Chapter 6: Application of Derivatives

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Exercise 6.1 — Exercise 6.1

Q1 Find the rate of change of the area of a circle with respect to its radius r when (a) r = 3 cm (b) r = 4 cm

SOLUTION

Area of a circle: $A = \pi r^2$.

Rate of change of area with respect to radius: $dA/dr = 2\pi r$.

(a) At r = 3 cm: $dA/dr = 2\pi(3) = 6\pi$ cm²/cm.

(b) At r = 4 cm: $dA/dr = 2\pi(4) = 8\pi$ cm²/cm.

Answer: (a) 6π cm²/cm (b) 8π cm²/cm

Q2 The volume of a cube is increasing at the rate of 8 cm³/s. How fast is the surface area increasing when the length of an edge is 12 cm?

SOLUTION

Let x be the edge length. Then $V = x^3$ and $S = 6x^2$.

$dV/dt = 3x^2 (dx/dt)$. Given $dV/dt = 8$ cm³/s.

At x = 12: $8 = 3(144)(dx/dt) \Rightarrow dx/dt = 8/432 = 1/54$ cm/s.

$dS/dt = 12x (dx/dt) = 12(12)(1/54) = 144/54 = 8/3$ cm²/s.

Answer: 8/3 cm²/s

Q3 The radius of a circle is increasing uniformly at the rate of 3 cm/s. Find the rate at which the area of the circle is increasing when the radius is 10 cm.

SOLUTION

$A = \pi r^2$, so $dA/dt = 2\pi r\,(dr/dt)$.

Given $dr/dt = 3$ cm/s. At $r = 10$ cm:

$dA/dt = 2\pi(10)(3) = 60\pi$ cm²/s.

Answer: 60π cm²/s

Q4 An edge of a variable cube is increasing at the rate of 3 cm/s. How fast is the volume of the cube increasing when the edge is 10 cm long?

SOLUTION

$V = x^3$, so $dV/dt = 3x^2\,(dx/dt)$.

Given $dx/dt = 3$ cm/s. At $x = 10$ cm:

$dV/dt = 3(100)(3) = 900$ cm³/s.

Answer: 900 cm³/s

Q5 A stone is dropped into a quiet lake and waves move in circles at the speed of 5 cm/s. At the instant when the radius of the circular wave is 8 cm, how fast is the enclosed area increasing?

SOLUTION

$A = \pi r^2$, so $dA/dt = 2\pi r\,(dr/dt)$.

Given $dr/dt = 5$ cm/s. At $r = 8$ cm:

$dA/dt = 2\pi(8)(5) = 80\pi$ cm²/s.

Answer: 80π cm²/s

Q6 The radius of a circle is increasing at the rate of 0.7 cm/s. What is the rate of increase of its circumference?

SOLUTION

Circumference $C = 2\pi r$, so $dC/dt = 2\pi\,(dr/dt)$.

Given $dr/dt = 0.7$ cm/s:

dC/dt = 2π(0.7) = 1.4π cm/s.

Answer: 1.4π cm/s

Q7 The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing at the rate of 4 cm/minute. When x = 8 cm and y = 6 cm, find the rates of change of (a) the perimeter, and (b) the area of the rectangle.

SOLUTION

Given: dx/dt = -5 cm/min, dy/dt = 4 cm/min.

(a) Perimeter P = 2(x + y).
dP/dt = 2(dx/dt + dy/dt) = 2(-5 + 4) = -2 cm/min.
The perimeter is decreasing at 2 cm/min.

(b) Area A = x · y.
dA/dt = y(dx/dt) + x(dy/dt) = 6(-5) + 8(4) = -30 + 32 = 2 cm²/min.
The area is increasing at 2 cm²/min.

Answer: (a) Perimeter decreasing at 2 cm/min (b) Area increasing at 2 cm²/min

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Q8 A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm.

SOLUTION

Volume of sphere: V = (4/3)π r³.

dV/dt = 4π r² (dr/dt).

Given dV/dt = 900 cm³/s. At r = 15 cm:

900 = 4π(225)(dr/dt) => dr/dt = 900/(900π) = 1/π cm/s.

Answer: 1/π cm/s

Q9 A balloon, which always remains spherical has a variable radius. Find the rate at which its volume is increasing with the radius when the radius is 10 cm.

SOLUTION

$V = (4/3)\pi r^3$.

$dV/dr = 4\pi r^2$.

At r = 10 cm: $dV/dr = 4\pi(100) = 400\pi$ cm³/cm.

Answer: 400π cm³/cm

Q10 A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2 cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall?

SOLUTION

Let x = distance of foot from wall, y = height on wall. Then $x^2 + y^2 = 25$ (in metres).

Differentiating: $2x(dx/dt) + 2y(dy/dt) = 0 \Rightarrow dy/dt = -(x/y)(dx/dt)$.

At x = 4 m: $y = \sqrt{(25 - 16)} = 3$ m.

dx/dt = 2 cm/s = 0.02 m/s.

$dy/dt = -(4/3)(0.02) = -0.08/3$ m/s = -8/3 cm/s.

The height is decreasing at 8/3 cm/s.

Answer: 8/3 cm/s

Q11 A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y-coordinate is changing 8 times as fast as the x-coordinate.

SOLUTION

Given: $6y = x^3 + 2$. Differentiating with respect to t: $6(dy/dt) = 3x^2(dx/dt)$.

Condition: $dy/dt = 8(dx/dt)$. Substituting:
$6 \cdot 8(dx/dt) = 3x^2(dx/dt) \Rightarrow 48 = 3x^2 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$.

At x = 4: $6y = 64 + 2 = 66 \Rightarrow y = 11$. Point: (4, 11).

At x = -4: $6y = -64 + 2 = -62 \Rightarrow y = -31/3$. Point: (-4, -31/3).

Answer: (4, 11) and (-4, -31/3)

Q12 The radius of an air bubble is increasing at the rate of 1/2 cm/s. At what rate is the volume of the bubble increasing when the radius is 1 cm?

SOLUTION

$V = (4/3)\pi r^3$, so $dV/dt = 4\pi r^2 (dr/dt)$.

Given $dr/dt = 1/2$ cm/s. At $r = 1$ cm:

$dV/dt = 4\pi(1)^2(1/2) = 2\pi$ cm³/s.

Answer: 2π cm³/s

Q13 A balloon, which always remains spherical, has a variable diameter $(3/2)(2x + 1)$. Find the rate of change of its volume with respect to x.

SOLUTION

Diameter $= (3/2)(2x + 1)$, so radius $r = (3/4)(2x + 1)$.

$V = (4/3)\pi r^3 = (4/3)\pi [(3/4)(2x+1)]^3 = (4/3)\pi \cdot (27/64)(2x+1)^3 = (9\pi/16)(2x+1)^3$.

$dV/dx = (9\pi/16) \cdot 3(2x+1)^2 \cdot 2 = (27\pi/8)(2x+1)^2$.

Answer: $dV/dx = (27\pi/8)(2x + 1)^2$

Q14 Sand is pouring from a pipe at the rate of 12 cm³/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?

SOLUTION

Given $h = r/6$, so $r = 6h$.

$V = (1/3)\pi r^2 h = (1/3)\pi(6h)^2 h = 12\pi h^3$.

$dV/dt = 36\pi h^2 (dh/dt)$.

Given $dV/dt = 12$ cm³/s, $h = 4$ cm:

$12 = 36\pi(16)(dh/dt) \Rightarrow dh/dt = 12/(576\pi) = 1/(48\pi)$ cm/s.

Answer: $1/(48\pi)$ cm/s

Q15 The total cost C(x) in Rupees associated with the production of x units of an item is given by $C(x) = 0.007x^3 – 0.003x^2 + 15x + 4000$. Find the marginal cost when 17 units are produced.

SOLUTION

Marginal cost $MC = dC/dx = 0.021x^2 – 0.006x + 15$.

At x = 17:

$MC = 0.021(289) – 0.006(17) + 15$

$= 6.069 – 0.102 + 15 = 20.967$.

Marginal cost when 17 units are produced is Rs 20.967 (approximately Rs 20.97).

Answer: Rs 20.967 (approx. Rs 20.97)

Q16 The total revenue in Rupees received from the sale of x units of a product is given by $R(x) = 13x^2 + 26x + 15$. Find the marginal revenue when x = 7.

SOLUTION

Marginal revenue $MR = dR/dx = 26x + 26$.

At x = 7: $MR = 26(7) + 26 = 182 + 26 = 208$.

Marginal revenue when x = 7 is Rs 208.

Answer: Rs 208

Q17 The rate of change of the area of a circle with respect to its radius r at r = 6 cm is

(A) 10π (B) 12π (C) 8π (D) 11π

SOLUTION

$A = \pi r^2$, so $dA/dr = 2\pi r$.

At r = 6: $dA/dr = 2\pi(6) = 12\pi$.

Answer: (B) 12π

Q18 The total revenue in Rupees received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$. The marginal revenue, when $x = 15$ is
(A) 116 (B) 96 (C) 90 (D) 126

SOLUTION

$MR = dR/dx = 6x + 36$.

At $x = 15$: $MR = 6(15) + 36 = 90 + 36 = 126$.

Answer: (D) 126

Exercise 6.2 — Exercise 6.2

Q1 Show that the function given by f(x) = 3x + 17 is increasing on R.

