 2 / √(1 - x²).

Q15 Find dy/dx if y = sec^{-1}(1 / (2x² - 1)), 0 < x < 1/√2.

SOLUTION

Put x = cos θ, so θ = cos^{-1} x (for 0 < x < 1/√2, we have π/4 < θ < π/2).
2x² - 1 = 2cos² θ - 1 = cos 2θ.
1/(2x²⁻1) = sec 2θ.
y = sec^{-1}(sec 2θ) = 2θ = 2 cos^{-1} x.
dy/dx = -2/√(1-x²).

Answer: dy/dx = -2 / √(1 - x²).

Exercise 5.4 — Exercise 5.4

Q1 Differentiate e^x / sin x with respect to x.

SOLUTION

By the quotient rule: d/dx[e^x/sin x] = (e^x sin x - e^x cos x) / sin² x
= e^x(sin x - cos x) / sin² x.

Answer: e^x(sin x - cos x) / sin² x.

Q2 Differentiate e^{sin^{-1} x} with respect to x.

SOLUTION

Let y = e^{sin^{-1} x}. Using chain rule:
dy/dx = e^{sin^{-1} x} · d/dx(sin^{-1} x) = e^{sin^{-1} x} / √(1-x²).

Answer: e^{sin^{-1} x} / √(1 - x²).

Q3 Differentiate e^{x³} with respect to x.

SOLUTION

d/dx[e^{x³}] = e^{x³} · 3x² = 3x² e^{x³}.

Answer: 3x² e^{x³}.

Q4 Differentiate sin(tan^{-1} e^{-x}) with respect to x.

SOLUTION

d/dx[sin(tan^{-1} e^{-x})] = cos(tan^{-1} e^{-x}) · d/dx[tan^{-1} e^{-x}]
= cos(tan^{-1} e^{-x}) · 1/(1+e^{-2x}) · (-e^{-x})
= -e^{-x} cos(tan^{-1} e^{-x}) / (1+e^{-2x}).

Answer: -e^{-x} cos(tan^{-1} e^{-x}) / (1 + e^{-2x}).

Q5 Differentiate log(cos e^x) with respect to x.

SOLUTION

d/dx[log(cos e^x)] = 1/cos(e^x) · (-sin(e^x)) · e^x = -e^x tan(e^x).

Answer: -e^x tan(e^x).

Q6 Differentiate $e^x + e^{x^2} + e^{x^3} + e^{x^4} + e^{x^5}$ with respect to x.

SOLUTION

$\frac{d}{dx}[e^x + e^{x^2} + e^{x^3} + e^{x^4} + e^{x^5}]$
$= e^x + 2x\, e^{x^2} + 3x^2\, e^{x^3} + 4x^3\, e^{x^4} + 5x^4\, e^{x^5}$.

Answer: $e^x + 2x\, e^{x^2} + 3x^2\, e^{x^3} + 4x^3\, e^{x^4} + 5x^4\, e^{x^5}$.

Q7 Differentiate $\sqrt{(e^x)}$ for $x > 0$ with respect to x.

SOLUTION

$\sqrt{(e^x)} = e^{x/2}$.
$\frac{d}{dx}[e^{x/2}] = (1/2)e^{x/2} = \sqrt{(e^x)}/2$.

Answer: $\sqrt{(e^x)}/2$.

Q8 Differentiate $\log(\log x)$, $x > 1$, with respect to x.

SOLUTION

$\frac{d}{dx}[\log(\log x)] = 1/\log x \cdot \frac{d}{dx}(\log x) = 1/(x \log x)$.

Answer: $1 / (x \log x)$.

Q9 Differentiate $\cos x / \log x$, $x > 0$, with respect to x.

SOLUTION

By quotient rule: $\frac{d}{dx}[\cos x / \log x]$
$= (-\sin x \cdot \log x - \cos x \cdot (1/x)) / (\log x)^2$
$= -(x \sin x \log x + \cos x) / (x (\log x)^2)$.

Answer: $-(x \sin x \log x + \cos x) / (x(\log x)^2)$.

Q10 Differentiate $\cos(\log x + e^x)$, $x > 0$, with respect to x.

SOLUTION

$\frac{d}{dx}[\cos(\log x + e^x)] = -\sin(\log x + e^x) \cdot \frac{d}{dx}(\log x + e^x)$
$= -\sin(\log x + e^x) \cdot (1/x + e^x)$.

Answer: $-(1/x + e^x) \sin(\log x + e^x)$.

Exercise 5.5 — Exercise 5.5

Q1 Differentiate cos x · cos 2x · cos 3x with respect to x.

SOLUTION

Let y = cos x · cos 2x · cos 3x. Take log: log y = log cos x + log cos 2x + log cos 3x.
Differentiate: (1/y) dy/dx = -tan x - 2 tan 2x - 3 tan 3x.
dy/dx = cos x · cos 2x · cos 3x · (-tan x - 2 tan 2x - 3 tan 3x).
= -(sin x cos 2x cos 3x + 2 cos x sin 2x cos 3x + 3 cos x cos 2x sin 3x).

Answer: -[sin x cos 2x cos 3x + 2 cos x sin 2x cos 3x + 3 cos x cos 2x sin 3x].

Q2 Differentiate √[(x-1)(x-2) / ((x-3)(x-4)(x-5))] with respect to x.

SOLUTION

Let y = √[(x-1)(x-2)/((x-3)(x-4)(x-5))].
log y = (1/2)[log(x-1)+log(x-2)-log(x-3)-log(x-4)-log(x-5)].
(1/y) dy/dx = (1/2)[1/(x-1)+1/(x-2)-1/(x-3)-1/(x-4)-1/(x-5)].
dy/dx = (y/2)[1/(x-1)+1/(x-2)-1/(x-3)-1/(x-4)-1/(x-5)]
= (1/2)√[(x-1)(x-2)/((x-3)(x-4)(x-5))] · [1/(x-1)+1/(x-2)-1/(x-3)-1/(x-4)-1/(x-5)].

Answer: (1/2)√[(x-1)(x-2)/((x-3)(x-4)(x-5))] · [1/(x-1) + 1/(x-2) - 1/(x-3) - 1/(x-4) - 1/(x-5)].

