Chapter 7: Integrals

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Exercise 7.1 — Integration by Inspection

Q1 Find the integral: $\int \sin 2x\, dx$

SOLUTION

We know $d/dx(\cos 2x) = -2 \sin 2x$, so $d/dx(-\cos 2x/2) = \sin 2x$.
Therefore $\int \sin 2x\, dx = -\cos 2x/2 + C$

Answer: $-(1/2) \cos 2x + C$

Q2 Find the integral: $\int \cos 3x\, dx$

SOLUTION

We know $d/dx(\sin 3x) = 3 \cos 3x$, so $d/dx(\sin 3x/3) = \cos 3x$.
Therefore $\int \cos 3x\, dx = \sin 3x/3 + C$

Answer: $(1/3) \sin 3x + C$

Q3 Find the integral: $\int e^{2x}\,dx$

SOLUTION

We know $d/dx(e^{2x}) = 2e^{2x}$, so $d/dx(e^{2x}/2) = e^{2x}$.
Therefore $\int e^{2x}\,dx = e^{2x}/2 + C$

Answer: $(1/2)\,e^{2x} + C$

Q4 Find the integral: $\int (ax + b)^2\,dx$

SOLUTION

We know $d/dx[(ax+b)^3] = 3a(ax+b)^2$, so $d/dx[(ax+b)^3/(3a)] = (ax+b)^2$.
Therefore $\int (ax + b)^2\,dx = (ax+b)^3/(3a) + C$

Answer: $(ax + b)^3 / (3a) + C$

Q5 Find the integral: $\int \sin 2x - 4e^{3x}\,dx$

SOLUTION

$\int \sin 2x\,dx = -\cos 2x/2$, and $\int e^{3x}\,dx = e^{3x}/3$.
$\int (\sin 2x - 4e^{3x})\,dx = -\cos 2x/2 - 4e^{3x}/3 + C$

Answer: $-(\cos 2x)/2 - (4/3)e^{3x} + C$

Q6 Find the integral: $\int (4e^{3x} + 1)\,dx$

SOLUTION

$\int 4e^{3x}\,dx = 4e^{3x}/3$ and $\int 1\,dx = x$.
Therefore $\int (4e^{3x} + 1)\,dx = 4e^{3x}/3 + x + C$

Answer: $(4/3)e^{3x} + x + C$

Q7 Find the integral: $\int x^2\,(1 - 1/x^2)\,dx$

SOLUTION

Expand: $x^2(1 - 1/x^2) = x^2 - 1$.
$\int (x^2 - 1)\,dx = x^3/3 - x + C$

Answer: $x^3/3 - x + C$

Q8 Find the integral: $\int (ax^2 + bx + c)\,dx$

SOLUTION

Integrate term by term:
$\int ax^2\,dx = ax^3/3$, $\int bx\,dx = bx^2/2$, $\int c\,dx = cx$.
Therefore the integral = $ax^3/3 + bx^2/2 + cx + C$

Answer: $ax^3/3 + bx^2/2 + cx + C$

Q9 Find the integral: $\int (2x^2 + e^x)\,dx$

SOLUTION

$\int 2x^2\,dx = 2x^3/3$ and $\int e^x\,dx = e^x$.
Therefore the integral = $2x^3/3 + e^x + C$

Answer: $(2/3)x^3 + e^x + C$

Q10 Find the integral: $\int (\sqrt{x} - 1/\sqrt{x})^2\,dx$

SOLUTION

Expand: $(\sqrt{x} - 1/\sqrt{x})^2 = x - 2 + 1/x$.
$\int (x - 2 + 1/x)\,dx = x^2/2 - 2x + \log|x| + C$

Answer: $x^2/2 - 2x + \log|x| + C$

Q11 Find the integral: $\int (x^3 + 5x^2 - 4) / x^2\,dx$

SOLUTION

Divide each term by x^2: $x + 5 - 4/x^2$.
$\int (x + 5 - 4x^{-2})\,dx = x^2/2 + 5x + 4/x + C$

Answer: $x^2/2 + 5x + 4/x + C$

Q12 Find the integral: $\int (x^3 + 3x + 4) / \sqrt{x}\,dx$

SOLUTION

Divide each term by $\sqrt{x} = x^{(1/2)}$:
$x^{(5/2)} + 3x^{(1/2)} + 4x^{(-1/2)}$.
$\int = 2x^{(7/2)}/7 + 2x^{(3/2)} + 8x^{(1/2)} + C$

Answer: (2/7)x^(7/2) + 2x^(3/2) + 8√x + C

Q13 Find the integral: ∫ (x³ − x² + x − 1) / (x − 1) dx

SOLUTION

Factor numerator: x³ − x² + x − 1 = x²(x−1) + (x−1) = (x−1)(x²+1).
So integrand = x² + 1.
∫ (x² + 1) dx = x³/3 + x + C

Answer: x³/3 + x + C

Q14 Find the integral: ∫ (1 − x) √x dx

SOLUTION

Expand: (1−x)√x = x^(1/2) − x^(3/2).
∫ (x^(1/2) − x^(3/2)) dx = 2x^(3/2)/3 − 2x^(5/2)/5 + C

Answer: (2/3)x^(3/2) − (2/5)x^(5/2) + C

Q15 Find the integral: ∫ √x (3x² + 2x + 3) dx

SOLUTION

Multiply each term by x^(1/2):
3x^(5/2) + 2x^(3/2) + 3x^(1/2).
∫ = 3 · 2x^(7/2)/7 + 2 · 2x^(5/2)/5 + 3 · 2x^(3/2)/3 + C
= 6x^(7/2)/7 + 4x^(5/2)/5 + 2x^(3/2) + C

Answer: (6/7)x^(7/2) + (4/5)x^(5/2) + 2x^(3/2) + C

Q16 Find the integral: ∫ (2x − 3 cos x + eˣ) dx

SOLUTION

∫ 2x dx = x², ∫ cos x dx = sin x, ∫ eˣ dx = eˣ.
∫ (2x − 3 cos x + eˣ) dx = x² − 3 sin x + eˣ + C

Answer: x² − 3 sin x + eˣ + C

Q17 Find the integral: $\int (2x^2 - 3\sin x + 5\sqrt{x})\,dx$

SOLUTION

$\int 2x^2\,dx = 2x^3/3$, $\int 3\sin x\,dx = -3\cos x$, $\int 5\sqrt{x}\,dx = 10x^{(3/2)}/3$.

Result = $2x^3/3 + 3\cos x + 10x^{(3/2)}/3 + C$

Answer: $(2/3)x^3 + 3\cos x + (10/3)x^{(3/2)} + C$

Q18 Find the integral: $\int \sec x(\sec x + \tan x)\,dx$

SOLUTION

Expand: $\sec^2 x + \sec x \tan x$.

$\int \sec^2 x\,dx = \tan x$, $\int \sec x \tan x\,dx = \sec x$.

Result = $\tan x + \sec x + C$

Answer: $\tan x + \sec x + C$

Q19 Find the integral: $\int (\sec^2 x) / \operatorname{cosec}^2 x\,dx$

SOLUTION

$\sec^2 x / \operatorname{cosec}^2 x = (1/\cos^2 x) \cdot \sin^2 x = \tan^2 x = \sec^2 x - 1$.