SOLUTION

f'(x) = 3 > 0 for all x in R.

By Theorem 1, f is strictly increasing on R.

Answer: f'(x) = 3 > 0 for all x ∈ R, so f is strictly increasing on R.

Q2 Show that the function given by f(x) = e^(2x) is increasing on R.

SOLUTION

f'(x) = 2e^(2x) > 0 for all x in R (since the exponential is always positive).

By Theorem 1, f is strictly increasing on R.

Answer: f'(x) = 2e^(2x) > 0 for all x ∈ R, so f is strictly increasing on R.

Q3 Show that the function given by f(x) = sin x is (a) increasing in (0, π/2) (b) decreasing in (π/2, π) (c) neither increasing nor decreasing in (0, π)

SOLUTION

f'(x) = cos x.

(a) For x in (0, π/2): cos x > 0, so f'(x) > 0. Hence f is increasing on (0, π/2).

(b) For x in (π/2, π): cos x < 0, so f'(x) < 0. Hence f is decreasing on (π/2, π).

(c) Since f is increasing on (0, π/2) and decreasing on (π/2, π), f is neither increasing nor decreasing on the whole interval (0, π).

Answer: (a) Increasing on (0, π/2) (b) Decreasing on (π/2, π) (c) Neither on (0, π)

Q4 Find the intervals in which the function f given by f(x) = 2x² – 3x is (a) increasing (b) decreasing

SOLUTION

f'(x) = 4x – 3.

f'(x) = 0 => x = 3/4.

The point x = 3/4 divides R into (-∞, 3/4) and (3/4, ∞).

(a) For $x > 3/4$: $f'(x) > 0$ => f is increasing on $(3/4, \infty)$.

(b) For $x < 3/4$: $f'(x) < 0$ => f is decreasing on $(-\infty, 3/4)$.

Answer: (a) Increasing on $(3/4, \infty)$ (b) Decreasing on $(-\infty, 3/4)$

Q5 Find the intervals in which the function f given by $f(x) = 2x^3 - 3x^2 - 36x + 7$ is (a) increasing (b) decreasing

SOLUTION

$f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6) = 6(x - 3)(x + 2)$.

$f'(x) = 0$ at $x = -2$ and $x = 3$.

Intervals: $(-\infty, -2)$, $(-2, 3)$, $(3, \infty)$.

In $(-\infty, -2)$: both factors negative => $f'(x) > 0$ => increasing.
In $(-2, 3)$: $(x+2)>0$, $(x-3)<0$ => $f'(x) < 0$ => decreasing.
In $(3, \infty)$: both factors positive => $f'(x) > 0$ => increasing.

(a) f is increasing on $(-\infty, -2)$ and $(3, \infty)$.
(b) f is decreasing on $(-2, 3)$.

Answer: (a) Increasing on $(-\infty, -2) \cup (3, \infty)$ (b) Decreasing on $(-2, 3)$

Q6 Find the intervals in which the following functions are strictly increasing or decreasing:
(a) $x^2 + 2x - 5$ (b) $10 - 6x - 2x^2$ (c) $-2x^3 - 9x^2 - 12x + 1$ (d) $6 - 9x - x^2$ (e) $(x + 1)^3 (x - 3)^3$

SOLUTION

(a) $f(x) = x^2 + 2x - 5$, $f'(x) = 2x + 2 = 2(x + 1)$. Critical: $x = -1$.
Strictly decreasing on $(-\infty, -1)$; strictly increasing on $(-1, \infty)$.

(b) $f(x) = 10 - 6x - 2x^2$, $f'(x) = -6 - 4x = -2(2x + 3)$. Critical: $x = -3/2$.
Strictly increasing on $(-\infty, -3/2)$; strictly decreasing on $(-3/2, \infty)$.

(c) $f(x) = -2x^3 - 9x^2 - 12x + 1$, $f'(x) = -6x^2 - 18x - 12 = -6(x+1)(x+2)$. Critical: $x = -2, -1$.
Strictly increasing on $(-2, -1)$; strictly decreasing on $(-\infty, -2)$ and $(-1, \infty)$.

(d) $f(x) = 6 - 9x - x^2$, $f'(x) = -9 - 2x = -(2x + 9)$. Critical: $x = -9/2$.
Strictly increasing on $(-\infty, -9/2)$; strictly decreasing on $(-9/2, \infty)$.

(e) $f(x) = (x+1)^3(x-3)^3$. By the product rule, $f'(x) = 3(x+1)^2(x-3)^3 + 3(x+1)^3(x-3)^2 = 3(x+1)^2(x-3)^2[(x-3) + (x+1)]$.
$f'(x) = 3(x+1)^2(x-3)^2(2x-2) = 6(x+1)^2(x-3)^2(x-1)$.
$f'(x) = 0$ at $x = -1, 1, 3$. Since $(x+1)^2 \geq 0$ and $(x-3)^2 \geq 0$, sign is determined by $(x-1)$.
For $x < 1$: $f'(x) \leq 0$ => strictly decreasing on $(-\infty, 1)$ [equality only at $x = -1$].
For $x > 1$: $f'(x) \geq 0$ => strictly increasing on $(1, \infty)$ [equality only at $x = 3$].
Strictly increasing on $(1, \infty)$; strictly decreasing on $(-\infty, 1)$.

Answer: (a) Dec: $(-\infty,-1)$, Inc: $(-1,\infty)$ (b) Inc: $(-\infty,-3/2)$, Dec: $(-3/2,\infty)$ (c) Inc: $(-2,-1)$, Dec: $(-\infty,-2) \cup (-1,\infty)$ (d) Inc: $(-\infty,-9/2)$, Dec: $(-9/2,\infty)$ (e) Dec: $(-\infty,1)$, Inc: $(1,\infty)$

Q7 Show that $y = \log(1 + x) - 2x/(2 + x)$, $x > -1$, is an increasing function of x throughout its domain.

SOLUTION

$dy/dx = 1/(1+x) - [2(2+x) - 2x]/(2+x)^2 = 1/(1+x) - 4/(2+x)^2$.

Combining over a common denominator:
$dy/dx = [(2+x)^2 - 4(1+x)] / [(1+x)(2+x)^2]$
$= [4 + 4x + x^2 - 4 - 4x] / [(1+x)(2+x)^2]$
$= x^2 / [(1+x)(2+x)^2]$.

For $x > -1$ and $x \neq 0$: $x^2 > 0$ and $(1+x) > 0$ and $(2+x)^2 > 0$, so $dy/dx > 0$.
At $x = 0$: $dy/dx = 0$ (isolated point, not an interval).

Hence y is increasing on $(-1, \infty)$.

Answer: $dy/dx = x^2/[(1+x)(2+x)^2] \geq 0$ for all $x > -1$, so y is an increasing function.

Q8 Find the values of x for which $y = [x(x - 2)]^2$ is an increasing function.

SOLUTION

$y = x^2(x-2)^2$.

$dy/dx = 2x(x-2)^2 + x^2 \cdot 2(x-2) = 2x(x-2)[(x-2) + x] = 2x(x-2)(2x-2) = 4x(x-1)(x-2)$.

$dy/dx = 0$ at $x = 0, 1, 2$.

Sign analysis on the four intervals:
$x < 0$: $4x < 0$, $(x-1) < 0$, $(x-2) < 0$ => product < 0 => decreasing.
$0 < x < 1$: $4x > 0$, $(x-1) < 0$, $(x-2) < 0$ => product > 0 => increasing.

$1 < x < 2$: $4x > 0$, $(x–1) > 0$, $(x–2) < 0$ => product < 0 => decreasing.

$x > 2$: all factors positive => increasing.

y is increasing on $[0, 1]$ and $[2, \infty)$.

Answer: y is increasing for $x \in [0, 1] \cup [2, \infty)$

Q9 Prove that $y = 4 \sin \theta/(2 + \cos \theta) – \theta$ is an increasing function of θ in $[0, \pi/2]$.

SOLUTION

$$dy/d\theta = [4 \cos \theta(2 + \cos \theta) – 4 \sin \theta(–\sin \theta)]/(2 + \cos \theta)^2 – 1$$
$$= [8 \cos \theta + 4 \cos^2 \theta + 4 \sin^2 \theta]/(2 + \cos \theta)^2 – 1$$
$$= [8 \cos \theta + 4]/(2 + \cos \theta)^2 – 1$$
$$= [4(2 + \cos \theta)]/(2 + \cos \theta)^2 – 1$$
$$= 4/(2 + \cos \theta) – 1$$
$$= (4 – 2 – \cos \theta)/(2 + \cos \theta)$$
$$= (2 – \cos \theta)/(2 + \cos \theta).$$

For θ in $[0, \pi/2]$: $\cos \theta \in [0, 1]$ so $2 – \cos \theta \geq 1 > 0$ and $2 + \cos \theta > 0$.

Hence $dy/d\theta > 0$ on $(0, \pi/2)$, so y is an increasing function on $[0, \pi/2]$.

Answer: $dy/d\theta = (2 – \cos \theta)/(2 + \cos \theta) > 0$ for $\theta \in [0, \pi/2]$, so y is increasing.

Q10 Prove that the logarithmic function is increasing on $(0, \infty)$.