Q3 Differentiate (log x)^{cos x} with respect to x.

SOLUTION

Let y = (log x)^{cos x}. Take log: log y = cos x · log(log x).
(1/y) dy/dx = -sin x · log(log x) + cos x · 1/(x log x).
dy/dx = (log x)^{cos x} [cos x/(x log x) - sin x log(log x)].

Answer: (log x)^{cos x} [cos x/(x log x) - sin x log(log x)].

Q4 Differentiate x^x - 2^{sin x} with respect to x.

SOLUTION

Let u = x^x and v = 2^{sin x}.
For u: log u = x log x => (1/u) du/dx = log x + 1 => du/dx = x^x(1 + log x).

For v: dv/dx = 2^{sin x} · log 2 · cos x.
dy/dx = x^x(1 + log x) - 2^{sin x} · cos x · log 2.

Answer: x^x(1 + log x) - 2^{sin x} cos x log 2.

Q5 Differentiate $(x+3)^2 \cdot (x+4)^3 \cdot (x+5)^4$ with respect to x.

SOLUTION

Let $y = (x+3)^2(x+4)^3(x+5)^4$. Take log:
$\log y = 2\log(x+3) + 3\log(x+4) + 4\log(x+5)$.
$(1/y)\, dy/dx = 2/(x+3) + 3/(x+4) + 4/(x+5)$.
$dy/dx = (x+3)^2(x+4)^3(x+5)^4 \cdot [2/(x+3) + 3/(x+4) + 4/(x+5)]$.
$= (x+3)(x+4)^2(x+5)^3 \cdot [2(x+4)(x+5) + 3(x+3)(x+5) + 4(x+3)(x+4)]$.
$= (x+3)(x+4)^2(x+5)^3(9x^2 + 70x + 133)$.

Answer: $(x+3)(x+4)^2(x+5)^3(9x^2 + 70x + 133)$.

Q6 Differentiate (x + 1/x)^x + x^{1+1/x} with respect to x.

SOLUTION

Let u = (x+1/x)^x and v = x^{1+1/x}.

For u: log u = x log(x+1/x) = x log((x²+1)/x).
(1/u) du/dx = log((x²+1)/x) + x · (2x/(x²+1) - 1/x)
= log(x+1/x) + (2x²/(x²+1)) - 1 = log(x+1/x) + (x²−1)/(x²+1).
du/dx = (x+1/x)^x [log(x+1/x) + (x²−1)/(x²+1)].

For v: log v = (1+1/x) log x.
(1/v) dv/dx = (-1/x²) log x + (1+1/x)/x = (x+1-log x)/x².
dv/dx = x^{1+1/x} · (x+1-log x)/x².

dy/dx = (x+1/x)^x [log(x+1/x) + (x²−1)/(x²+1)] + x^{1+1/x}(x+1-log x)/x².

Answer: (x+1/x)^x [log(x+1/x) + (x²−1)/(x²+1)] + x^{1+1/x}(x+1-log x)/x².

Q7 Differentiate (log x)^x + x^{log x} with respect to x.

SOLUTION

Let u = (log x)^x and v = x^{log x}.

For u: log u = x log(log x).

(1/u) du/dx = log(log x) + x · 1/(x log x) = log(log x) + 1/log x.
du/dx = (log x)^x [log(log x) + 1/log x].

For v: log v = (log x)².
(1/v) dv/dx = 2 log x / x.
dv/dx = x^{log x} · 2 log x / x.

dy/dx = (log x)^x [log(log x) + 1/log x] + x^{log x} · 2 log x/x.

Answer: (log x)^x [log(log x) + 1/log x] + 2 x^{log x} log x / x.

Q8 Differentiate (sin x)^x + sin^{-1}(√x) with respect to x.

SOLUTION

Let u = (sin x)^x and v = sin^{-1}(√x).

For u: log u = x log(sin x).
(1/u) du/dx = log(sin x) + x · cos x/sin x = log(sin x) + x cot x.
du/dx = (sin x)^x [x cot x + log(sin x)].

For v: dv/dx = 1/√(1-x) · 1/(2√x) = 1/(2√(x(1-x))).

dy/dx = (sin x)^x [x cot x + log(sin x)] + 1/(2√(x(1-x))).

Answer: (sin x)^x [x cot x + log(sin x)] + 1/(2√(x-x²)).

Q9 Differentiate x^{sin x} + (sin x)^{cos x} with respect to x.

SOLUTION

Let u = x^{sin x} and v = (sin x)^{cos x}.

For u: log u = sin x · log x.
(1/u) du/dx = cos x log x + sin x/x.
du/dx = x^{sin x} [cos x log x + sin x/x].

For v: log v = cos x · log(sin x).
(1/v) dv/dx = -sin x log(sin x) + cos x · (cos x/sin x).
= -sin x log(sin x) + cos x cot x.
dv/dx = (sin x)^{cos x} [cos x cot x - sin x log(sin x)].

dy/dx = x^{sin x}[cos x log x + sin x/x] + (sin x)^{cos x}[cos x cot x - sin x log(sin x)].

Answer: x^{sin x}[cos x log x + sin x/x] + (sin x)^{cos x}[cos x cot x - sin x log(sin x)].

Q10 Differentiate x^x cos x + (x²+1)/(x²−1) with respect to x.

SOLUTION

Let u = x^x cos x and v = (x²+1)/(x²−1).

For u: write x^x cos x = w · cos x where w = x^x.
dw/dx = x^x(1+log x).
du/dx = x^x(1+log x) cos x - x^x sin x = x^x[(1+log x) cos x - sin x].

For v: dv/dx = [2x(x²−1) - (x²+1)·2x]/(x²−1)² = -4x/(x²−1)².

dy/dx = x^x[(1+log x) cos x - sin x] - 4x/(x²−1)².

Answer: x^x[(1+log x) cos x - sin x] - 4x/(x²−1)².

Q11 Differentiate (x cos x)^x + (x sin x)^{1/x} with respect to x.

SOLUTION

Let u = (x cos x)^x and v = (x sin x)^{1/x}.

For u: log u = x log(x cos x) = x[log x + log cos x].
(1/u) du/dx = log(x cos x) + x[1/x - tan x] = log(x cos x) + 1 - x tan x.
du/dx = (x cos x)^x [1 + log(x cos x) - x tan x].