$\int (\sec^2 x - 1)\,dx = \tan x - x + C$

Answer: $\tan x - x + C$

Q20 Find the integral: $\int (2 - 3\sin x) / \cos^2 x\,dx$

SOLUTION

Rewrite: $2/\cos^2 x - 3\sin x/\cos^2 x = 2\sec^2 x - 3\sec x \tan x$.

$\int = 2\tan x - 3\sec x + C$

Answer: $2\tan x - 3\sec x + C$

Q21 Choose the correct answer: The anti derivative of (√x + 1/√x) equals (A) (1/3)x^(1/3) + 2x^(1/2) + C (B) (2/3)x^(2/3) + (1/2)x^(1/2) + C (C) (2/3)x^(3/2) + 2x^(1/2) + C (D) (3/2)x^(3/2) + (1/2)x^(1/2) + C

SOLUTION

Write the integrand with fractional powers: √x + 1/√x = x^(1/2) + x^(−1/2).
∫ x^(1/2) dx = x^(3/2)/(3/2) = (2/3)x^(3/2).
∫ x^(−1/2) dx = x^(1/2)/(1/2) = 2x^(1/2).
Therefore the anti derivative is (2/3)x^(3/2) + 2x^(1/2) + C, which is option (C).

Answer: Answer (C): (2/3)x^(3/2) + 2x^(1/2) + C

Q22 Choose the correct answer: If d/dx f(x) = 4x³ − 3/x⁴ such that f(2) = 0, then f(x) is (A) x⁴ + 1/x³ − 129/8 (B) x³ + 1/x⁴ + 129/8 (C) x⁴ + 1/x³ + 129/8 (D) x³ + 1/x⁴ − 129/8

SOLUTION

Integrate: f(x) = ∫ (4x³ − 3x⁻⁴) dx = x⁴ + x⁻³ + C.
Apply f(2) = 0: 16 + 1/8 + C = 0 → C = −129/8.
Therefore f(x) = x⁴ + 1/x³ − 129/8.

Answer: Answer (A): f(x) = x⁴ + 1/x³ − 129/8

Exercise 7.2 — Integration by Substitution

Q1 Find the integral: $\int 2x / (1 + x^2)\, dx$

SOLUTION

Let $t = 1 + x^2$, then $dt = 2x\, dx$.

$\int dt/t = \log|t| + C = \log(1 + x^2) + C$

Answer: $\log(1 + x^2) + C$

Q2 Find the integral: $\int (\log x)^2 / x\, dx$

SOLUTION

Let $t = \log x$, then $dt = dx/x$.

$\int t^2\, dt = t^3/3 + C = (\log x)^3/3 + C$

Answer: $(\log x)^3/3 + C$

Q3 Find the integral: $\int 1 / (x + x \log x)\, dx$

SOLUTION

Factor: $1/(x(1 + \log x))$.

Let $t = 1 + \log x$, then $dt = dx/x$.

$\int dt/t = \log|t| + C = \log|1 + \log x| + C$

Answer: $\log|1 + \log x| + C$

Q4 Find the integral: $\int \sin x \cdot \sin(\cos x)\, dx$

SOLUTION

Let $t = \cos x$, then $dt = -\sin x\, dx$.

$\int \sin t \cdot (-dt) = \cos t + C = \cos(\cos x) + C$

Answer: $\cos(\cos x) + C$

Q5 Find the integral: $\int \sin(ax + b)\cos(ax + b)\,dx$

SOLUTION

Write as $(1/2)\sin(2ax + 2b)$.

Let $t = 2ax + 2b$, $dt = 2a\,dx$.

$\int (1/2)\sin t \cdot dt/(2a) = -\cos t/(4a) + C = -\cos(2ax + 2b)/(4a) + C$

Answer: $-\cos(2ax + 2b)/(4a) + C$ [equivalently: $\sin^2(ax + b)/(2a) + C$]

Q6 Find the integral: $\int \sqrt{(ax + b)}\,dx$

SOLUTION

Let $t = ax + b$, $dt = a\,dx$.

$\int t^{(1/2)}\,dt/a = (2t^{(3/2)})/(3a) + C = 2(ax+b)^{(3/2)}/(3a) + C$

Answer: $(2/3a)(ax + b)^{(3/2)} + C$

Q7 Find the integral: $\int x\sqrt{(x + 2)}\,dx$

SOLUTION

Let $t = x + 2$, so $x = t - 2$, $dx = dt$.

$\int (t-2)\sqrt{t}\,dt = \int (t^{(3/2)} - 2t^{(1/2)})\,dt$

$= 2t^{(5/2)}/5 - 4t^{(3/2)}/3 + C$

$= (2/5)(x+2)^{(5/2)} - (4/3)(x+2)^{(3/2)} + C$

Answer: $(2/5)(x+2)^{(5/2)} - (4/3)(x+2)^{(3/2)} + C$

Q8 Find the integral: $\int x\sqrt{(1 + 2x^2)}\,dx$

SOLUTION

Let $t = 1 + 2x^2$, $dt = 4x\,dx$, so $x\,dx = dt/4$.

$\int \sqrt{t} \cdot dt/4 = (1/4) \cdot (2t^{(3/2)}/3) + C = t^{(3/2)}/6 + C$

$= (1 + 2x^2)^{(3/2)}/6 + C$

Answer: $(1 + 2x^2)^{(3/2)}/6 + C$

Q9 Find the integral: $\int (4x + 2)\sqrt{(x^2 + x + 1)}\,dx$

SOLUTION

Note: d/dx(x² + x + 1) = 2x + 1, and 4x + 2 = 2(2x + 1).
Let t = x² + x + 1, dt = (2x + 1) dx.
$\int 2\sqrt{t}\,dt = (4/3)t^{(3/2)} + C = (4/3)(x^2 + x + 1)^{(3/2)} + C$

Answer: $(4/3)(x^2 + x + 1)^{(3/2)} + C$

Q10 Find the integral: $\int 1 / (x - \sqrt{x})\,dx$

SOLUTION

Factor: $1/(\sqrt{x}(\sqrt{x} - 1))$. Let t = √x − 1, dt = dx/(2√x).
$\int 2dt/t = 2\log|t| + C = 2\log|\sqrt{x} - 1| + C$

Answer: $2\log|\sqrt{x} - 1| + C$

Q11 Find the integral: $\int x / \sqrt{(x + 4)}\,dx$ (x > 0)

SOLUTION

Let t = x + 4, x = t − 4, dx = dt.
$\int (t-4)/\sqrt{t}\,dt = \int (t^{(1/2)} - 4t^{(-1/2)})\,dt$
$= (2/3)t^{(3/2)} - 8t^{(1/2)} + C$
$= (2/3)(x+4)^{(3/2)} - 8\sqrt{(x+4)} + C$

Answer: $(2/3)(x+4)^{(3/2)} - 8\sqrt{(x+4)} + C$

Q12 Find the integral: $\int (x^3 - 1)^{(1/3)} x^5\,dx$

SOLUTION

Write $x^5 = x^3 \cdot x^2$, and let t = x³ − 1, dt = 3x² dx.
$x^3 = t + 1$, so $\int t^{(1/3)}(t+1) \cdot dt/3 = (1/3)\int (t^{(4/3)} + t^{(1/3)})\,dt$
$= (1/3)[3t^{(7/3)}/7 + 3t^{(4/3)}/4] + C$
$= (x^3-1)^{(7/3)}/7 + (x^3-1)^{(4/3)}/4 + C$

Answer: $(x^3-1)^{(7/3)}/7 + (x^3-1)^{(4/3)}/4 + C$

Q13 Find the integral: $\int x^2 / (2 + 3x^3)^3 \, dx$

SOLUTION

Let $t = 2 + 3x^3$, $dt = 9x^2 \, dx$, so $x^2 \, dx = dt/9$.