SOLUTION

Let $f(x) = \log x$, defined for $x > 0$.

$f'(x) = 1/x > 0$ for all x in $(0, \infty)$.

By Theorem 1, f is increasing on $(0, \infty)$.

Answer: $f'(x) = 1/x > 0$ for all $x > 0$, so log x is increasing on $(0, \infty)$.

Q11 Prove that the function f given by $f(x) = x^2 – x + 1$ is neither strictly increasing nor decreasing on $(–1, 1)$.

SOLUTION

$f'(x) = 2x – 1$.

$f'(x) = 0$ at $x = 1/2$.

For x in $(–1, 1/2)$: $f'(x) < 0$ => f is decreasing.

For x in (1/2, 1): f'(x) > 0 => f is increasing.

Since f is neither always increasing nor always decreasing on (–1, 1), it is neither strictly increasing nor strictly decreasing on (–1, 1).

Answer: f is decreasing on (–1, 1/2) and increasing on (1/2, 1), so neither strictly increasing nor decreasing on (–1, 1).

Q12 Which of the following functions are decreasing on (0, π/2)?
(A) cos x (B) cos 2x (C) cos 3x (D) tan x

SOLUTION

(A) d/dx(cos x) = –sin x < 0 on (0, π/2) => decreasing. ✓

(B) d/dx(cos 2x) = –2 sin 2x. On (0, π/2): sin 2x > 0 => –2 sin 2x < 0 => decreasing. ✓

(C) d/dx(cos 3x) = –3 sin 3x. At x = π/3: sin π = 0; for x in (π/3, π/2): sin 3x < 0, so –3 sin 3x > 0 => not decreasing on whole (0, π/2). ✗

(D) d/dx(tan x) = $\sec^2 x$ > 0 on (0, π/2) => increasing. ✗

Functions decreasing on (0, π/2): (A) cos x and (B) cos 2x.

Answer: (A) and (B)

Q13 On which of the following intervals is the function f given by f(x) = x^{100} + sin x – 1 decreasing?
(A) (0, 1) (B) (π/2, π) (C) (0, π/2) (D) None of these

SOLUTION

f'(x) = 100x^{99} + cos x.

(A) On (0, 1): x^{99} > 0 and cos x > 0, so f'(x) > 0 => f is increasing, not decreasing.

(B) On (π/2, π): cos x < 0, but 100x^{99} is large and positive (x > π/2 > 1), so f'(x) > 0 => increasing.

(C) On (0, π/2): 100x^{99} > 0 and cos x > 0 => f'(x) > 0 => increasing.

f is increasing on all given intervals, not decreasing on any of them.

Answer: (D) None of these

Q14 For what values of a the function f given by $f(x) = x^2 + ax + 1$ is increasing on [1, 2]?

SOLUTION

$f'(x) = 2x + a$.

For f to be increasing on [1, 2], we need $f'(x) \geq 0$ for all x in [1, 2].

The minimum of f'(x) on [1, 2] occurs at x = 1 (since 2x + a is increasing in x).

$f'(1) = 2 + a \geq 0 \Rightarrow a \geq -2$.

Therefore f is increasing on [1, 2] for all $a \geq -2$.

Answer: $a \geq -2$

Q15 Let I be any interval disjoint from [–1, 1]. Prove that the function f given by $f(x) = x + 1/x$ is increasing on I.

SOLUTION

$f'(x) = 1 - 1/x^2 = (x^2 - 1)/x^2$.

For x in I where I is disjoint from [–1, 1]: $|x| > 1$, so $x^2 > 1$.
Thus $x^2 - 1 > 0$ and $x^2 > 0$, giving $f'(x) > 0$.

By Theorem 1, f is increasing on every interval I disjoint from [–1, 1].

Answer: $f'(x) = (x^2 - 1)/x^2 > 0$ for $|x| > 1$, so f is increasing on I.

Q16 Prove that the function f given by $f(x) = \log \sin x$ is increasing on $(0, \pi/2)$ and decreasing on $(\pi/2, \pi)$.

SOLUTION

$f'(x) = \cos x / \sin x = \cot x$.

On $(0, \pi/2)$: $\cot x > 0 \Rightarrow f'(x) > 0 \Rightarrow$ f is increasing.

On $(\pi/2, \pi)$: $\cot x < 0 \Rightarrow f'(x) < 0 \Rightarrow$ f is decreasing.

Answer: $f'(x) = \cot x > 0$ on $(0,\pi/2)$ and < 0 on $(\pi/2,\pi)$, so increasing on $(0,\pi/2)$ and decreasing on $(\pi/2,\pi)$.

Q17 Prove that the function f given by f(x) = log |cos x| is decreasing on (0, π/2) and increasing on (3π/2, 2π).

SOLUTION

On (0, π/2): cos x > 0, so f(x) = log(cos x).
f'(x) = –sin x/cos x = –tan x.
On (0, π/2): tan x > 0 => f'(x) < 0 => f is decreasing.

On (3π/2, 2π): cos x > 0, so f(x) = log(cos x).
f'(x) = –sin x/cos x = –tan x.
On (3π/2, 2π): sin x < 0 so tan x < 0, hence –tan x > 0 => f is increasing.

Answer: f'(x) = –tan x; decreasing on (0,π/2) where tan x > 0; increasing on (3π/2,2π) where tan x < 0.

Q18 Prove that the function given by f(x) = $x^3 - 3x^2 + 3x - 100$ is increasing in R.

SOLUTION

f'(x) = $3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x - 1)^2$.

Since $3(x - 1)^2 \geq 0$ for all x in R, and equals 0 only at x = 1 (an isolated point), f'(x) > 0 for all $x \neq 1$.

Hence f is increasing on R.

Answer: f'(x) = $3(x-1)^2 \geq 0$ for all $x \in R$, so f is increasing in R.

Q19 The interval in which y = x^2 e^(–x) is increasing is
(A) (–∞, ∞) (B) (–2, 0) (C) (2, ∞) (D) (0, 2)

SOLUTION

dy/dx = 2x e^(–x) + x^2(–e^(–x)) = x e^(–x)(2 – x).

e^(–x) > 0 always, so sign of dy/dx is determined by x(2 – x).

x(2 – x) > 0 when 0 < x < 2.

y is increasing on (0, 2).

Answer: (D) (0, 2)

Exercise 6.3 — Exercise 6.3

Q1 Find the maximum and minimum values, if any, of the following functions given by
(i) $f(x) = (2x - 1)^2 + 3$ (ii) $f(x) = 9x^2 + 12x + 2$ (iii) $f(x) = -(x - 1)^2 + 10$ (iv) $g(x) = x^3 + 1$

SOLUTION

(i) $f(x) = (2x-1)^2 + 3 \geq 3$ for all x, with equality at $x = 1/2$.
Minimum value = 3 at $x = 1/2$. No maximum value.

(ii) $f(x) = 9x^2 + 12x + 2 = (3x+2)^2 - 2$. Minimum value = –2 at $x = -2/3$. No maximum.

(iii) $f(x) = -(x-1)^2 + 10 \leq 10$ for all x, with equality at $x = 1$.
Maximum value = 10 at $x = 1$. No minimum value.

(iv) $g(x) = x^3 + 1$. $g'(x) = 3x^2 \geq 0$ (never changes sign). g is strictly increasing, so it has neither a maximum nor a minimum.

Answer: (i) Min = 3, no max (ii) Min = –2, no max (iii) Max = 10, no min (iv) Neither max nor min

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Q2 Find the maximum and minimum values, if any, of the following functions given by
(i) $f(x) = |x + 2| - 1$ (ii) $g(x) = -|x + 1| + 3$ (iii) $h(x) = \sin(2x) + 5$ (iv) $f(x) = |\sin 4x + 3|$ (v) $h(x) = x + 1, x \in (-1, 1)$

SOLUTION

(i) $f(x) = |x+2| - 1 \geq -1$, minimum = –1 at $x = -2$. No maximum.

(ii) $g(x) = -|x+1| + 3 \leq 3$, maximum = 3 at $x = -1$. No minimum.

(iii) $h(x) = \sin(2x) + 5$. Since $-1 \leq \sin(2x) \leq 1$: maximum = 6, minimum = 4.

(iv) $f(x) = |\sin 4x + 3|$. Since $-1 \leq \sin 4x \leq 1$: $\sin 4x + 3 \in [2, 4]$, which is always positive.
So $f(x) = \sin 4x + 3$: maximum = 4, minimum = 2.

(v) $h(x) = x + 1, x \in (-1, 1)$. This is strictly increasing on an open interval.
Neither maximum nor minimum value exists (endpoints not attained).

Answer: (i) Min=–1, no max (ii) Max=3, no min (iii) Max=6, Min=4 (iv) Max=4, Min=2 (v) Neither max nor min

Q3 Find the local maxima and local minima, if any, of the following functions. Find also the local maximum and minimum values:
(i) $f(x) = x^2$ (ii) $g(x) = x^3 - 3x$ (iii) $h(x) = \sin x + \cos x$, $0 < x < \pi/2$ (iv) $f(x) = \sin x - \cos x$, $0 < x < 2\pi$ (v) $f(x) = x^3 - 6x^2 + 9x + 15$ (vi) $g(x) = x/2 + 2/x$, $x > 0$ (vii) $g(x) = 1/(x^2 + 2)$ (viii) $f(x) = x\sqrt{(1-x)}$, $0 < x < 1$

SOLUTION

(i) $f'(x) = 2x = 0$ at $x = 0$. $f''(x) = 2 > 0$ => local minimum at $x = 0$. Local min value = $f(0) = 0$.