For v: log v = (1/x) log(x sin x) = (log x + log sin x)/x.
(1/v) dv/dx = [x(1/x + cot x) - (log x + log sin x)] / x²
= (1 + x cot x - log(x sin x)) / x².
dv/dx = (x sin x)^{1/x} · (1 + x cot x - log(x sin x)) / x².

dy/dx = (x cos x)^x[1 + log(x cos x) - x tan x] + (x sin x)^{1/x}(1 + x cot x - log(x sin x))/x².

Answer: (x cos x)^x[1 + log(x cos x) - x tan x] + (x sin x)^{1/x}[1 + x cot x - log(x sin x)]/x².

Q12 Find dy/dx if x^y + y^x = 1.

SOLUTION

Let u = x^y and v = y^x, so u + v = 1 => du/dx + dv/dx = 0.

For u: log u = y log x => (1/u) du/dx = (dy/dx) log x + y/x.
du/dx = x^y [y/x + log x · dy/dx].

For v: log v = x log y => (1/v) dv/dx = log y + (x/y) dy/dx.

dv/dx = y^x [log y + (x/y) dy/dx].

x^y[y/x + log x · dy/dx] + y^x[log y + (x/y) dy/dx] = 0.
dy/dx [x^y log x + x y^{x-1}] = -(y x^{y-1} + y^x log y).
dy/dx = -(y x^{y-1} + y^x log y) / (x^y log x + x y^{x-1}).

Answer: dy/dx = -(y x^{y-1} + y^x log y) / (x^y log x + x y^{x-1}).

Q13 Find dy/dx if y^x = x^y.

SOLUTION

Take log of both sides: x log y = y log x.
Differentiate w.r.t. x: log y + (x/y) dy/dx = (dy/dx) log x + y/x.
dy/dx (x/y - log x) = y/x - log y.
dy/dx = (y/x - log y)/(x/y - log x) = y(y - x log y) / (x(x - y log x)).

Answer: dy/dx = y(y - x log y) / (x(x - y log x)).

Q14 Find dy/dx if (cos x)^y = (cos y)^x.

SOLUTION

Take log: y log cos x = x log cos y.
Differentiate: dy/dx · log cos x + y(-sin x/cos x) = log cos y + x(-sin y/cos y) dy/dx.
dy/dx [log cos x + x tan y] = log cos y + y tan x.
dy/dx = (log cos y + y tan x) / (log cos x + x tan y).

Answer: dy/dx = (log cos y + y tan x) / (log cos x + x tan y).

Q15 Find dy/dx if x^y = e^{x-y}.

SOLUTION

Take log: y log x = x - y.
Differentiate: dy/dx · log x + y/x = 1 - dy/dx.
dy/dx (log x + 1) = 1 - y/x = (x-y)/x.
dy/dx = (x-y) / (x(1 + log x)).

Answer: dy/dx = (x - y) / (x(1 + log x)).

Q16 Find the derivative of f(x) = (1+x)(1+x²)(1+x⁴)(1+x⁸) and hence find f'(1).

SOLUTION

Take log: log f = log(1+x)+log(1+x²)+log(1+x⁴)+log(1+x⁸).
(1/f) f' = 1/(1+x) + 2x/(1+x²) + 4x³/(1+x⁴) + 8x⁷/(1+x⁸).
f'(x) = f(x)[1/(1+x) + 2x/(1+x²) + 4x³/(1+x⁴) + 8x⁷/(1+x⁸)].

At x=1: f(1) = 2·2·2·2 = 16.
Bracket = 1/(1+1) + 2·1/(1+1) + 4·1/(1+1) + 8·1/(1+1) = 1/2 + 1 + 2 + 4 = 15/2.
f'(1) = 16 · 15/2 = 120.

Answer: f'(x) = f(x)[1/(1+x) + 2x/(1+x²) + 4x³/(1+x⁴) + 8x⁷/(1+x⁸)]; f'(1) = 120.

Q17 Differentiate (x² - 5x + 8)(x³ + 7x + 9) in three ways: (i) by product rule, (ii) by expanding, (iii) by logarithmic differentiation. Do they all give the same answer?

SOLUTION

Let u = x²−5x+8, v = x³+7x+9.

(i) Product rule: dy/dx = u'v + uv'
= (2x-5)(x³+7x+9) + (x²−5x+8)(3x²+7).
(2x-5)(x³+7x+9) = 2x⁴−5x³+14x²−17x-45.
(x²−5x+8)(3x²+7) = 3x⁴−15x³+31x²−35x+56.
Sum = 5x⁴−20x³+45x²−52x+11.

(ii) Expanding: y = (x²−5x+8)(x³+7x+9) = x⁵+7x³+9x²−5x⁴−35x²−45x+8x³+56x+72
= x⁵−5x⁴+15x³−26x²+11x+72.
dy/dx = 5x⁴−20x³+45x²−52x+11.

(iii) Log diff gives the same result. Yes, all three methods give the same answer.

Answer: dy/dx = 5x⁴ - 20x³ + 45x² - 52x + 11. All three methods give the same answer.

Q18 If u, v and w are functions of x, show that d(uvw)/dx = (du/dx)vw + u(dv/dx)w + uv(dw/dx) in two ways: first by repeated application of product rule, second by logarithmic differentiation.

SOLUTION

Method 1 (product rule):

d(uvw)/dx = d(u·(vw))/dx = (du/dx)(vw) + u · d(vw)/dx

= (du/dx)vw + u[(dv/dx)w + v(dw/dx)]

= (du/dx)vw + u(dv/dx)w + uv(dw/dx). ✓

Method 2 (logarithmic differentiation):

Let y = uvw. Then log y = log u + log v + log w.

(1/y) dy/dx = (1/u)(du/dx) + (1/v)(dv/dx) + (1/w)(dw/dx).

dy/dx = uvw[(1/u)(du/dx) + (1/v)(dv/dx) + (1/w)(dw/dx)]

= (du/dx)vw + u(dv/dx)w + uv(dw/dx). ✓

Answer: d(uvw)/dx = (du/dx)vw + u(dv/dx)w + uv(dw/dx). Both methods give the same result.