$\int t^{-3} \, dt/9 = (1/9) \cdot t^{-2}/(-2) + C = -1/(18(2 + 3x^3)^2) + C$

Answer: $-1 / [18(2 + 3x^3)^2] + C$

Q14 Find the integral: $\int 1 / (x(\log x)^m) \, dx, \ (m \neq 1)$

SOLUTION

Let $t = \log x$, $dt = dx/x$.

$\int t^{(-m)} \, dt = t^{(1-m)}/(1-m) + C = (\log x)^{(1-m)}/(1-m) + C$

Answer: $(\log x)^{(1-m)} / (1-m) + C$

Q15 Find the integral: $\int x / (9 - 4x^2) \, dx$

SOLUTION

Let $t = 9 - 4x^2$, $dt = -8x \, dx$, so $x \, dx = -dt/8$.

$\int (-dt/8)/t = -(1/8) \log|t| + C = -(1/8) \log|9 - 4x^2| + C$

Answer: $-(1/8) \log|9 - 4x^2| + C$

Q16 Find the integral: $\int e^{2x+3} \, dx$

SOLUTION

Let $t = 2x + 3$, $dt = 2 \, dx$.

$\int e \, dt/2 = e/2 + C = e^{(2x+3)}/2 + C$

Answer: $e^{(2x+3)}/2 + C$

Q17 Find the integral: $\int x / e^{(x^2)} \, dx$

SOLUTION

Let $t = -x^2$, $dt = -2x \, dx$, so $x \, dx = -dt/2$.

$\int e \, (-dt/2) = -e/2 + C = -e^{(-x^2)}/2 + C$

Answer: $-(1/2)e^{(-x^2)} + C$

Q18 Find the integral: $\int e^{(\tan^{-1} x)} / (1 + x^2)\, dx$

SOLUTION

Let $t = \tan^{-1} x$, so $dt = 1/(1 + x^2)\, dx$.

$\int e^{(\tan^{-1} x)} \cdot 1/(1+x^2)\, dx = \int e\, dt = e + C = e^{(\tan^{-1} x)} + C$

Answer: $e^{(\tan^{-1} x)} + C$

Q19 Find the integral: $\int (e^{(2x)} - 1) / (e^{(2x)} + 1)\, dx$

SOLUTION

Divide numerator and denominator by e^x: $(e^{(2x)} - 1)/(e^{(2x)} + 1) = (e^x - e^{-x})/(e^x + e^{-x})$.

Let $t = e^x + e^{-x}$, so $dt = (e^x - e^{-x})\, dx$.

$\int dt/t = \log|t| + C = \log(e^x + e^{-x}) + C$.

Answer: $\log(e^x + e^{-x}) + C$

Q20 Find the integral: $\int (e^{(2x)} - e^{(-2x)}) / (e^{(2x)} + e^{(-2x)})\, dx$

SOLUTION

Let $t = e^{(2x)} + e^{(-2x)}$, so $dt = (2e^{(2x)} - 2e^{(-2x)})\, dx = 2(e^{(2x)} - e^{(-2x)})\, dx$.

Then $(e^{(2x)} - e^{(-2x)})\, dx = dt/2$.

$\int dt/(2t) = (1/2) \log|t| + C = (1/2) \log(e^{(2x)} + e^{(-2x)}) + C$.

Answer: $(1/2) \log(e^{(2x)} + e^{(-2x)}) + C$

Q21 Find the integral: $\int \tan^2(2x - 3)\, dx$

SOLUTION

Use $\tan^2\theta = \sec^2\theta - 1$, so $\tan^2(2x - 3) = \sec^2(2x - 3) - 1$.

$\int \sec^2(2x - 3)\, dx = (1/2) \tan(2x - 3)$, and $\int 1\, dx = x$.

Therefore $\int \tan^2(2x - 3)\, dx = (1/2) \tan(2x - 3) - x + C$.

Answer: $(1/2) \tan(2x - 3) - x + C$

Q22 Find the integral: $\int \sec^2(7 - 4x)\, dx$

SOLUTION

Let $t = 7 - 4x$, so $dt = -4\, dx$, i.e. $dx = -dt/4$.

$\int \sec^2(7 - 4x)\, dx = -(1/4) \int \sec^2 t\, dt = -(1/4) \tan t + C$.

Therefore the integral is $-(1/4) \tan(7 - 4x) + C$.

Answer: $-(1/4) \tan(7 - 4x) + C$

Q23 Find the integral: $\int \sin^{-1} x / \sqrt{(1 - x^2)}\, dx$

SOLUTION

Let $t = \sin^{-1} x$, so $dt = 1/\sqrt{(1 - x^2)}\, dx$.

$\int \sin^{-1} x \cdot 1/\sqrt{(1 - x^2)}\, dx = \int t\, dt = t^2/2 + C$.

Therefore the integral is $(1/2)(\sin^{-1} x)^2 + C$.

Answer: $(1/2)(\sin^{-1} x)^2 + C$

Q24 Find the integral: $\int (2 \cos x - 3 \sin x) / (6 \cos x + 4 \sin x)\, dx$

SOLUTION

Note that $6 \cos x + 4 \sin x = 2(3 \cos x + 2 \sin x)$, and $d/dx(2 \sin x + 3 \cos x) = 2 \cos x - 3 \sin x$.

So write the integral as $\int (2 \cos x - 3 \sin x) / [2(2 \sin x + 3 \cos x)]\, dx$.

Let $t = 2 \sin x + 3 \cos x$, so $dt = (2 \cos x - 3 \sin x)\, dx$.

$\int dt/(2t) = (1/2) \log|t| + C = (1/2) \log|2 \sin x + 3 \cos x| + C$.

Answer: $(1/2) \log|2 \sin x + 3 \cos x| + C$

Q25 Find the integral: $\int 1 / (\cos^2 x (1 - \tan x)^2)\, dx$

SOLUTION

Write $1/\cos^2 x = \sec^2 x$, so the integrand is $\sec^2 x / (1 - \tan x)^2$.

Let $t = 1 - \tan x$, so $dt = -\sec^2 x\, dx$.