(ii) $g'(x) = 3x^2 - 3 = 3(x-1)(x+1) = 0$ at $x = \pm 1$.
$g''(x) = 6x$. $g''(-1) = -6 < 0$ => local max at $x = -1$, value = $g(-1) = 2$.
$g''(1) = 6 > 0$ => local min at $x = 1$, value = $g(1) = -2$.

(iii) $h'(x) = \cos x - \sin x = 0$ on $(0, \pi/2)$ at $x = \pi/4$.
$h''(\pi/4) = -\sin(\pi/4) - \cos(\pi/4) = -\sqrt{2} < 0$ => local max at $x = \pi/4$.
Local max value = $h(\pi/4) = \sin(\pi/4) + \cos(\pi/4) = \sqrt{2}$.

(iv) $f'(x) = \cos x + \sin x = 0$ => $\tan x = -1$ => $x = 3\pi/4$ and $x = 7\pi/4$ in $(0, 2\pi)$.
$f''(x) = -\sin x + \cos x$.
$f''(3\pi/4) = -\sin(3\pi/4) + \cos(3\pi/4) = -1/\sqrt{2} - 1/\sqrt{2} = -\sqrt{2} < 0$ => local max at $x = 3\pi/4$, value = $\sqrt{2}$.
$f''(7\pi/4) = -\sin(7\pi/4) + \cos(7\pi/4) = 1/\sqrt{2} + 1/\sqrt{2} = \sqrt{2} > 0$ => local min at $x = 7\pi/4$, value = $-\sqrt{2}$.

(v) $f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3) = 0$ at $x = 1, 3$.
$f''(x) = 6x - 12$. $f''(1) = -6 < 0$ => local max at $x = 1$, value = $f(1) = 19$.
$f''(3) = 6 > 0$ => local min at $x = 3$, value = $f(3) = 15$.

(vi) $g'(x) = 1/2 - 2/x^2 = (x^2 - 4)/(2x^2) = 0$ at $x = 2$ (for $x > 0$).
$g''(x) = 4/x^3 > 0$ for $x > 0$ => local min at $x = 2$, value = $g(2) = 1 + 1 = 2$.

(vii) $g'(x) = -2x/(x^{2+}2)^2 = 0$ at $x = 0$.
$g''(0) = -1/2 < 0$ => local max at $x = 0$, value = $g(0) = 1/2$.

(viii) $f'(x) = \sqrt{(1-x)} + x\cdot(-1)/(2\sqrt{(1-x)}) = (2(1-x) - x)/(2\sqrt{(1-x)}) = (2-3x)/(2\sqrt{(1-x)})$.
$f'(x) = 0$ at $x = 2/3$.
$f''(2/3) < 0$ => local max at $x = 2/3$, value = $f(2/3) = (2/3)\cdot\sqrt{(1/3)} = 2\sqrt{3}/9$.

Answer: (i) Local min = 0 at x=0 (ii) Local max = 2 at x=–1; local min = –2 at x=1 (iii) Local max = √2 at x=π/4 (iv) Local max = √2 at x=3π/4; local min = –√2 at x=7π/4 (v) Local max = 19 at x=1; local min = 15 at x=3 (vi) Local min = 2 at x=2 (vii) Local max = 1/2 at x=0 (viii) Local max = 2√3/9 at x=2/3

Q4 Prove that the following functions do not have maxima or minima:
(i) f(x) = e^x (ii) g(x) = log x (iii) h(x) = $x^3 + x^2 + x + 1$

SOLUTION

(i) f'(x) = e^x > 0 for all x ∈ R. f is strictly increasing, so no maxima or minima.

(ii) g'(x) = 1/x > 0 for all x > 0. g is strictly increasing on (0,∞), so no maxima or minima.

(iii) h'(x) = $3x^2 + 2x + 1$. Discriminant = 4 – 12 = –8 < 0 and leading coefficient positive,
so h'(x) > 0 for all x ∈ R. h is strictly increasing, so no maxima or minima.

Answer: All three have strictly positive derivatives everywhere in their domains, so none has a local maximum or minimum.

Q5 Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals:
(i) $f(x) = x^3$, x ∈ [–2, 2] (ii) f(x) = sin x + cos x, x ∈ [0, π] (iii) $f(x) = 4x – x^2/2$, x ∈ [–2, 9/2] (iv) $f(x) = (x–1)^2 + 3$, x ∈ [–3, 1]

SOLUTION

(i) $f'(x) = 3x^2 = 0$ at x = 0. Evaluate at x = –2, 0, 2:
f(–2) = –8, f(0) = 0, f(2) = 8.
Absolute max = 8 at x = 2; absolute min = –8 at x = –2.

(ii) f'(x) = cos x – sin x = 0 => x = π/4 in [0, π]. Evaluate at x = 0, π/4, π:
f(0) = 1, f(π/4) = √2, f(π) = –1.
Absolute max = √2 at x = π/4; absolute min = –1 at x = π.

(iii) f'(x) = 4 – x = 0 at x = 4. Evaluate at x = –2, 4, 9/2:
f(–2) = –8–2 = –10, f(4) = 16–8 = 8, f(9/2) = 18 – 81/8 = 63/8.
Absolute max = 8 at x = 4; absolute min = –10 at x = –2.

(iv) f'(x) = 2(x–1) = 0 at x = 1. Evaluate at x = –3, 1:
$f(–3) = (–4)^2 + 3 = 19$, f(1) = 0 + 3 = 3.

Absolute max = 19 at x = –3; absolute min = 3 at x = 1.

Answer: (i) Max=8 at x=2, Min=–8 at x=–2 (ii) Max=√2 at x=π/4, Min=–1 at x=π (iii) Max=8 at x=4, Min=–10 at x=–2 (iv) Max=19 at x=–3, Min=3 at x=1

Q6 Find the maximum profit that a company can make, if the profit function is given by $p(x) = 41 – 72x – 18x^2$

SOLUTION

$p'(x) = –72 – 36x = –36(x + 2)$. Setting $p'(x) = 0$: $x = –2$.

$p''(x) = –36 < 0$, so $x = –2$ gives a maximum.

Maximum profit = $p(–2) = 41 – 72(–2) – 18(4) = 41 + 144 – 72 = 113$.

Answer: Maximum profit = 113

Q7 Find both the maximum value and the minimum value of $3x^4 – 8x^3 + 12x^2 – 48x + 25$ on the interval [0, 3].

SOLUTION

$f(x) = 3x^4 – 8x^3 + 12x^2 – 48x + 25$.
$f'(x) = 12x^3 – 24x^2 + 24x – 48 = 12(x^3 – 2x^2 + 2x – 4) = 12(x–2)(x^{2}+2)$.

$f'(x) = 0$ at $x = 2$ (since $x^2 + 2 > 0$ always). $x = 2$ is in [0, 3].

Evaluate at critical point and endpoints:
$f(0) = 25$, $f(2) = 48 – 64 + 48 – 96 + 25 = –39$, $f(3) = 243 – 216 + 108 – 144 + 25 = 16$.

Maximum value = 25 at $x = 0$; minimum value = –39 at $x = 2$.

Answer: Maximum = 25 at x = 0; minimum = –39 at x = 2

Q8 At what points in the interval [0, 2π], does the function sin 2x attain its maximum value?

SOLUTION

$f(x) = \sin 2x$. The maximum value of sin 2x is 1, attained when $2x = \pi/2 + 2k\pi$.

In [0, 2π]: $2x = \pi/2 \Rightarrow x = \pi/4$, and $2x = 5\pi/2 \Rightarrow x = 5\pi/4$.

The function attains its maximum value 1 at $x = \pi/4$ and $x = 5\pi/4$.

Answer: Maximum value 1 attained at $x = \pi/4$ and $x = 5\pi/4$

Q9 What is the maximum value of the function $\sin x + \cos x$?

SOLUTION

$f(x) = \sin x + \cos x = \sqrt{2} \sin(x + \pi/4)$.

Maximum value of $\sqrt{2} \sin(x + \pi/4)$ is $\sqrt{2}$, attained when $\sin(x + \pi/4) = 1$.

Maximum value = $\sqrt{2}$.

Answer: $\sqrt{2}$

Q10 Find the maximum value of $2x^3 - 24x + 107$ in the interval [1, 3]. Find the maximum value of the same function in [–3, –1].

SOLUTION

$f(x) = 2x^3 - 24x + 107$, $f'(x) = 6x^2 - 24 = 6(x^2 - 4) = 0$ at $x = \pm 2$.

On [1, 3]: critical point $x = 2$. Evaluate:
$f(1) = 2 - 24 + 107 = 85$, $f(2) = 16 - 48 + 107 = 75$, $f(3) = 54 - 72 + 107 = 89$.
Maximum value on [1, 3] = 89 at $x = 3$.

On [–3, –1]: critical point $x = -2$. Evaluate:
$f(-3) = -54 + 72 + 107 = 125$, $f(-2) = -16 + 48 + 107 = 139$, $f(-1) = -2 + 24 + 107 = 129$.
Maximum value on [–3, –1] = 139 at $x = -2$.