Exercise 5.6 — Exercise 5.6

Q1 If x and y are connected parametrically by $x = 2at^2$, $y = at^4$, find dy/dx without eliminating the parameter.

SOLUTION

$dx/dt = 4at$, $dy/dt = 4at^3$.

$dy/dx = (dy/dt)/(dx/dt) = 4at^3/(4at) = t^2$.

Answer: $dy/dx = t^2$.

Q2 If x and y are connected parametrically by $x = a \cos \theta$, $y = b \cos \theta$, find dy/dx without eliminating the parameter.

SOLUTION

$dx/d\theta = -a \sin \theta$, $dy/d\theta = -b \sin \theta$.

$dy/dx = (-b \sin \theta)/(-a \sin \theta) = b/a$.

Answer: $dy/dx = b/a$.

Q3 If x and y are connected parametrically by $x = \sin t$, $y = \cos 2t$, find dy/dx without eliminating the parameter.

SOLUTION

$dx/dt = \cos t$, $dy/dt = -2 \sin 2t = -4 \sin t \cos t$.

$dy/dx = -4 \sin t \cos t / \cos t = -4 \sin t$.

Answer: $dy/dx = -4 \sin t$.

Q4 If x and y are connected parametrically by $x = 4t$, $y = 4/t$, find dy/dx without eliminating the parameter.

SOLUTION

$dx/dt = 4$, $dy/dt = -4/t^2$.

$dy/dx = (-4/t^2)/4 = -1/t^2$.

Answer: $dy/dx = -1/t^2$.

Q5 If x and y are connected parametrically by x = cos θ - cos 2θ, y = sin θ - sin 2θ, find dy/dx without eliminating the parameter.

SOLUTION

dx/dθ = -sin θ + 2 sin 2θ.
dy/dθ = cos θ - 2 cos 2θ.
dy/dx = (cos θ - 2 cos 2θ)/(2 sin 2θ - sin θ).

Answer: dy/dx = (cos θ - 2 cos 2θ) / (2 sin 2θ - sin θ).

Q6 If x and y are connected parametrically by x = a(θ - sin θ), y = a(1 + cos θ), find dy/dx without eliminating the parameter.

SOLUTION

dx/dθ = a(1 - cos θ), dy/dθ = -a sin θ.
dy/dx = -a sin θ / (a(1-cos θ)) = -sin θ/(1-cos θ).
Using half-angle: -2 sin(θ/2)cos(θ/2)/(2 sin²(θ/2)) = -cos(θ/2)/sin(θ/2) = -cot(θ/2).

Answer: dy/dx = -cot(θ/2).

Q7 If x and y are connected parametrically by x = sin³ t / √(cos 2t), y = cos³ t / √(cos 2t), find dy/dx without eliminating the parameter.

SOLUTION

x = sin³ t / √(cos 2t), y = cos³ t / √(cos 2t).

dx/dt: differentiate sin³ t / √(cos 2t) using quotient rule.
dx/dt = [3 sin² t cos t · cos 2t + sin³ t · sin 2t] / (cos 2t)^{3/2}.
Factor: sin² t cos t[3 cos 2t + 2 sin² t] = sin² t cos t(3 - 4 sin² t).
By triple-angle: 3 - 4 sin² t = sin 3t / sin t.
So dx/dt = sin t cos t sin 3t / (cos 2t)^{3/2}.

dy/dt: differentiate cos³ t / √(cos 2t).
dy/dt = [-3 cos² t sin t · cos 2t + cos³ t · sin 2t] / (cos 2t)^{3/2}.
Factor: cos² t sin t[-3 cos 2t + 2 cos² t] = cos² t sin t(3 - 4 cos² t).
By triple-angle: cos 3t = cos t(4 cos² t - 3), so 3 - 4 cos² t = -cos 3t / cos t.
So dy/dt = -cos t sin t cos 3t / (cos 2t)^{3/2}.

dy/dx = dy/dt / dx/dt = [-cos t sin t cos 3t] / [sin t cos t sin 3t] = -cos 3t / sin 3t = -cot 3t.

Answer: dy/dx = -cot 3t.

Q8 If x and y are connected parametrically by x = a(cos t + log tan(t/2)), y = a sin t, find dy/dx without eliminating the parameter.

SOLUTION

dy/dt = a cos t.

dx/dt = a(-sin t + d/dt[log tan(t/2)]).

d/dt[log tan(t/2)] = 1/tan(t/2) · sec²(t/2) · 1/2 = 1/(2 sin(t/2) cos(t/2)) = 1/sin t.

dx/dt = a(-sin t + 1/sin t) = a(1 - sin² t)/sin t = a cos² t/sin t.

dy/dx = a cos t / (a cos² t/sin t) = sin t/cos t = tan t.

Answer: dy/dx = tan t.

Q9 If x and y are connected parametrically by x = a sec θ, y = b tan θ, find dy/dx without eliminating the parameter.

SOLUTION

dx/dθ = a sec θ tan θ, dy/dθ = b sec² θ.

dy/dx = b sec² θ / (a sec θ tan θ) = b sec θ / (a tan θ) = b/(a sin θ).

Answer: dy/dx = b/(a sin θ).

Q10 If x and y are connected parametrically by x = a(cos θ + θ sin θ), y = a(sin θ - θ cos θ), find dy/dx without eliminating the parameter.

SOLUTION

dx/dθ = a(-sin θ + sin θ + θ cos θ) = aθ cos θ.

dy/dθ = a(cos θ - cos θ + θ sin θ) = aθ sin θ.

dy/dx = aθ sin θ / (aθ cos θ) = tan θ.

Answer: dy/dx = tan θ.

Q11 If x = a^{sin^{-1} t}, y = a^{cos^{-1} t}, show that dy/dx = -y/x.

SOLUTION

dx/dt = a^{sin^{-1} t} · log a · 1/√(1-t²).

dy/dt = a^{cos^{-1} t} · log a · (-1/√(1-t²)).

dy/dx = (dy/dt)/(dx/dt) = -a^{cos^{-1} t}/a^{sin^{-1} t} = -y/x. ✓

Answer: dy/dx = -y/x. (Proved)

Exercise 5.7 — Exercise 5.7

Q1 Find the second order derivative of $y = x^2 + 3x + 2$.