$\int \sec^2 x/(1 - \tan x)^2\, dx = \int -dt/t^2 = 1/t + C = 1/(1 - \tan x) + C$.

Answer: $1/(1 - \tan x) + C$

Q26 Find the integral: $\int \cos\sqrt{x} / \sqrt{x}\, dx$

SOLUTION

Let $t = \sqrt{x}$, so $dt = 1/(2\sqrt{x})\, dx$, i.e. $1/\sqrt{x}\, dx = 2\, dt$.
$\int \cos\sqrt{x} \cdot 1/\sqrt{x}\, dx = \int \cos t \cdot 2\, dt = 2 \sin t + C$.
Therefore the integral is $2 \sin\sqrt{x} + C$.

Answer: $2 \sin\sqrt{x} + C$

Q27 Find the integral: $\int \sqrt{(\sin 2x)} \cos 2x\, dx$

SOLUTION

Let $t = \sin 2x$, $dt = 2 \cos 2x\, dx$.
$\int \sqrt{t} \cdot dt/2 = (1/2) \cdot (2t^{(3/2)}/3) + C = t^{(3/2)}/3 + C = (\sin 2x)^{(3/2)}/3 + C$

Answer: $(\sin 2x)^{(3/2)}/3 + C$

Q28 Find the integral: $\int \cos x / \sqrt{(1 + \sin x)}\, dx$

SOLUTION

Let $t = 1 + \sin x$, so $dt = \cos x\, dx$.
$\int t^{(-1/2)}\, dt = 2\sqrt{t} + C = 2\sqrt{(1 + \sin x)} + C$.

Answer: $2\sqrt{(1 + \sin x)} + C$

Q29 Find the integral: $\int \cot x \log(\sin x)\, dx$

SOLUTION

Let $t = \log(\sin x)$, $dt = (\cos x/\sin x)\, dx = \cot x\, dx$.
$\int t\, dt = t^2/2 + C = (\log \sin x)^2/2 + C$

Answer: $(\log \sin x)^2/2 + C$

Q30 Find the integral: $\int \sin x / (1 + \cos x)\, dx$

SOLUTION

Let $t = 1 + \cos x$, $dt = -\sin x\, dx$.
$\int (-dt)/t = -\log|t| + C = -\log(1 + \cos x) + C$

Answer: $-\log(1 + \cos x) + C$

Q31 Find the integral: $\int \sin x / (1 + \cos x)^2 \, dx$

SOLUTION

Let $t = 1 + \cos x$, $dt = -\sin x \, dx$.

$\int (-dt)/t^2 = 1/t + C = 1/(1 + \cos x) + C$

Answer: $1/(1 + \cos x) + C$

Q32 Find the integral: $\int 1 / (1 + \cot x) \, dx$

SOLUTION

$1/(1 + \cot x) = \sin x/(\sin x + \cos x)$.

Write $\sin x = (1/2)[(\sin x + \cos x) + (\sin x - \cos x)]$, so $\sin x/(\sin x + \cos x) = 1/2 + (1/2)(\sin x - \cos x)/(\sin x + \cos x)$.

Since d/dx(sin x + cos x) = cos x − sin x = −(sin x − cos x), ∫ (sin x − cos x)/(sin x + cos x) dx = −log	sin x + cos x	.
Therefore the integral = x/2 − (1/2) log	sin x + cos x	+ C.

Answer: $x/2 - (1/2) \log|\sin x + \cos x| + C$

Q33 Find the integral: $\int 1 / (1 - \tan x) \, dx$

SOLUTION

$1/(1 - \tan x) = \cos x/(\cos x - \sin x)$.

Write $\cos x = (1/2)[(\cos x - \sin x) + (\cos x + \sin x)]$, so $\cos x/(\cos x - \sin x) = 1/2 + (1/2)(\cos x + \sin x)/(\cos x - \sin x)$.

Since d/dx(cos x − sin x) = −(sin x + cos x), ∫ (cos x + sin x)/(cos x − sin x) dx = −log	cos x − sin x	.
Therefore the integral = x/2 − (1/2) log	cos x − sin x	+ C.

Answer: $x/2 - (1/2) \log|\cos x - \sin x| + C$

Q34 Find the integral: $\int \sqrt{(\tan x)} / (\sin x \cos x)\, dx$

SOLUTION

Since $\sin x \cos x = (\tan x)/\sec^2 x$, we have $\sqrt{(\tan x)}/(\sin x \cos x) = \sqrt{(\tan x)} \cdot \sec^2 x / \tan x = \sec^2 x/\sqrt{(\tan x)}$.

Let $t = \tan x$, so $dt = \sec^2 x\, dx$.

$\int dt/\sqrt{t} = 2\sqrt{t} + C = 2\sqrt{(\tan x)} + C$.

Answer: $2\sqrt{(\tan x)} + C$

Q35 Find the integral: $\int (1 + \log x)^2 / x\, dx$

SOLUTION

Let $t = 1 + \log x$, $dt = dx/x$.

$\int t^2\, dt = t^3/3 + C = (1 + \log x)^3/3 + C$

Answer: $(1 + \log x)^3/3 + C$

Q36 Find the integral: $\int (x + 1)(x + \log x)^2 / x\, dx$

SOLUTION

Rewrite: $(x+1)/x \cdot (x+\log x)^2 = (1+1/x)(x+\log x)^2$.

Note $d/dx(x+\log x) = 1+1/x$. Let $t = x+\log x$, $dt = (1+1/x)dx$.

$\int t^2\, dt = t^3/3 + C = (x + \log x)^3/3 + C$

Answer: $(x + \log x)^3/3 + C$

Q37 Find the integral: $\int x^3 \sin(\tan^{-1}(x^4)) / (1 + x^8)\, dx$

SOLUTION

Let $t = \tan^{-1}(x^4)$, then $dt = 4x^3/(1+x^8)\, dx$.

$\int \sin t \cdot dt/4 = -\cos t/4 + C = -\cos(\tan^{-1}(x^4))/4 + C$

Answer: $-(1/4) \cos(\tan^{-1}(x^4)) + C$

Q38 Choose the correct answer: $\int (10x^9 + 10^x \log 10) / (10^x + x^{10})\, dx$ equals (A) $10^x - x^{10} + C$ (B) $10^x + x^{10} + C$ (C) $(10^x - x^{10})^{-1} + C$ (D) $\log(10^x + x^{10}) + C$

SOLUTION

$d/dx(10^x + x^{10}) = 10^x \log 10 + 10x^9$.

So numerator = $d/dx(10^x + x^{10})$.

$\int d(t)/t = \log|t| + C = \log(10^x + x^{10}) + C$

Answer: Answer (D): $\log(10^x + x^{10}) + C$

Q39 Choose the correct answer: $\int dx / (\sin^2 x \cos^2 x)$ equals (A) $\tan x + \cot x + C$ (B) $\tan x - \cot x + C$ (C) $\tan x \cdot \cot x + C$ (D) $\tan x - \cot 2x + C$

SOLUTION

$1/(\sin^2 x \cos^2 x) = (\sin^2 x + \cos^2 x)/(\sin^2 x \cos^2 x) = 1/\cos^2 x + 1/\sin^2 x = \sec^2 x + \operatorname{cosec}^2 x$.