Answer: Maximum on [1, 3] = 89 (at $x = 3$); maximum on [–3, –1] = 139 (at $x = -2$)

Q11 It is given that at $x = 1$, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value on the interval [0, 2]. Find the value of a.

SOLUTION

$f(x) = x^4 - 62x^2 + ax + 9$.
$f'(x) = 4x^3 - 124x + a$.

Since f attains maximum at $x = 1$, we have $f'(1) = 0$:
$4(1) - 124(1) + a = 0 \Rightarrow 4 - 124 + a = 0 \Rightarrow a = 120$.

Answer: $a = 120$

Q12 Find the maximum and minimum values of $x + \sin 2x$ on $[0, 2\pi]$.

SOLUTION

$f(x) = x + \sin 2x$, $f'(x) = 1 + 2\cos 2x$.

$f'(x) = 0 \Rightarrow \cos 2x = -1/2 \Rightarrow 2x = 2\pi/3, 4\pi/3, 8\pi/3, 10\pi/3$
$\Rightarrow x = \pi/3, 2\pi/3, 4\pi/3, 5\pi/3$.

Evaluate f at critical points and endpoints:
$f(0) = 0$,
$f(\pi/3) = \pi/3 + \sin(2\pi/3) = \pi/3 + \sqrt{3}/2$,
$f(2\pi/3) = 2\pi/3 + \sin(4\pi/3) = 2\pi/3 - \sqrt{3}/2$,
$f(4\pi/3) = 4\pi/3 + \sin(8\pi/3) = 4\pi/3 + \sqrt{3}/2$,
$f(5\pi/3) = 5\pi/3 + \sin(10\pi/3) = 5\pi/3 - \sqrt{3}/2$,
$f(2\pi) = 2\pi$.

Maximum value $= 2\pi$ at $x = 2\pi$; minimum value $= 0$ at $x = 0$.

Answer: Maximum $= 2\pi$ at $x = 2\pi$; minimum $= 0$ at $x = 0$

Q13 Find two numbers whose sum is 24 and whose product is as large as possible.

SOLUTION

Let the two numbers be x and $(24 - x)$. Product $P = x(24-x) = 24x - x^2$.

$P'(x) = 24 - 2x = 0 \Rightarrow x = 12$.
$P''(x) = -2 < 0 \Rightarrow x = 12$ is a point of maximum.

The two numbers are 12 and 12, and maximum product = 144.

Answer: 12 and 12; maximum product = 144

Q14 Find two positive numbers x and y such that $x + y = 60$ and xy^3 is maximum.

SOLUTION

$y = 60 - x$. Let $P = x \cdot y^3 = x(60-x)^3$.

$P'(x) = (60-x)^3 + x \cdot 3(60-x)^2 \cdot (-1) = (60-x)^2[(60-x) - 3x] = (60-x)^2(60-4x)$.

$P'(x) = 0$ gives $x = 60$ (trivial) or $x = 15$.
$P''(15) < 0$, so $x = 15$ is a maximum.

$x = 15$, $y = 45$.

Answer: x = 15, y = 45

Q15 Find two positive numbers x and y such that their sum is 35 and the product $x^2 y^5$ is a maximum.

SOLUTION

$y = 35 - x$. $P = x^2(35-x)^5$.

$P'(x) = 2x(35-x)^5 + x^2 \cdot 5(35-x)^4 \cdot (-1) = x(35-x)^4[2(35-x) - 5x] = x(35-x)^4(70 - 7x)$. $= 7x(35-x)^4(10-x)$.

$P'(x) = 0$ at $x = 0, 10, 35$. For x in (0, 35): maximum at $x = 10$.

$x = 10$, $y = 25$.

Answer: x = 10, y = 25

Q16 Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.

SOLUTION

Let the numbers be x and (16–x). $S = x^3 + (16-x)^3$.

$S'(x) = 3x^2 - 3(16-x)^2 = 3(x^2 - 256 + 32x - x^2) = 3(32x - 256) = 96(x-8)$.

$S'(x) = 0$ at $x = 8$. $S''(x) = 96 > 0$ => minimum.

The two numbers are 8 and 8; sum of cubes = 2(512) = 1024.

Answer: 8 and 8; minimum sum of cubes = 1024

Q17 A square piece of tin of side 18 cm is to be made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible?

SOLUTION

Let the side of the cut square be x cm. Then the box has:
height = x, length = width = (18 – 2x).

$V(x) = x(18-2x)^2 = x(324 - 72x + 4x^2) = 4x^3 - 72x^2 + 324x$.

$V'(x) = 12x^2 - 144x + 324 = 12(x^2 - 12x + 27) = 12(x-3)(x-9)$.

$V'(x) = 0$ at $x = 3$ or $x = 9$. Since x must be less than 9 (so 18–2x > 0), we take x =

3.

V''(x) = 24x – 144. V''(3) = 72 – 144 = –72 < 0 => maximum at x = 3.

Maximum volume = V(3) = 3(12)² = 432 cm³.

Answer: Side of square to cut = 3 cm; maximum volume = 432 cm³

Q18 A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum?

SOLUTION

Let side of cut square = x. Then height = x, length = (45 – 2x), width = (24 – 2x). Constraint: 0 < x < 12.

V(x) = x(45–2x)(24–2x).

By the product rule across three factors:
V'(x) = (45–2x)(24–2x) – 2x(24–2x) – 2x(45–2x)
= (24–2x)(45–4x) – 2x(45–2x)
Expanding: = 1080 – 96x – 90x + 8x² – 90x + 4x² = 12x² – 276x + 1080 = 12(x² – 23x + 90).
= 12(x–5)(x–18).

V'(x) = 0 at x = 5 or x = 18. Since 0 < x < 12, we take x = 5.

V''(x) = 12(2x–23). V''(5) = 12(–13) = –156 < 0 => maximum at x = 5.

Maximum volume = V(5) = 5(35)(14) = 2450 cm³.

Answer: Side of square to cut = 5 cm; maximum volume = 2450 cm³

Q19 Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.

SOLUTION

Let the circle have radius r. A rectangle inscribed in the circle with one side 2a has other side 2b where a² + b² = r².

Area A = 4ab. Maximise A² = 16a² b² subject to a² + b² = r².

Let u = a², v = b². Maximise uv subject to u + v = r².
By AM–GM: uv ≤ (u+v)²/4 = r⁴/4, with equality when u = v, i.e., a = b.

So area is maximum when a = b, which means the rectangle is a square (all sides = a√2).

Answer: The rectangle of maximum area inscribed in a circle is the square (sides equal, a = b).

Q20 Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base.

SOLUTION

Let radius = r, height = h. Given surface area S = $2\pi r^2 + 2\pi rh$ (fixed).
So $h = (S - 2\pi r^2)/(2\pi r)$.

Volume $V = \pi r^2 h = \pi r^2 \cdot (S - 2\pi r^2)/(2\pi r) = r(S - 2\pi r^2)/2 = (Sr - 2\pi r^3)/2$.

$dV/dr = (S - 6\pi r^2)/2 = 0 \Rightarrow S = 6\pi r^2$.

But $S = 2\pi r^2 + 2\pi rh$, so $6\pi r^2 = 2\pi r^2 + 2\pi rh \Rightarrow 4\pi r^2 = 2\pi rh \Rightarrow h = 2r$.

$d^2V/dr^2 = -6\pi r < 0 \Rightarrow$ maximum. Hence at maximum volume, h = 2r = diameter.

Answer: At maximum volume h = 2r, so height = diameter of the base.

Q21 Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimetres, find the dimensions of the can which has the minimum surface area.

SOLUTION

Let radius = r, height = h. Volume: $\pi r^2 h = 100 \Rightarrow h = 100/(\pi r^2)$.

Surface area: $S = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 2\pi r \cdot 100/(\pi r^2) = 2\pi r^2 + 200/r$.

$dS/dr = 4\pi r - 200/r^2 = 0 \Rightarrow 4\pi r^3 = 200 \Rightarrow r^3 = 50/\pi \Rightarrow r = (50/\pi)^{(1/3)}$.

$d^2S/dr^2 = 4\pi + 400/r^3 > 0 \Rightarrow$ minimum surface area.

$h = 100/(\pi r^2)$. Substituting: $h/r = 100/(\pi r^3) = 100/(\pi \cdot 50/\pi) = 100/50 = 2 \Rightarrow h = 2r$.

Radius = $(50/\pi)^{(1/3)}$ cm; height = $2(50/\pi)^{(1/3)}$ cm (i.e., h = 2r).

Answer: $r = (50/\pi)^{(1/3)}$ cm, $h = 2r = 2(50/\pi)^{(1/3)}$ cm

Q22 A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?

SOLUTION

Let side of square = a, so perimeter = 4a. Remaining length = 28 – 4a = circumference of circle = 2πr => r = (28–4a)/(2π).

Total area A = $a^2 + \pi r^2 = a^2 + \pi\cdot(28–4a)^2/(4\pi^2) = a^2 + (28–4a)^2/(4\pi)$.

$dA/da = 2a + 2(28–4a)(–4)/(4\pi) = 2a – 2(28–4a)/\pi$.