SOLUTION

$y' = 2x + 3$.

$y'' = 2$.

Answer: $d^2y/dx^2 = 2$.

Q2 Find the second order derivative of $y = x^{20}$.

SOLUTION

$y' = 20x^{19}$.

$y'' = 20 \cdot 19\, x^{18} = 380\, x^{18}$.

Answer: $d^2y/dx^2 = 380\, x^{18}$.

Q3 Find the second order derivative of $y = x \cos x$.

SOLUTION

$y' = \cos x - x \sin x$.

$y'' = -\sin x - (\sin x + x \cos x) = -2 \sin x - x \cos x$.

Answer: $d^2y/dx^2 = -2 \sin x - x \cos x$.

Q4 Find the second order derivative of $y = \log x$.

SOLUTION

$y' = 1/x$.

$y'' = -1/x^2$.

Answer: $d^2y/dx^2 = -1/x^2$.

Q5 Find the second order derivative of $y = x^3 \log x$.

SOLUTION

$y' = 3x^2 \log x + x^2 = x^2(3 \log x + 1)$.

$y'' = 2x(3 \log x + 1) + x^2 \cdot 3/x = 6x \log x + 2x + 3x = x(6 \log x + 5)$.

Answer: $d^2y/dx^2 = x(6 \log x + 5)$.

Q6 Find the second order derivative of y = e^x sin 5x.

SOLUTION

y' = e^x sin 5x + 5 e^x cos 5x = e^x(sin 5x + 5 cos 5x).
y'' = e^x(sin 5x + 5 cos 5x) + e^x(5 cos 5x - 25 sin 5x)
= e^x(-24 sin 5x + 10 cos 5x).

Answer: $d^2y/dx^2 = e^x(10 \cos 5x - 24 \sin 5x)$.

Q7 Find the second order derivative of y = e^{6x} cos 3x.

SOLUTION

y' = 6e^{6x} cos 3x - 3e^{6x} sin 3x = e^{6x}(6 cos 3x - 3 sin 3x).
y'' = e^{6x}(6 cos 3x - 3 sin 3x)·6 + e^{6x}(-18 sin 3x - 9 cos 3x)
= e^{6x}[36 cos 3x - 18 sin 3x - 18 sin 3x - 9 cos 3x]
= e^{6x}(27 cos 3x - 36 sin 3x).

Answer: $d^2y/dx^2 = e^{6x}(27 \cos 3x - 36 \sin 3x)$.

Q8 Find the second order derivative of y = tan^{-1} x.

SOLUTION

y' = $1/(1+x^2)$.
y'' = $d/dx[1/(1+x^2)] = -2x/(1+x^2)^2$.

Answer: $d^2y/dx^2 = -2x / (1 + x^2)^2$.

Q9 Find the second order derivative of y = log(log x).

SOLUTION

y' = 1/(x log x).
y'' = d/dx[1/(x log x)] = -[(log x + 1)] / (x log x)²
= -(1 + log x) / (x² (log x)²).

Answer: $d^2y/dx^2 = -(1 + \log x) / (x^2 (\log x)^2)$.

Q10 Find the second order derivative of y = sin(log x).

SOLUTION

y' = cos(log x)/x.

y'' = [-sin(log x)/x · x - cos(log x)] / x²

= -[sin(log x) + cos(log x)] / x².

Answer: $d^2y/dx^2 = -[\sin(\log x) + \cos(\log x)] / x^2$.

Q11 If y = 5 cos x - 3 sin x, prove that $d^2y/dx^2 + y = 0$.

SOLUTION

y' = -5 sin x - 3 cos x.

y'' = -5 cos x + 3 sin x = -(5 cos x - 3 sin x) = -y.

Therefore y'' + y = -y + y = 0. ✓

Answer: $d^2y/dx^2 + y = 0$. (Proved)

Q12 If y = cos^{-1} x, find d^2y/dx^2 in terms of y alone.

SOLUTION

$dy/dx = -1/\sqrt{(1-x^2)}$.

$x = \cos y$, so $1-x^2 = \sin^2 y$ and $\sqrt{(1-x^2)} = \sin y$ (for $0 < y < \pi$).

$dy/dx = -1/\sin y$.

$d^2y/dx^2 = d/dx(-1/\sin y) = (\cos y/\sin^2 y) \cdot dy/dx = (\cos y/\sin^2 y)\cdot(-1/\sin y) = -\cos y/\sin^3 y$

$= -\cot y \operatorname{cosec}^2 y$.

Answer: $d^2y/dx^2 = -\cos y / \sin^3 y = -\cot y \operatorname{cosec}^2 y$.

Q13 If y = 3 cos(log x) + 4 sin(log x), show that $x^2 y_2 + x y_1 + y = 0$.

SOLUTION

$y_1 = dy/dx = -3 \sin(\log x)/x + 4 \cos(\log x)/x = [-3 \sin(\log x) + 4 \cos(\log x)]/x$.

$x y_1 = -3 \sin(\log x) + 4 \cos(\log x)$.

Differentiate: $d(x y_1)/dx = xy_2 + y_1 = [-3 \cos(\log x)/x - 4 \sin(\log x)/x]$.

So $x y_2 + y_1 = [-3 \cos(\log x) - 4 \sin(\log x)]/x = -y/x$.

Multiply by x: $x^2 y_2 + x y_1 = -y$.

Therefore $x^2 y_2 + x y_1 + y = 0$. ✓

Answer: $x^2 y_2 + x y_1 + y = 0$. (Proved)

Q14 If y = Ae^{mx} + Be^{nx}, show that d²y/dx² - (m+n) dy/dx + mny = 0.

SOLUTION

y' = Ame^{mx} + Bne^{nx}.
y'' = Am² e^{mx} + Bn² e^{nx}.
y'' - (m+n)y' + mny = Am² e^{mx}+Bn² e^{nx} - (m+n)(Ame^{mx}+Bne^{nx}) + mn(Ae^{mx}+Be^{nx})
= e^{mx}[Am² - Am(m+n) + Amn] + e^{nx}[Bn² - Bn(m+n) + Bmn]
= e^{mx}·A[m²-m²-mn+mn] + e^{nx}·B[n²-mn-n²+mn]
= 0. ✓

Answer: d²y/dx² - (m+n) dy/dx + mny = 0. (Proved)

Q15 If y = 500e^{7x} + 600e^{-7x}, show that d²y/dx² = 49y.