$\int (\sec^2 x + \operatorname{cosec}^2 x)\, dx = \tan x - \cot x + C$

Answer: Answer (B): $\tan x - \cot x + C$

Exercise 7.3 — Trigonometric Identities in Integration

Q1 Find the integral: $\int \sin^2(2x + 5)\,dx$

SOLUTION

Use identity: $\sin^2\theta = (1 - \cos 2\theta)/2$.

$\int (1 - \cos(4x+10))/2\,dx = x/2 - \sin(4x+10)/8 + C$

Answer: $x/2 - \sin(4x + 10)/8 + C$

Q2 Find the integral: $\int \sin 3x \cos 4x\,dx$

SOLUTION

Use product-to-sum: $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$.

$2 \sin 3x \cos 4x = \sin 7x + \sin(-x) = \sin 7x - \sin x$.

$\int (\sin 7x - \sin x)/2\,dx = -\cos 7x/14 + \cos x/2 + C$

Answer: $-\cos 7x/14 + \cos x/2 + C$

Q3 Find the integral: $\int \cos 2x \cos 4x \cos 6x\,dx$

SOLUTION

First, $2 \cos 2x \cos 4x = \cos 2x + \cos 6x$, so $\cos 2x \cos 4x = (1/2)(\cos 2x + \cos 6x)$.

Multiply by $\cos 6x$: $(1/2)(\cos 2x \cos 6x + \cos^2 6x) = (1/2)[(1/2)(\cos 4x + \cos 8x) + (1 + \cos 12x)/2]$.

So $\cos 2x \cos 4x \cos 6x = 1/4 + (1/4)\cos 4x + (1/4)\cos 8x + (1/4)\cos 12x$.

$\int = x/4 + \sin 4x/16 + \sin 8x/32 + \sin 12x/48 + C$.

Answer: $x/4 + \sin 4x/16 + \sin 8x/32 + \sin 12x/48 + C$

Q4 Find the integral: $\int \sin^3(2x + 1)\,dx$

SOLUTION

Use $\sin^3\theta = (3 \sin \theta - \sin 3\theta)/4$ with $\theta = 2x + 1$, or write $\sin^3(2x+1) = \sin(2x+1)(1 - \cos^2(2x+1))$.

Let $u = \cos(2x + 1)$, $du = -2 \sin(2x + 1)\,dx$.

$\int \sin(2x+1)(1 - \cos^2(2x+1))\,dx = -(1/2)\int(1 - u^2)\,du = -(1/2)(u - u^3/3)$.

$= -(1/2)\cos(2x + 1) + (1/6)\cos^3(2x + 1) + C$.

Answer: $-(1/2)\cos(2x+1) + (1/6)\cos^3(2x+1) + C$

Q5 Find the integral: $\int \sin^3x \cos^3x\, dx$

SOLUTION

Write $\sin^3x\cos^3x = \sin^3x\cos^2x \cdot \cos x = \sin^3x(1-\sin^2x)\cos x$.
Let $t = \sin x$, $dt = \cos x\, dx$.
$\int t^3(1-t^2)\, dt = \int (t^3 - t^5)\, dt = t^4/4 - t^6/6$.
$= (1/4)\sin^4x - (1/6)\sin^6x + C$.

Answer: $(1/4)\sin^4x - (1/6)\sin^6x + C$

Q6 Find the integral: $\int \sin x \sin 2x \sin 3x\, dx$

SOLUTION

Group $\sin x \sin 2x = (1/2)(\cos x - \cos 3x)$, then multiply by $\sin 3x$.
$\sin x \sin 2x \sin 3x = (1/2)(\sin 3x \cos x - \sin 3x \cos 3x) = (1/2)\cdot(1/2)(\sin 4x + \sin 2x) - (1/2)\cdot(1/2)\sin 6x$.
$= (1/4)\sin 4x + (1/4)\sin 2x - (1/4)\sin 6x$.
$\int = -(1/8)\cos 2x - (1/16)\cos 4x + (1/24)\cos 6x + C$.

Answer: $-(1/8)\cos 2x - (1/16)\cos 4x + (1/24)\cos 6x + C$

Q7 Find the integral: $\int \sin 4x \sin 8x\, dx$

SOLUTION

$2\sin A \sin B = \cos(A-B) - \cos(A+B)$, so $2\sin 4x \sin 8x = \cos 4x - \cos 12x$.
$\sin 4x \sin 8x = (1/2)(\cos 4x - \cos 12x)$.
$\int = (1/2)(\sin 4x/4 - \sin 12x/12) = (1/8)\sin 4x - (1/24)\sin 12x + C$.

Answer: $(1/8)\sin 4x - (1/24)\sin 12x + C$

Q8 Find the integral: $\int (1-\cos x)/(1+\cos x)\, dx$

SOLUTION

Use $1 - \cos x = 2\sin^2(x/2)$ and $1 + \cos x = 2\cos^2(x/2)$.
$(1-\cos x)/(1+\cos x) = \tan^2(x/2) = \sec^2(x/2) - 1$.
$\int (\sec^2(x/2) - 1)\, dx = 2\tan(x/2) - x + C$.

Answer: $2\tan(x/2) - x + C$

Q9 Find the integral: $\int \cos x / (1 + \cos x)\,dx$

SOLUTION

Write $\cos x/(1 + \cos x) = (1 + \cos x - 1)/(1 + \cos x) = 1 - 1/(1 + \cos x)$.
Since $1 + \cos x = 2\cos^2(x/2)$, $1/(1 + \cos x) = (1/2)\sec^2(x/2)$.
$\int [1 - (1/2)\sec^2(x/2)]\,dx = x - (1/2)\cdot 2\tan(x/2) = x - \tan(x/2) + C$.

Answer: $x - \tan(x/2) + C$

Q10 Find the integral: $\int \sin^4 x\,dx$

SOLUTION

$\sin^2 x = (1 - \cos 2x)/2$, so $\sin^4 x = [(1 - \cos 2x)/2]^2 = (1 - 2\cos 2x + \cos^2 2x)/4$.
$\cos^2 2x = (1 + \cos 4x)/2$, so $\sin^4 x = 3/8 - (\cos 2x)/2 + (\cos 4x)/8$.
$\int = 3x/8 - \sin 2x/4 + \sin 4x/32 + C$.

Answer: $3x/8 - \sin 2x/4 + \sin 4x/32 + C$

Q11 Find the integral: $\int \cos^4 2x\,dx$

SOLUTION

$\cos^2 2x = (1 + \cos 4x)/2$, so $\cos^4 2x = [(1 + \cos 4x)/2]^2 = (1 + 2\cos 4x + \cos^2 4x)/4$.
$\cos^2 4x = (1 + \cos 8x)/2$, so $\cos^4 2x = 3/8 + (\cos 4x)/2 + (\cos 8x)/8$.
$\int = 3x/8 + \sin 4x/8 + \sin 8x/64 + C$.