Setting $dA/da = 0$: $2a = 2(28–4a)/\pi$ => $\pi a = 28–4a$ => $a(\pi+4) = 28$ => $a = 28/(\pi+4)$.

$d^2A/da^2 = 2 + 8/\pi > 0$ => minimum.

Length for square = $4a = 112/(\pi+4)$ m; length for circle = $28 – 112/(\pi+4) = 28\pi/(\pi+4)$ m.

Answer: Length for square = $112/(\pi+4)$ m; length for circle = $28\pi/(\pi+4)$ m

Q23 Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is 8/27 of the volume of the sphere.

SOLUTION

Let the cone have base radius r, height h. The sphere has radius R with centre O. The cone's apex is at the top of the sphere; the base is at distance (h – R) from the centre.
So $r^2 = R^2 – (h–R)^2 = 2Rh – h^2$.

Volume of cone: $V = (1/3)\pi r^2 h = (\pi/3)(2Rh – h^2)h = (\pi/3)(2Rh^2 – h^3)$.

$dV/dh = (\pi/3)(4Rh – 3h^2) = (\pi h/3)(4R – 3h) = 0$ => $h = 4R/3$.

$d^2V/dh^2 = (\pi/3)(4R – 6h)$. At $h = 4R/3$: $= (\pi/3)(4R – 8R) = –4\pi R/3 < 0$ => maximum.

At $h = 4R/3$: $r^2 = 2R(4R/3) – (4R/3)^2 = 8R^2/3 – 16R^2/9 = 24R^2/9 – 16R^2/9 = 8R^2/9$.

$V_{max} = (\pi/3)(8R^2/9)(4R/3) = (\pi/3)(32R^3/27) = 32\pi R^3/81$.

Volume of sphere = $(4/3)\pi R^3 = 108\pi R^3/81$.

Ratio = $V_{max} / V_{sphere} = 32/108 = 8/27$.

Hence the volume of the largest inscribed cone is 8/27 of the volume of the

sphere.

Answer: Maximum cone volume = $32\pi R^3/81$ = (8/27) of the sphere volume

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Q24 Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ times the radius of the base.

SOLUTION

Let radius = r, height = h, slant height $l = \sqrt{(r^2 + h^2)}$.
Curved surface area $S = \pi r l = \pi r\sqrt{(r^2 + h^2)}$.
Volume $V = (1/3)\pi r^2 h$ is fixed => $h = 3V/(\pi r^2)$.

$S^2 = \pi^2 r^2(r^2 + h^2) = \pi^2 r^2(r^2 + 9V^2/(\pi^2 r^4)) = \pi^2 r^4 + 9V^2/r^2$.

Let $f(r) = \pi^2 r^4 + 9V^2/r^2$.
$df/dr = 4\pi^2 r^3 - 18V^2/r^3 = 0 \Rightarrow 4\pi^2 r^6 = 18V^2 \Rightarrow r^6 = 9V^2/(2\pi^2)$.

Since $V = (1/3)\pi r^2 h$: $V^2 = \pi^2 r^4 h^2/9$.
$r^6 = 9\cdot\pi^2 r^4 h^2/(9\cdot 2\pi^2) = r^4 h^2/2 \Rightarrow r^2 = h^2/2 \Rightarrow h = r\sqrt{2}$.

Hence the altitude $h = \sqrt{2} \cdot r$.

Answer: Altitude $h = \sqrt{2} \cdot r$ (radius of base)

Q25 Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is tan^(–1) $\sqrt{2}$.

SOLUTION

Let the semi-vertical angle be α, slant height l (fixed). Then $r = l \sin\alpha$, $h = l \cos\alpha$.

$V = (1/3)\pi r^2 h = (1/3)\pi\, l^2 \sin^2\alpha \cdot l \cos\alpha = (\pi l^3/3) \sin^2\alpha \cos\alpha$.

$dV/d\alpha = (\pi l^3/3)(2 \sin\alpha \cos\alpha \cdot \cos\alpha + \sin^2\alpha \cdot (-\sin\alpha))$
$= (\pi l^3/3) \sin\alpha\,(2\cos^2\alpha - \sin^2\alpha) = 0$.

Since $\sin\alpha \neq 0$: $2\cos^2\alpha = \sin^2\alpha \Rightarrow \tan^2\alpha = 2 \Rightarrow \tan\alpha = \sqrt{2} \Rightarrow \alpha = \tan^{-1}(\sqrt{2})$.

$d^2V/d\alpha^2 < 0$ at this α, confirming maximum.

Answer: Semi-vertical angle $\alpha = \tan^{-1}(\sqrt{2})$

Q26 Show that semi-vertical angle of right circular cone of given surface area and maximum volume is sin^(–1)(1/3).

SOLUTION

Let radius = r, semi-vertical angle = α, slant height l. Surface area S = πrl + πr² = πr(l + r) (fixed).
r = l sin α, so l = r/sin α. Then S = πr(r/sin α + r) = πr²(1/sin α + 1) = πr²(1 + sin α)/sin α.
=> r² = S sin α / (π(1 + sin α)).

Height h = l cos α = r cos α/sin α = r cot α.
V = (1/3)πr² h = (1/3)πr² · r cot α = (1/3)πr³ cot α.

Substituting r²: V² proportional to r⁶ · cot² α ∝ [S sin α/(π(1+sin α))]³ · cot² α.

Maximising f(α) = sin³ α · cot² α / (1+sin α)³ = sin³ α · cos² α/(sin² α·(1+sin α)³) = sin α cos² α/(1+sin α)³.

df/dα = [cos³ α – 2 sin² α cos α](1+sin α)³ – sin α cos² α · 3(1+sin α)² cos α] / (1+sin α)⁶.
Setting numerator = 0 and dividing by cos α(1+sin α)²:
(cos² α – 2 sin² α)(1 + sin α) = 3 sin α cos² α.
(1 – sin² α – 2 sin² α)(1 + sin α) = 3 sin α(1 – sin² α).
(1 – 3 sin² α)(1 + sin α) = 3 sin α(1 – sin α)(1 + sin α).
Dividing by (1 + sin α): 1 – 3 sin² α = 3 sin α(1 – sin α) = 3 sin α – 3 sin² α.
1 = 3 sin α => sin α = 1/3 => α = sin^(–1)(1/3).

Answer: Semi-vertical angle = sin^(–1)(1/3)

Q27 The point on the curve x² = 2y which is nearest to the point (0, 5) is
(A) (2√2, 4) (B) (2√2, 0) (C) (0, 0) (D) (2, 2)

SOLUTION

Let (x, y) be on x² = 2y, so y = x²/2.
Distance squared from (0, 5): D² = x² + (x²/2 – 5)².

Let f(x) = x² + (x²/2 – 5)². f'(x) = 2x + 2(x²/2 – 5)·x = 2x(1 + x²/2 – 5) = 2x(x²/2 – 4) = x(x² – 8).

f'(x) = 0 at x = 0 or x = ±2√2.
f''(x) = x² – 8 + 2x² = 3x² – 8. At x = ±2√2: f'' = 24 – 8 = 16 > 0 => minimum.

At x = 2√2: y = (2√2)²/2 = 8/2 = 4. Point: (2√2, 4).

Answer: (A) (2√2, 4)

Q28 For all real values of x, the minimum value of $(1 – x + x^2)/(1 + x + x^2)$ is
(A) 0 (B) 1 (C) 3 (D) 1/3

SOLUTION

Let $f(x) = (1 – x + x^2)/(1 + x + x^2)$.

$f'(x) = [(–1 + 2x)(1+x+x^2) – (1–x+x^2)(1+2x)] / (1+x+x^2)^2$.

Numerator $= (–1+2x)(1+x+x^2) – (1–x+x^2)(1+2x)$.
Expanding: $(–1–x–x^2+2x+2x^2+2x^3) – (1+2x–x–2x^2+x^2+2x^3)$
$= (–1+x+x^2+2x^3) – (1+x–x^2+2x^3) = –1+x+x^2 – 1–x+x^2 = 2x^2 – 2$.

$f'(x) = 0 \Rightarrow 2x^2 – 2 = 0 \Rightarrow x = \pm 1$.
$f(1) = 1/3$, $f(–1) = 3$.

Minimum value = 1/3 at x = 1.

Answer: (D) 1/3

Q29 The maximum value of $[x(x – 1) + 1]^{(1/3)}$, $0 \le x \le 1$ is
(A) $(1/3)^{(1/3)}$ (B) 1/2 (C) 1 (D) 0

SOLUTION

Let $f(x) = [x(x–1) + 1]^{(1/3)} = (x^2 – x + 1)^{(1/3)}$.

$f'(x) = (1/3)(x^2–x+1)^{(–2/3)} \cdot (2x–1) = 0$ at $x = 1/2$.

$f(0) = (0 – 0 + 1)^{(1/3)} = 1$.
$f(1/2) = (1/4 – 1/2 + 1)^{(1/3)} = (3/4)^{(1/3)} < 1$.
$f(1) = (1 – 1 + 1)^{(1/3)} = 1$.

Maximum value = 1, attained at x = 0 and x = 1.

Answer: (C) 1

Misc — Miscellaneous Exercise

Q1 Show that the function given by $f(x) = (\log x)/x$ has maximum at $x = e$.