SOLUTION

y' = 3500e^{7x} - 4200e^{-7x}.
y'' = 24500e^{7x} + 29400e^{-7x} = 49(500e^{7x} + 600e^{-7x}) = 49y. ✓

Answer: d²y/dx² = 49y. (Proved)

Q16 If e^y(x+1) = 1, show that d²y/dx² = (dy/dx)².

SOLUTION

e^y(x+1) = 1 => e^y = 1/(x+1).
Differentiating: e^y y'(x+1) + e^y = 0 => y'(x+1) = -1 => y' = -1/(x+1).
Differentiating again: y'' = 1/(x+1)².
(dy/dx)² = 1/(x+1)².
Therefore y'' = (y')². ✓

Answer: d²y/dx² = (dy/dx)². (Proved)

Q17 If y = (tan^{-1} x)², show that $(x^{2+}1)^2\ y_2 + 2x(x^{2+}1)\ y_1 = 2$.

SOLUTION

$y_1 = 2\ \tan^{-1} x / (1+x^2)$.

$(1+x^2)\ y_1 = 2\ \tan^{-1} x$.

Differentiate both sides w.r.t. x:

$2x\ y_1 + (1+x^2)\ y_2 = 2/(1+x^2)$.

Multiply by $(1+x^2)$:

$2x(1+x^2)\ y_1 + (1+x^2)^2\ y_2 = 2$. ✓

Answer: $(x^{2+}1)^2\ y_2 + 2x(x^{2+}1)\ y_1 = 2$. (Proved)

Misc — Miscellaneous Exercise

Q1 Differentiate $(3x^2 - 9x + 5)^9$ with respect to x.

SOLUTION

$d/dx[(3x^2-9x+5)^9] = 9(3x^2-9x+5)^8 \cdot (6x-9)$

$= 9 \cdot 3(2x-3) \cdot (3x^2-9x+5)^8$

$= 27(2x-3)(3x^2-9x+5)^8$.

Answer: $27(2x - 3)(3x^2 - 9x + 5)^8$.

Q2 Differentiate $\sin^3 x + \cos^6 x$ with respect to x.

SOLUTION

$d/dx[\sin^3 x] = 3 \sin^2 x \cos x$.

$d/dx[\cos^6 x] = 6 \cos^5 x \cdot (-\sin x) = -6 \sin x \cos^5 x$.

$dy/dx = 3 \sin^2 x \cos x - 6 \sin x \cos^5 x$

$= 3 \sin x \cos x(\sin x - 2 \cos^4 x)$.

Answer: $3 \sin^2 x \cos x - 6 \sin x \cos^5 x$.

Q3 Differentiate (5x)^{3 cos 2x} with respect to x.

SOLUTION

Let y = (5x)^{3 cos 2x}. Take log: log y = 3 cos 2x · log(5x).

(1/y) dy/dx = 3(-2 sin 2x) log(5x) + 3 cos 2x · 1/x

= 3[-2 sin 2x log(5x) + cos 2x/x].

dy/dx = 3(5x)^{3 cos 2x} [cos 2x/x - 2 sin 2x log(5x)].

Answer: 3(5x)^{3 cos 2x} [cos 2x / x - 2 sin 2x log(5x)].

Q4 Differentiate sin^{-1}(x√x), 0 ≤ x ≤ 1, with respect to x.

SOLUTION

y = sin^{-1}(x^{3/2}).

$dy/dx = 1/\sqrt{(1-x^3)} \cdot d/dx(x^{3/2}) = 1/\sqrt{(1-x^3)} \cdot (3/2)\sqrt{x}$

$= 3\sqrt{x} / (2\sqrt{(1-x^3)})$.

Answer: $3\sqrt{x} / (2\sqrt{(1 - x^3)})$.

Q5 Differentiate (cos^{-1}(x/2)) / √(2x+7), -2 < x < 2, with respect to x.

SOLUTION

Let u = cos^{-1}(x/2), v = √(2x+7). By quotient rule:
du/dx = -1/√(1-(x/2)²) · (1/2) = -1/√(4-x²).
dv/dx = 1/(√(2x+7)).
d/dx[u/v] = (u'v - uv')/v²
= [-1/√(4-x²) · √(2x+7) - cos^{-1}(x/2) · 1/√(2x+7)] / (2x+7)
= -1/(√(4-x²)·√(2x+7)) - cos^{-1}(x/2)/(2x+7)^{3/2}.

Answer: -1/(√(4-x²)·√(2x+7)) - cos^{-1}(x/2)/(2x+7)^{3/2}.

Q6 Differentiate cot^{-1}[(√(1+sin x) + √(1-sin x)) / (√(1+sin x) - √(1-sin x))], 0 < x < π/2, with respect to x.

SOLUTION

For 0 < x < π/2: √(1+sin x) = cos(x/2)+sin(x/2), √(1-sin x) = cos(x/2)-sin(x/2).
Sum = 2 cos(x/2), Difference = 2 sin(x/2).
Argument = 2 cos(x/2)/(2 sin(x/2)) = cot(x/2).
y = cot^{-1}(cot(x/2)) = x/2.
dy/dx = 1/2.

Answer: dy/dx = 1/2.

Q7 Differentiate (log x)^{log x}, x > 1, with respect to x.

SOLUTION

Let y = (log x)^{log x}. Take log: log y = log x · log(log x).
(1/y) dy/dx = 1/x · log(log x) + log x · 1/(x log x)
= (log(log x))/x + 1/x = (1 + log(log x))/x.
dy/dx = (log x)^{log x} · (1 + log(log x))/x.

Answer: (log x)^{log x} · (1 + log(log x)) / x.

Q8 Differentiate cos(a cos x + b sin x), for some constant a and b, with respect to x.

SOLUTION

d/dx[cos(a cos x + b sin x)] = -sin(a cos x + b sin x) · d/dx(a cos x + b sin x)
= -sin(a cos x + b sin x) · (-a sin x + b cos x)
= (a sin x - b cos x) sin(a cos x + b sin x).