Answer: $3x/8 + \sin 4x/8 + \sin 8x/64 + C$

Q12 Find the integral: $\int \sin^2 x / (1 + \cos x)\,dx$

SOLUTION

$\sin^2 x = 1 - \cos^2 x = (1 - \cos x)(1 + \cos x)$.
So $\sin^2 x/(1 + \cos x) = 1 - \cos x$.
$\int (1 - \cos x)\,dx = x - \sin x + C$.

Answer: $x - \sin x + C$

Q13 Find the integral: $\int (\cos 2x - \cos 2\alpha) / (\cos x - \cos \alpha)\, dx$

SOLUTION

$\cos 2x - \cos 2\alpha = 2\cos^2 x - 1 - (2\cos^2 \alpha - 1) = 2(\cos^2 x - \cos^2 \alpha) = 2(\cos x + \cos \alpha)(\cos x - \cos \alpha)$.

Dividing by $(\cos x - \cos \alpha)$: $2(\cos x + \cos \alpha)$.

$\int 2(\cos x + \cos \alpha)\, dx = 2\sin x + 2x \cos \alpha + C$

Answer: $2 \sin x + 2x \cos \alpha + C$

Q14 Find the integral: $\int (\cos x - \sin x) / (1 + \sin 2x)\, dx$

SOLUTION

$1 + \sin 2x = \sin^2 x + \cos^2 x + 2 \sin x \cos x = (\sin x + \cos x)^2$.

$(\cos x - \sin x)/(\sin x + \cos x)^2 = (\cos x - \sin x)/(\sin x + \cos x)^2$.

Let $t = \sin x + \cos x$, $dt = (\cos x - \sin x)dx$.

$\int dt/t^2 = -1/t + C = -1/(\sin x + \cos x) + C$

Answer: $-1/(\sin x + \cos x) + C$

Q15 Find the integral: $\int \tan^3 2x \sec 2x\, dx$

SOLUTION

$\tan^3 2x \sec 2x = \tan^2 2x \cdot \tan 2x \sec 2x = (\sec^2 2x - 1) \tan 2x \sec 2x$.

$\int = \int \sec^2 2x \tan 2x \sec 2x\, dx - \int \tan 2x \sec 2x\, dx$.

Let $t = \sec 2x$ for first: $\int t^2\, dt/2 = t^3/6$; second: $\sec 2x/2$.

Result = $\sec^3 2x/6 - \sec 2x/2 + C$

Answer: $\sec^3(2x)/6 - \sec(2x)/2 + C$

Q16 Find the integral: $\int \tan^4 x\, dx$

SOLUTION

$\tan^4 x = \tan^2 x(\sec^2 x - 1) = \tan^2 x \sec^2 x - \tan^2 x$.

$\int \tan^2 x \sec^2 x\, dx$: let $t = \tan x$, $= t^3/3$.

$\int \tan^2 x\, dx = \int (\sec^2 x - 1)dx = \tan x - x$.

Result = $\tan^3 x/3 - \tan x + x + C$

Answer: $\tan^3 x/3 - \tan x + x + C$

Q17 Find the integral: $\int (\sin^3x + \cos^3x) / (\sin^2x \cos^2x)\, dx$

SOLUTION

Split: $\sin^3x/(\sin^2x \cos^2x) + \cos^3x/(\sin^2x \cos^2x) = \sin x/\cos^2x + \cos x/\sin^2x = \sec x \tan x + \operatorname{cosec} x \cot x$.

$\int (\sec x \tan x + \operatorname{cosec} x \cot x)\, dx = \sec x - \operatorname{cosec} x + C$

Answer: sec x − cosec x + C

Q18 Find the integral: $\int \cos 2x + 2\sin^2x / \cos^2x\, dx$

SOLUTION

$\cos 2x + 2\sin^2x = \cos 2x + (1 - \cos 2x) = 1$.

$\int 1/\cos^2x\, dx = \int \sec^2x\, dx = \tan x + C$

Answer: tan x + C

Q19 Find the integral: $\int 1 / (\sin x \cos^3x)\, dx$

SOLUTION

Multiply and divide by cos x: $1/(\sin x \cos^3x) = \sec^4x/\tan x$.

$\sec^4x/\tan x = (1+\tan^2x)\sec^2x/\tan x$.

Let t=tanx, dt=sec²x dx: $\int(1+t^2)/t\, dt = \int(1/t+t)dt = \log$	t	$+t^2/2$.
= log	tanx	+ tan²x/2 + C

Answer: log|tan x| + tan²x/2 + C

Q20 Find the integral: $\int (\cos 2x) / (\cos x + \sin x)^2\, dx$

SOLUTION

$\cos 2x = \cos^2x - \sin^2x = (\cos x - \sin x)(\cos x + \sin x)$.

So $\cos 2x/(\cos x + \sin x)^2 = (\cos x - \sin x)/(\cos x + \sin x)$.

Let $t = \cos x + \sin x$, so $dt = (\cos x - \sin x)\, dx$.

$\int dt/t = \log|t| + C = \log|\cos x + \sin x| + C$.

Answer: log|cos x + sin x| + C

Q21 Find the integral: $\int \sin^{-1}(\cos x)\,dx$

SOLUTION

For $0 \le x \le \pi$, $\cos x = \sin(\pi/2 - x)$, so $\sin^{-1}(\cos x) = \pi/2 - x$.
$\int (\pi/2 - x)\,dx = (\pi/2)x - x^2/2 + C$.

Answer: $\pi x/2 - x^2/2 + C$

Q22 Find the integral: $\int 1 / (\cos(x - a)\cos(x - b))\,dx$

SOLUTION

Multiply and divide by $\sin(a - b)$: write $\sin(a - b) = \sin((x - b) - (x - a)) = \sin(x - b)\cos(x - a) - \cos(x - b)\sin(x - a)$.
Then $1/(\cos(x - a)\cos(x - b)) = (1/\sin(a - b))[\sin(x - b)/\cos(x - b) - \sin(x - a)/\cos(x - a)] = (1/\sin(a - b))[\tan(x - b) - \tan(x - a)]$.

$\int$ = $(1/\sin(a - b))[-\log$	$\cos(x - b)$	$+ \log$	$\cos(x - a)$	$] + C.$
$= (1/\sin(a - b))\log$	$\cos(x - a)/\cos(x - b)$	$+ C.$		

Answer: $(1/\sin(a - b))\log|\cos(x - a)/\cos(x - b)| + C$

Q23 Choose the correct answer: $\int (\sin^2x - \cos^2x) / (\sin^2x \cos^2x)\,dx$ equals (A) $\tan x + \cot x + C$ (B) $\tan x + \operatorname{cosec} x + C$ (C) $-\tan x + \cot x + C$ (D) $\tan x + \sec x + C$

SOLUTION

Split the fraction: $(\sin^2x - \cos^2x)/(\sin^2x \cos^2x) = \sin^2x/(\sin^2x \cos^2x) - \cos^2x/(\sin^2x \cos^2x) = 1/\cos^2x - 1/\sin^2x$.
So the integrand is $\sec^2x - \operatorname{cosec}^2x$.
$\int (\sec^2x - \operatorname{cosec}^2x)\,dx = \tan x + \cot x + C$, which is option (A).