SOLUTION

$f(x) = (\log x)/x$, domain $x > 0$.

$f'(x) = [1/x \cdot x - \log x \cdot 1] / x^2 = (1 - \log x)/x^2$.

$f'(x) = 0 \Rightarrow \log x = 1 \Rightarrow x = e$.

$f''(x) = [-(1/x)\cdot x^2 - (1-\log x)\cdot 2x] / x^4 = [-x - 2x(1-\log x)] / x^4 = (-3 + 2\log x) / x^3$.

$f''(e) = (-3 + 2)/e^3 = -1/e^3 < 0 \Rightarrow x = e$ is a point of local maximum.

Since this is the only critical point and $f \to 0$ as $x \to 0^+$ and $x \to \infty$, it is also the global maximum.
Maximum value = $f(e) = (\log e)/e = 1/e$.

Answer: $f'(e) = 0$ and $f''(e) = -1/e^3 < 0$, so f has maximum at $x = e$ with maximum value $1/e$.

Q2 The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 cm per second. How fast is the area decreasing when the two equal sides are equal to the base?

SOLUTION

Let the equal sides have length a and the base be b (fixed).
Height $h = \sqrt{(a^2 - b^2/4)}$.

Area $A = (1/2) \cdot b \cdot h = (b/2)\sqrt{(a^2 - b^2/4)}$.

$dA/da = (b/2) \cdot a/\sqrt{(a^2 - b^2/4)}$.

When $a = b$: $\sqrt{(b^2 - b^2/4)} = \sqrt{(3b^2/4)} = b\sqrt{3}/2$.
$dA/da|_{(a=b)} = (b/2) \cdot b / (b\sqrt{3}/2) = b/\sqrt{3}$.

$dA/dt = (dA/da)(da/dt) = (b/\sqrt{3})(-3) = -3b/\sqrt{3} = -\sqrt{3}\ b$ cm²/s.

The area is decreasing at the rate of $\sqrt{3}\ b$ cm²/s.

Answer: Area is decreasing at $\sqrt{3}\ b$ cm²/s

Q3 Find the intervals in which the function f given by $f(x) = (4 \sin x – 2x – x \cos x)/(2 + \cos x)$ is (i) increasing (ii) decreasing.

SOLUTION

$f(x) = (4 \sin x – 2x – x \cos x)/(2 + \cos x)$.

Using the quotient rule:
Numerator' = $4 \cos x – 2 – \cos x + x \sin x = 3 \cos x – 2 + x \sin x$.
(Denominator)' = $–\sin x$.

$f'(x) = [(3 \cos x – 2 + x \sin x)(2 + \cos x) – (4 \sin x – 2x – x \cos x)(–\sin x)] / (2 + \cos x)^2$.

Expanding the numerator:
$(3 \cos x – 2 + x \sin x)(2 + \cos x) + \sin x(4 \sin x – 2x – x \cos x)$
$= 6 \cos x + 3 \cos^2 x – 4 – 2 \cos x + 2x \sin x + x \sin x \cos x + 4 \sin^2 x – 2x \sin x – x \sin x \cos x$
$= 4 \cos x + 3 \cos^2 x + 4 \sin^2 x – 4$
$= 4 \cos x + 3 \cos^2 x + 4(1 – \cos^2 x) – 4$
$= 4 \cos x – \cos^2 x$
$= \cos x(4 – \cos x)$.

So $f'(x) = \cos x(4 – \cos x)/(2 + \cos x)^2$.

Since $4 – \cos x \geq 3 > 0$ and $(2 + \cos x)^2 > 0$, sign of f'(x) = sign of cos x.

(i) $f'(x) > 0$ when $\cos x > 0$, i.e., $x \in (0, \pi/2) \cup (3\pi/2, 2\pi)$ in one period.
(ii) $f'(x) < 0$ when $\cos x < 0$, i.e., $x \in (\pi/2, 3\pi/2)$ in one period.

Answer: (i) Increasing on $(0, \pi/2) \cup (3\pi/2, 2\pi)$ (ii) Decreasing on $(\pi/2, 3\pi/2)$

Q4 Find the intervals in which the function f given by $f(x) = x^3 + 1/x^3$, $x \neq 0$ is (i) increasing (ii) decreasing.

SOLUTION

$f'(x) = 3x^2 – 3/x^4 = 3(x^6 – 1)/x^4$.

$f'(x) = 0$ when $x^6 = 1$, i.e., $x = \pm 1$.
$f'(x) > 0$ when $x^6 > 1$, i.e., $x < –1$ or $x > 1$.
$f'(x) < 0$ when $x^6 < 1$, i.e., $–1 < x < 0$ or $0 < x < 1$.

(i) f is increasing on $(–\infty, –1)$ and $(1, \infty)$.
(ii) f is decreasing on $(–1, 0)$ and $(0, 1)$.

Answer: (i) Increasing on $(-\infty, -1) \cup (1, \infty)$ (ii) Decreasing on $(-1, 0) \cup (0, 1)$

Q5 Find the maximum area of an isosceles triangle inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$ with its vertex at one end of the major axis.

SOLUTION

Let the vertex be at (a, 0) (right end of major axis). A general point on the ellipse is (a cos t, b sin t).
The other two vertices are symmetric: (a cos t, b sin t) and (a cos t, –b sin t).

Base = 2b sin t (vertical), height = a – a cos t = a(1 – cos t).
Area A = (1/2)(2b sin t)(a(1 – cos t)) = ab sin t(1 – cos t).

dA/dt = ab[cos t(1 – cos t) + sin t · sin t] = ab[cos t – cos² t + sin² t]
= ab[cos t – cos² t + 1 – cos² t] = ab[1 + cos t – 2 cos² t].

Setting dA/dt = 0: 2 cos² t – cos t – 1 = 0 => (2 cos t + 1)(cos t – 1) = 0.
cos t = –1/2 (since cos t = 1 gives A = 0) => t = 2π/3.

At t = 2π/3: sin t = √3/2, 1 – cos t = 3/2.
A_max = ab · (√3/2)(3/2) = 3√3 ab/4.

Answer: Maximum area = $3\sqrt{3}\, ab/4$

Q6 A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m³. If building of tank costs Rs 70 per sq metre for the base and Rs 45 per square metre for sides. What is the cost of least expensive tank?

SOLUTION

Let the base dimensions be x m × y m. Depth = 2 m. Volume: 2xy = 8 => xy = 4 => y = 4/x.

Cost = 70(xy) + 45[2(2x) + 2(2y)] = 70(4) + 45·4(x + y) = 280 + 180(x + 4/x).

Let C(x) = 280 + 180x + 720/x.
C'(x) = 180 – 720/x² = 0 => x² = 4 => x = 2 (taking positive value).

C''(x) = 1440/x³ > 0 => minimum at x = 2.

At x = 2: y = 4/2 = 2.
C(2) = 280 + 180(2) + 720/2 = 280 + 360 + 360 = 1000.

The cost of the least expensive tank is Rs 1000.

Answer: Rs 1000 (dimensions: 2 m × 2 m base, depth 2 m)

Q7 The sum of the perimeter of a circle and square is k, where k is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle.

SOLUTION

Let radius of circle = r, side of square = s. Then $2\pi r + 4s = k \Rightarrow s = (k - 2\pi r)/4$.

Total area $A = \pi r^2 + s^2 = \pi r^2 + [(k - 2\pi r)/4]^2$.

$dA/dr = 2\pi r + 2\cdot(k-2\pi r)/4\cdot(-2\pi/4) = 2\pi r - \pi(k-2\pi r)/4$.

Setting $dA/dr = 0$: $2\pi r = \pi(k-2\pi r)/4 \Rightarrow 8r = k - 2\pi r \Rightarrow r(8+2\pi) = k \Rightarrow r = k/(2\pi+8)$.

At this r: $s = (k - 2\pi r)/4 = [k - 2\pi k/(2\pi+8)]/4 = k[1 - 2\pi/(2\pi+8)]/4 = k\cdot 8/[(2\pi+8)\cdot 4] = 2k/(2\pi+8) = 2r$.

So s = 2r, i.e., side of square = 2 × radius of circle.

$d^2A/dr^2 = 2\pi + \pi^2/2 > 0 \Rightarrow$ minimum.

Answer: Sum of areas is minimum when s = 2r (side of square = twice the radius of the circle).

Q8 A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

SOLUTION

Let width = 2r, height of rectangle = h. The semicircle has radius r.
Perimeter $= 2h + 2r + \pi r = 10 \Rightarrow h = (10 - 2r - \pi r)/2 = (10 - r(2+\pi))/2$.

Area (light admitted) $= 2rh + (1/2)\pi r^2$.
$A = 2r\cdot(10 - r(2+\pi))/2 + \pi r^2/2 = r(10 - r(2+\pi)) + \pi r^2/2$
$= 10r - r^2(2+\pi) + \pi r^2/2 = 10r - 2r^2 - \pi r^2 + \pi r^2/2 = 10r - 2r^2 - \pi r^2/2$.

$dA/dr = 10 - 4r - \pi r = 10 - r(4+\pi) = 0 \Rightarrow r = 10/(4+\pi)$.

$d^2A/dr^2 = -(4+\pi) < 0 \Rightarrow$ maximum.