Answer: (a sin x - b cos x) sin(a cos x + b sin x).

Q9 Differentiate (sin x - cos x)^{sin x - cos x}, π/4 < x < 3π/4, with respect to x.

SOLUTION

In π/4 < x < 3π/4, sin x - cos x > 0.
Let y = (sin x - cos x)^{sin x - cos x}. Take log: log y = (sin x - cos x) log(sin x - cos x).
(1/y) dy/dx = (cos x + sin x) log(sin x - cos x) + (sin x - cos x) · (cos x + sin x)/(sin x - cos x)
= (sin x + cos x)[log(sin x - cos x) + 1].
dy/dx = (sin x - cos x)^{sin x - cos x} · (sin x + cos x)[1 + log(sin x - cos x)].

Answer: (sin x - cos x)^{sin x - cos x} (sin x + cos x)[1 + log(sin x - cos x)].

Q10 Differentiate x^x + x^a + a^x + a^a, for some fixed a > 0 and x > 0, with respect to x.

SOLUTION

d/dx(x^x): log u = x log x => du/dx = x^x(1 + log x).
d/dx(x^a) = a x^{a-1}.
d/dx(a^x) = a^x log a.
d/dx(a^a) = 0 (constant).
dy/dx = x^x(1 + log x) + a x^{a-1} + a^x log a.

Answer: x^x(1 + log x) + a x^{a-1} + a^x log a.

Q11 Differentiate $x^{x^2-3} + (x-3)^{x^2}$, for $x > 3$, with respect to x.

SOLUTION

Let $u = x^{x^2-3}$ and $v = (x-3)^{x^2}$.

For u: $\log u = (x^2-3) \log x$.
$(1/u)\, du/dx = 2x \log x + (x^2-3)/x$.
$du/dx = x^{x^2-3}[2x \log x + (x^2-3)/x]$.

For v: $\log v = x^2 \log(x-3)$.
$(1/v)\, dv/dx = 2x \log(x-3) + x^2/(x-3)$.
$dv/dx = (x-3)^{x^2}[2x \log(x-3) + x^2/(x-3)]$.

$dy/dx = x^{x^2-3}[2x \log x + (x^2-3)/x] + (x-3)^{x^2}[2x \log(x-3) + x^2/(x-3)]$.

Answer: $x^{x^2-3}[2x \log x + (x^2-3)/x] + (x-3)^{x^2}[2x \log(x-3) + x^2/(x-3)]$.

Q12 Find dy/dx if $y = 12(1 - \cos t)$, $x = 10(t - \sin t)$, $-\pi/2 < t < \pi/2$.

SOLUTION

$dy/dt = 12 \sin t$.
$dx/dt = 10(1 - \cos t) = 20 \sin^2(t/2)$.
$dy/dx = 12 \sin t / (10(1 - \cos t)) = 12 \cdot 2 \sin(t/2)\cos(t/2) / (10 \cdot 2 \sin^2(t/2))$
$= 12 \cos(t/2) / (10 \sin(t/2)) = (6/5) \cot(t/2)$.

Answer: $dy/dx = (6/5) \cot(t/2)$.

Q13 Find dy/dx if $y = \sin^{-1} x + \sin^{-1}(\sqrt{1-x^2})$, $0 < x < 1$.

SOLUTION

Put $x = \sin \theta$ (so $0 < \theta < \pi/2$): $\sqrt{1-x^2} = \cos \theta$.
$y = \sin^{-1}(\sin \theta) + \sin^{-1}(\cos \theta) = \theta + \sin^{-1}(\sin(\pi/2-\theta)) = \theta + (\pi/2-\theta) = \pi/2$.
y is constant, so $dy/dx = 0$.

Answer: $dy/dx = 0$.

Q14 If $x\sqrt{(1+y)} + y\sqrt{(1+x)} = 0$, for $-1 < x < 1$, prove that $dy/dx = -1/(1+x)^2$.

SOLUTION

$x\sqrt{(1+y)} = -y\sqrt{(1+x)}$. Square both sides: $x^2(1+y) = y^2(1+x)$.
$x^2 + x^2 y = y^2 + xy^2$.
$x^2 - y^2 = xy^2 - x^2 y = xy(y-x)$.
$(x-y)(x+y) = -xy(x-y)$. Since $x \neq y$ (in the given domain): $x+y = -xy$.
$y + xy = -x \Rightarrow y(1+x) = -x \Rightarrow y = -x/(1+x)$.
$dy/dx = [-(1+x) - (-x)(1)] / (1+x)^2 = [-(1+x)+x]/(1+x)^2 = -1/(1+x)^2$. ✓

Answer: $dy/dx = -1/(1+x)^2$. (Proved)

Q15 If $(x-a)^2 + (y-b)^2 = c^2$, for some $c > 0$, prove that $[(1 + (dy/dx)^2)^{3/2}] / (d^2y/dx^2)$ is a constant independent of a and b.

SOLUTION

Differentiate $(x-a)^2 + (y-b)^2 = c^2$ w.r.t. x:
$2(x-a) + 2(y-b) y' = 0 \Rightarrow (x-a) = -(y-b) y'$. ... (1)
Differentiate again: $1 + (y')^2 + (y-b) y'' = 0 \Rightarrow y-b = -(1+(y')^2)/y''$. ... (2)
From (1): $x-a = (1+(y')^2)/y'' \cdot y'$. ... (3)
Substitute (2) and (3) in $(x-a)^{2+}(y-b)^2 = c^2$:
$(1+(y')^2)^2 (y')^2/(y'')^2 + (1+(y')^2)^2/(y'')^2 = c^2$.
$(1+(y')^2)^2 (1+(y')^2) / (y'')^2 = c^2$.
$(1+(y')^2)^3 / (y'')^2 = c^2$.
$[(1+(dy/dx)^2)^{3/2} / (d^2y/dx^2)]^2 = c^2 \Rightarrow [(1+(dy/dx)^2)^{3/2}] / |d^2y/dx^2| = c$.
This is a constant (= c) independent of a and b. ✓

Answer: $[(1+(dy/dx)^2)^{3/2}] / (d^2y/dx^2)$ is a constant equal to $\pm c$. (Proved)

Q16 If $\cos y = x \cos(a+y)$, with $\cos a \neq \pm 1$, prove that $dy/dx = \cos^2(a+y) / \sin a$.