Answer: Answer (A): $\tan x + \cot x + C$

Q24 Choose the correct answer: $\int e^x(1 + x) / \cos^2(e^x x)\,dx$ equals (A) $-\cot(x e^x) + C$ (B) $\tan(x e^x) + C$ (C) $\tan(e^x) + C$ (D) $\cot(e^x) + C$

SOLUTION

Let $t = x e^x$, so $dt = (e^x + x e^x)\,dx = e^x(1 + x)\,dx$.
$\int e^x(1 + x)/\cos^2(e^x x)\,dx = \int dt/\cos^2t = \int \sec^2t\,dt = \tan t + C$.
$= \tan(x e^x) + C$, which is option (B).

Answer: Answer (B): $\tan(x\,e^{x}) + C$

Exercise 7.4 — Integrals of Particular Functions

Q1 Find the integral: $\int 3x^2 / (x^6 + 1)\, dx$

SOLUTION

Let $t = x^3$, so $dt = 3x^2\, dx$.

$\int 3x^2/(x^6 + 1)\, dx = \int dt/(t^2 + 1) = \tan^{-1} t + C.$

$= \tan^{-1}(x^3) + C.$

Answer: $\tan^{-1}(x^3) + C$

Q2 Find the integral: $\int 1 / \sqrt{(1 + 4x^2)}\, dx$

SOLUTION

Write $1 + 4x^2 = (2x)^2 + 1$. Let $t = 2x$, $dt = 2\, dx$.

$\int 1/\sqrt{(1 + 4x^2)}\, dx = (1/2)\int dt/\sqrt{(t^2 + 1)} = (1/2) \log$	$t + \sqrt{(t^2 + 1)}$	$+ C.$
$= (1/2) \log$	$2x + \sqrt{(1 + 4x^2)}$	$+ C.$

Answer: $(1/2) \log|2x + \sqrt{(1 + 4x^2)}| + C$

Q3 Find the integral: $\int 1 / \sqrt{((2 - x)^2 + 1)}\, dx$

SOLUTION

Let $t = x - 2$, so $dt = dx$ and $(2 - x)^2 = t^2$.

$\int dt/\sqrt{(t^2 + 1)} = \log$	$t + \sqrt{(t^2 + 1)}$	$+ C.$
$= \log$	$(x - 2) + \sqrt{((2 - x)^2 + 1)}$	$+ C.$

Answer: $\log|(x - 2) + \sqrt{((2 - x)^2 + 1)}| + C$

Q4 Find the integral: $\int 1 / \sqrt{(9 - 25x^2)}\, dx$

SOLUTION

Write $9 - 25x^2 = 9(1 - (5x/3)^2)$.

$\int 1/(3\sqrt{(1 - (5x/3)^2)})\, dx = (1/3)\cdot(3/5) \sin^{-1}(5x/3) + C.$

$= (1/5) \sin^{-1}(5x/3) + C.$

Answer: $(1/5) \sin^{-1}(5x/3) + C$

Q5 Find the integral: $\int 3x / (1 + 2x^4)\, dx$

SOLUTION

Let $t = x^2$, so $dt = 2x\, dx$, i.e. $x\, dx = dt/2$.

$\int 3x/(1 + 2x^4)\, dx = (3/2)\int dt/(1 + 2t^2) = (3/2)\cdot(1/\sqrt{2})\tan^{-1}(\sqrt{2}\, t) + C.$

$= (3/(2\sqrt{2}))\tan^{-1}(\sqrt{2}\, x^2) + C = (3\sqrt{2}/4)\tan^{-1}(\sqrt{2}\, x^2) + C.$

Answer: $(3\sqrt{2}/4)\tan^{-1}(\sqrt{2}\, x^2) + C$

Q6 Find the integral: $\int x^2 / (1 - x^6)\, dx$

SOLUTION

Let $t = x^3$, so $dt = 3x^2\, dx$, i.e. $x^2\, dx = dt/3$.

$\int x^2/(1 - x^6)\, dx = (1/3)\int dt/(1 - t^2) = (1/3)\cdot(1/2)\log$	$(1 + t)/(1 - t)$	$+ C.$
$= (1/6)\log$	$(1 + x^3)/(1 - x^3)$	$+ C.$

Answer: $(1/6)\log|(1 + x^3)/(1 - x^3)| + C$

Q7 Find the integral: $\int (x - 1) / \sqrt{(x^2 - 1)}\, dx$

SOLUTION

Split: $\int x/\sqrt{(x^2 - 1)}\, dx - \int 1/\sqrt{(x^2 - 1)}\, dx$.

First: let $t = x^2 - 1$, $dt = 2x\, dx \rightarrow \int dt/(2\sqrt{t}) = \sqrt{t} = \sqrt{(x^2 - 1)}$.

Second: $\int dx/\sqrt{(x^2 - 1)} = \log$	$x + \sqrt{(x^2 - 1)}$	.
Result = $\sqrt{(x^2 - 1)} - \log$	$x + \sqrt{(x^2 - 1)}$	$+ C.$

Answer: $\sqrt{(x^2 - 1)} - \log|x + \sqrt{(x^2 - 1)}| + C$

Q8 Find the integral: $\int x^2 / \sqrt{(x^6 + a^6)}\, dx$

SOLUTION

Let $t = x^3$, so $dt = 3x^2\, dx$, i.e. $x^2\, dx = dt/3$.

$\int x^2/\sqrt{(x^6 + a^6)}\, dx = (1/3)\int dt/\sqrt{(t^2 + (a^3)^2)} = (1/3)\log$	$t + \sqrt{(t^2 + a^6)}$	$+ C.$
$= (1/3)\log$	$x^3 + \sqrt{(x^6 + a^6)}$	$+ C.$

Answer: $(1/3)\log|x^3 + \sqrt{(x^6 + a^6)}| + C$

Q9 Find the integral: $\int \sec^2 x / \sqrt{(\tan^2 x + 4)}\, dx$

SOLUTION

Let $t = \tan x$, so $dt = \sec^2 x\, dx$.

$\int dt/\sqrt{(t^2 + 4)} = \log$	$t + \sqrt{(t^2 + 4)}$	+ C.
$= \log$	$\tan x + \sqrt{(\tan^2 x + 4)}$	+ C.

Answer: $\log|\tan x + \sqrt{(\tan^2 x + 4)}| + C$

Q10 Find the integral: $\int 1 / \sqrt{(x^2 + 2x + 2)}\, dx$

SOLUTION

Complete the square: $x^2 + 2x + 2 = (x + 1)^2 + 1$.

$\int dx/\sqrt{((x + 1)^2 + 1)} = \log|(x + 1) + \sqrt{(x^2 + 2x + 2)}| + C$.

Answer: $\log|(x + 1) + \sqrt{(x^2 + 2x + 2)}| + C$

Q11 Find the integral: $\int 1 / (9x^2 + 6x + 5)\, dx$

SOLUTION

$9x^2 + 6x + 5 = (3x + 1)^2 + 4$.