$h = (10 - r(2+\pi))/2 = [10 - 10(2+\pi)/(4+\pi)]/2 = [10\cdot(4+\pi-2-\pi)/(4+\pi)]/2 = [10\cdot 2/(4+\pi)]/2 = 10/(4+\pi) = r$.

Width $= 2r = 20/(4+\pi)$; height $h = 10/(4+\pi) = r$.

Answer: Width = 20/(π+4) m, height = 10/(π+4) m

Q9 A point on the hypotenuse of a triangle is at distance a and b from the sides of the triangle. Show that the minimum length of the hypotenuse is (a^(2/3) + b^(2/3))^(3/2).

SOLUTION

Let the right angle be at C, with sides CA and CB. Let P be a point on the hypotenuse AB.
Let angle at A = θ. Then AP = a/sin θ (since the perpendicular from P to CA has length a = AP sin θ) and PB = b/cos θ.

Length AB = AP + PB = a/sin θ + b/cos θ = a csc θ + b sec θ.

dL/dθ = –a cos θ/sin² θ + b sin θ/cos² θ = 0.
=> b sin³ θ = a cos³ θ => tan³ θ = a/b => tan θ = (a/b)^(1/3).

At this angle: sin θ = a^(1/3)/√(a^(2/3)+b^(2/3)), cos θ = b^(1/3)/√(a^(2/3)+b^(2/3)).

L_min = a · √(a^(2/3)+b^(2/3))/a^(1/3) + b · √(a^(2/3)+b^(2/3))/b^(1/3)
= √(a^(2/3)+b^(2/3)) · (a^(2/3) + b^(2/3))
= (a^(2/3) + b^(2/3))^(3/2).

Answer: Minimum hypotenuse length = (a^(2/3) + b^(2/3))^(3/2)

Q10 Find the points at which the function f given by f(x) = (x – 2)⁴ (x + 1)³ has (i) local maxima (ii) local minima (iii) point of inflexion

SOLUTION

f'(x) = 4(x–2)³(x+1)³ + 3(x–2)⁴(x+1)² = (x+1)²(x–2)³[4(x+1) + 3(x–2)]
= (x+1)²(x–2)³(7x–2).

Critical points: x = –1, x = 2/7, x = 2.

Sign analysis (note (x+1)² ≥ 0):
x < 2/7 (excluding x = –1): (x–2)³ < 0, (7x–2) < 0 => f'(x) > 0 (positive × negative × negative).
2/7 < x < 2: (x–2)³ < 0, (7x–2) > 0 => f'(x) < 0.
x > 2: (x–2)³ > 0, (7x–2) > 0 => f'(x) > 0.

(i) f' changes + to – at x = 2/7 => local maximum at x = 2/7.

(ii) f' changes – to + at x = 2 => local minimum at x = 2, f(2) = 0.
(iii) At x = –1: f' does not change sign (equals 0 but $(x+1)^2$ makes it touch zero) => point of inflexion at x = –1.

Answer: (i) Local max at x = 2/7 (ii) Local min at x = 2 (value = 0) (iii) Point of inflexion at x = –1

Q11 Find the absolute maximum and minimum values of the function f given by $f(x) = \cos^2 x + \sin x$, $x \in [0, \pi]$.

SOLUTION

$f(x) = \cos^2 x + \sin x = 1 - \sin^2 x + \sin x$.

$f'(x) = -2 \sin x \cos x + \cos x = \cos x(1 - 2 \sin x)$.

$f'(x) = 0$ when $\cos x = 0 \Rightarrow x = \pi/2$, or $\sin x = 1/2 \Rightarrow x = \pi/6, 5\pi/6$.

Evaluate at critical points and endpoints:
$f(0) = 1 + 0 = 1$,
$f(\pi/6) = \cos^2(\pi/6) + \sin(\pi/6) = 3/4 + 1/2 = 5/4$,
$f(\pi/2) = 0 + 1 = 1$,
$f(5\pi/6) = \cos^2(5\pi/6) + \sin(5\pi/6) = 3/4 + 1/2 = 5/4$,
$f(\pi) = 1 + 0 = 1$.

Absolute maximum = 5/4 at $x = \pi/6$ and $x = 5\pi/6$.
Absolute minimum = 1 at $x = 0, \pi/2$, and π.

Answer: Absolute maximum = 5/4 (at $x = \pi/6$ and $5\pi/6$); absolute minimum = 1 (at $x = 0, \pi/2, \pi$)

Q12 Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is 4r/3.

SOLUTION

Let the cone have base radius R, height h. With sphere of radius r centred at O: The base is at distance (h – r) from centre, so $R^2 = r^2 - (h-r)^2 = 2rh - h^2$.

Volume $V = (1/3)\pi R^2 h = (\pi/3)(2rh - h^2)h = (\pi/3)(2rh^2 - h^3)$.

$dV/dh = (\pi/3)(4rh - 3h^2) = (\pi h/3)(4r - 3h) = 0$ (for $h > 0$).
$\Rightarrow h = 4r/3$.

$d^2V/dh^2 = (\pi/3)(4r - 6h)$. At $h = 4r/3$: $= (\pi/3)(4r - 8r) = -4\pi r/3 < 0 \Rightarrow$ maximum.

Altitude of cone of maximum volume = 4r/3.

Answer: Altitude = 4r/3

Q13 Let f be a function defined on [a, b] such that f'(x) > 0, for all x ∈ (a, b). Then prove that f is an increasing function on (a, b).

SOLUTION

Let $x_1, x_2 \in (a, b)$ with $x_1 < x_2$.

By the Mean Value Theorem, there exists $c \in (x_1, x_2) \subseteq (a, b)$ such that:
$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$.

Since f'(c) > 0 (given) and $x_2 - x_1 > 0$:
$f(x_2) - f(x_1) > 0$, i.e., $f(x_2) > f(x_1)$.

Since $x_1 < x_2$ implies $f(x_1) < f(x_2)$ for all $x_1, x_2 \in (a, b)$, f is increasing on (a, b).

Answer: By MVT: $f(x_2) - f(x_1) = f'(c)(x_2 - x_1) > 0$ for $x_1 < x_2$, so f is increasing.

Q14 Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $2R/\sqrt{3}$. Also find the maximum volume.

SOLUTION

Let the cylinder have radius r and height h. Then $r^2 + (h/2)^2 = R^2 \Rightarrow r^2 = R^2 - h^2/4$.

$V = \pi r^2 h = \pi(R^2 - h^2/4)h = \pi(R^2 h - h^3/4)$.

$dV/dh = \pi(R^2 - 3h^2/4) = 0 \Rightarrow h^2 = 4R^2/3 \Rightarrow h = 2R/\sqrt{3}$.

$d^2V/dh^2 = -3\pi h/2 < 0 \Rightarrow$ maximum.

At $h = 2R/\sqrt{3}$: $r^2 = R^2 - R^2/3 = 2R^2/3$.
$V_{max} = \pi(2R^2/3)(2R/\sqrt{3}) = 4\pi R^3/(3\sqrt{3}) = 4\sqrt{3}\,\pi R^3/9$.

Answer: Height = $2R/\sqrt{3}$; maximum volume = $4\sqrt{3}\,\pi R^3/9$

Q15 Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height h and semi vertical angle α is one-third that of the cone and the greatest volume of cylinder is $(4/27)\pi h^3 \tan^2 \alpha$.

SOLUTION

Let the cone have height h, semi-vertical angle α. Base radius of cone = h tan α. Let cylinder have radius r and height H.

By similar triangles: $H/h = (h \tan \alpha - r)/(h \tan \alpha) \Rightarrow H = h(1 - r/(h \tan \alpha)) = h - r/\tan \alpha = h - r \cot \alpha$.

$V = \pi r^2 H = \pi r^2(h - r \cot \alpha) = \pi(hr^2 - r^3 \cot \alpha)$.

$dV/dr = \pi(2hr - 3r^2 \cot \alpha) = \pi r(2h - 3r \cot \alpha) = 0$.

$r \neq 0$: $r = 2h \tan \alpha/3$.

$H = h - (2h \tan \alpha/3) \cot \alpha = h - 2h/3 = h/3$.

$d^2V/dr^2 = \pi(2h - 6r \cot \alpha)$. At $r = 2h \tan \alpha/3$: $= \pi(2h - 4h) = -2\pi h < 0 \Rightarrow$ maximum.

$V_{max} = \pi(2h \tan \alpha/3)^2 \cdot (h/3) = \pi \cdot 4h^2 \tan^2 \alpha/9 \cdot h/3 = 4\pi h^3 \tan^2 \alpha/27$.

Answer: Cylinder height = h/3; maximum volume = $(4/27)\pi h^3 \tan^2 \alpha$

Q16 A cylindrical tank of radius 10 m is being filled with wheat at the rate of 314 cubic metre per hour. Then the depth of the wheat is increasing at the rate of
(A) 1 m/h (B) 0.1 m/h (C) 1.1 m/h (D) 0.5 m/h

SOLUTION

$V = \pi r^2 h = \pi(10)^2 h = 100\pi h$.

$dV/dt = 100\pi(dh/dt)$.

Given $dV/dt = 314$ m³/h:

$314 = 100\pi(dh/dt) \Rightarrow dh/dt = 314/(100\pi) \approx 314/314.16 \approx 1$ m/h.

Answer: (A) 1 m/h