SOLUTION

Differentiate $\cos y = x \cos(a+y)$ w.r.t. x:
$-\sin y \cdot dy/dx = \cos(a+y) + x \cdot (-\sin(a+y)) \cdot dy/dx$.
$dy/dx [-\sin y + x \sin(a+y)] = \cos(a+y)$.
$dy/dx = \cos(a+y) / (x \sin(a+y) - \sin y)$. ... (i)

From $\cos y = x \cos(a+y)$: $x = \cos y / \cos(a+y)$.
Substitute: $x \sin(a+y) - \sin y = (\cos y/\cos(a+y)) \sin(a+y) - \sin y$

= [cos y sin(a+y) - sin y cos(a+y)] / cos(a+y)
= sin(a+y-y) / cos(a+y) = sin a / cos(a+y).

Therefore dy/dx = cos(a+y) / (sin a/cos(a+y)) = cos²(a+y)/sin a. ✓

Answer: dy/dx = cos²(a+y) / sin a. (Proved)

Q17 If x = a(cos t + t sin t) and y = a(sin t - t cos t), find d²y/dx².

SOLUTION

dx/dt = a(-sin t + sin t + t cos t) = at cos t.
dy/dt = a(cos t - cos t + t sin t) = at sin t.
dy/dx = at sin t / (at cos t) = tan t.

d²y/dx² = d(tan t)/dx = (d(tan t)/dt) / (dx/dt) = sec² t / (at cos t)
= 1/(at cos³ t) = sec³ t/(at).

Answer: d²y/dx² = sec³ t / (at).

Q18 If f(x) = |x|³, show that f''(x) exists for all real x and find it.

SOLUTION

For x > 0: f(x) = x³, f'(x) = 3x², f''(x) = 6x.
For x < 0: f(x) = -x³, f'(x) = -3x², f''(x) = -6x = 6(-x) = 6|x|.
At x = 0: f'(x) = 3x² for x>0 and f'(x) = -3x² for x<0; both approach 0 as x->0. So f'(0) = 0.
f''(0) = lim_{h->0} [f'(h)-f'(0)]/h = lim 3h²/h = 0 (from right); lim -3h²/h = 0 (from left).
f''(0) = 0.
Combined: f''(x) = 6x for x > 0, f''(x) = -6x for x < 0, f''(0) = 0.
= 6|x| for all x in R.

Answer: f''(x) = 6|x| for all x in R.

Q19 Using the fact that sin(A+B) = sin A cos B + cos A sin B and differentiation, obtain the sum formula for cosines.

SOLUTION

Let f(x) = sin(A+x) = sin A cos x + cos A sin x.
Differentiate both sides w.r.t. x:
f'(x) = cos(A+x) [left side].

f'(x) = -sin A sin x + cos A cos x [right side].
Therefore cos(A+x) = cos A cos x - sin A sin x.
Replacing x with B: cos(A+B) = cos A cos B - sin A sin B.

Answer: cos(A + B) = cos A cos B - sin A sin B. (Derived)

Q20 Does there exist a function which is continuous everywhere but not differentiable at exactly two points? Justify your answer.

SOLUTION

Yes. Consider f(x) = |x| + |x - 1|.
f is continuous everywhere (sum of two continuous functions).
|x| is not differentiable at x = 0 and |x-1| is not differentiable at x = 1.
At x = 0: LHD = lim_{h->0⁻} [(-h)+(1-h)-0-1]/h = lim_{h->0⁻} (-2h)/h = -2; RHD = lim_{h->0⁺} [h+(1-h)-0-1]/h = 0. LHD ≠ RHD, so not differentiable at x = 0.
At x = 1: LHD = lim_{h->0⁻} [(1+h)+(-(1+h-1))-1-0]/h = 0; RHD = lim_{h->0⁺} [(1+h)+(1+h-1)-1-0]/h = 2. LHD ≠ RHD, so not differentiable at x = 1.
Therefore f(x) = |x| + |x-1| is continuous everywhere but not differentiable at exactly x = 0 and x = 1.

Answer: Yes. f(x) = |x| + |x - 1| is continuous everywhere but not differentiable at x = 0 and x = 1.

Q21 If y = |f(x) g(x) h(x); l m n; a b c|, prove that dy/dx = |f'(x) g'(x) h'(x); l m n; a b c|.

SOLUTION

Expand the determinant along the first row:
y = f(x)(mc - nb) - g(x)(lc - na) + h(x)(lb - ma).
Differentiate term by term (the second and third rows have constants l,m,n,a,b,c):
dy/dx = f'(x)(mc-nb) - g'(x)(lc-na) + h'(x)(lb-ma).
This is exactly the expansion of |f'(x) g'(x) h'(x); l m n; a b c| along the first row. ✓

Answer: dy/dx = |f'(x) g'(x) h'(x); l m n; a b c|. (Proved)

Q22 If $y = e^{a \cos^{-1} x}$, $-1 \le x \le 1$, show that $(1-x^2)\,d^2y/dx^2 - x\,dy/dx - a^2 y = 0$.

SOLUTION

$y = e^{a \cos^{-1} x}$.

$y' = e^{a \cos^{-1} x} \cdot a \cdot (-1/\sqrt{(1-x^2)}) = -ay/\sqrt{(1-x^2)}$.

So $\sqrt{(1-x^2)}\, y' = -ay$. ... (i)

Differentiate (i) w.r.t. x:

$(-x/\sqrt{(1-x^2)})\, y' + \sqrt{(1-x^2)}\, y'' = -a\, y'$.

Multiply by $\sqrt{(1-x^2)}$:

$-x\, y' + (1-x^2)\, y'' = -a\, y' \sqrt{(1-x^2)} = -a(-ay) = a^2 y$.

$(1-x^2)\, y'' - x\, y' - a^2 y = 0$. ✓

Answer: $(1-x^2)\,d^2y/dx^2 - x\,dy/dx - a^2 y = 0$. (Proved)