$\int dx/((3x + 1)^2 + 4)$. Let $t = 3x + 1$, $dt = 3\, dx$.

$= (1/3)\int dt/(t^2 + 2^2) = (1/3)\cdot(1/2) \tan^{-1}(t/2) + C = (1/6) \tan^{-1}((3x + 1)/2) + C$.

Answer: $(1/6) \tan^{-1}((3x + 1)/2) + C$

Q12 Find the integral: $\int 1 / \sqrt{(7 - 6x - x^2)}\, dx$

SOLUTION

$7 - 6x - x^2 = 16 - (x + 3)^2$.

$\int dx/\sqrt{(4^2 - (x + 3)^2)} = \sin^{-1}((x + 3)/4) + C$.

Answer: $\sin^{-1}((x + 3)/4) + C$

Q13 Find the integral: $\int 1 / \sqrt{(x-1)(x-2)}\, dx$

SOLUTION

$(x-1)(x-2) = x^2 - 3x + 2 = (x - 3/2)^2 - 1/4$.

$\int dx/\sqrt{(x - 3/2)^2 - (1/2)^2} = \log|(x - 3/2) + \sqrt{(x^2 - 3x + 2)}| + C$.

Answer: $\log|(x - 3/2) + \sqrt{(x-1)(x-2)}| + C$

Q14 Find the integral: $\int 1 / \sqrt{(8 + 3x - x^2)}\, dx$

SOLUTION

$8 + 3x - x^2 = 41/4 - (x - 3/2)^2$.

$\int dx/\sqrt{(\sqrt{41}/2)^2 - (x - 3/2)^2} = \sin^{-1}((x - 3/2)/(\sqrt{41}/2)) + C = \sin^{-1}((2x - 3)/\sqrt{41}) + C$.

Answer: $\sin^{-1}((2x - 3)/\sqrt{41}) + C$

Q15 Find the integral: $\int 1 / \sqrt{(x-a)(x-b)}\, dx$

SOLUTION

$(x-a)(x-b) = x^2 - (a+b)x + ab = (x - (a+b)/2)^2 - ((a-b)/2)^2$.

$\int dx/\sqrt{(x - (a+b)/2)^2 - ((a-b)/2)^2} = \log|(x - (a+b)/2) + \sqrt{(x-a)(x-b)}| + C$.

Answer: $\log|(x - (a+b)/2) + \sqrt{(x-a)(x-b)}| + C$

Q16 Find the integral: $\int (4x + 1) / \sqrt{(2x^2 + x - 3)}\, dx$

SOLUTION

Note $d/dx(2x^2 + x - 3) = 4x + 1$, exactly the numerator.

Let $t = 2x^2 + x - 3$, $dt = (4x + 1)\, dx$.

$\int dt/\sqrt{t} = 2\sqrt{t} + C = 2\sqrt{(2x^2 + x - 3)} + C$.

Answer: $2\sqrt{(2x^2 + x - 3)} + C$

Q17 Find the integral: $\int (x + 2) / \sqrt{(x^2 - 1)}\, dx$

SOLUTION

Write $x + 2 = (1/2) \cdot d/dx(x^2 - 1) + 2 = (1/2)(2x) + 2$.

$\int = (1/2)\int 2x/\sqrt{(x^2 - 1)}\, dx + 2\int dx/\sqrt{(x^2 - 1)}$.

$= \sqrt{(x^2 - 1)} + 2 \log|x + \sqrt{(x^2 - 1)}| + C$.

Answer: $\sqrt{(x^2 - 1)} + 2\log|x + \sqrt{(x^2 - 1)}| + C$

Q18 Find the integral: $\int (5x - 2) / (1 + 2x + 3x^2)\, dx$

SOLUTION

Write $5x - 2 = A \cdot d/dx(3x^2 + 2x + 1) + B = A(6x + 2) + B$, so $6A = 5$, $2A + B = -2 \rightarrow A = 5/6$, $B = -11/3$.

$\int = (5/6)\int (6x + 2)/(3x^2 + 2x + 1)\, dx - (11/3)\int dx/(3x^2 + 2x + 1)$.

First = (5/6) log	$3x^2 + 2x + 1$	. For the second, $3x^2 + 2x + 1 = 3[(x + 1/3)^2 + 2/9]$, so $\int dx/(3x^2 + 2x + 1) = (1/\sqrt{2}) \tan^{-1}((3x + 1)/\sqrt{2})$.
Result = (5/6) log	$3x^2 + 2x + 1$	$- (11/(3\sqrt{2})) \tan^{-1}((3x + 1)/\sqrt{2}) + C$.

Answer: $(5/6)\log|3x^2 + 2x + 1| - (11/(3\sqrt{2})) \tan^{-1}((3x + 1)/\sqrt{2}) + C$

Q19 Find the integral: $\int (6x + 7) / \sqrt{((x - 5)(x - 4))}\, dx$

SOLUTION

$(x - 5)(x - 4) = x^2 - 9x + 20$. Write $6x + 7 = 3 \cdot d/dx(x^2 - 9x + 20) + B = 3(2x - 9) + B \rightarrow B = 34$.

$\int = 3\int (2x - 9)/\sqrt{(x^2 - 9x + 20)}\, dx + 34\int dx/\sqrt{((x - 9/2)^2 - 1/4)}$.

$= 6\sqrt{(x^2 - 9x + 20)} + 34\log|(x - 9/2) + \sqrt{(x^2 - 9x + 20)}| + C$.

Answer: $6\sqrt{(x^2 - 9x + 20)} + 34\log|(x - 9/2) + \sqrt{((x - 5)(x - 4))}| + C$

Q20 Find the integral: $\int (x + 2) / \sqrt{(4x - x^2)}\, dx$

SOLUTION

$4x - x^2 = 4 - (x - 2)^2$. Write $x + 2 = -(1/2) \cdot d/dx(4x - x^2) + 4 = -(1/2)(4 - 2x) + 4$.

$\int = -(1/2)\int (4 - 2x)/\sqrt{(4x - x^2)}\, dx + 4\int dx/\sqrt{(4 - (x - 2)^2)}$.

$= -\sqrt{(4x - x^2)} + 4 \sin^{-1}((x - 2)/2) + C$.

Answer: $-\sqrt{(4x - x^2)} + 4 \sin^{-1}((x - 2)/2) + C$

Q21 Find the integral: $\int (x + 2) / \sqrt{(x^2 + 2x + 3)}\, dx$

SOLUTION

Write $x + 2 = (1/2)\cdot d/dx(x^2 + 2x + 3) + 1 = (1/2)(2x + 2) + 1$.
$\int = (1/2)\int (2x + 2)/\sqrt{(x^2 + 2x + 3)}\, dx + \int dx/\sqrt{((x + 1)^2 + 2)}$.
$= \sqrt{(x^2 + 2x + 3)} + \log|(x + 1) + \sqrt{(x^2 + 2x + 3)}| + C$.

Answer: $\sqrt{(x^2 + 2x + 3)} + \log|(x + 1) + \sqrt{(x^2 + 2x + 3)}| + C$

Q22 Find the integral: $\int (x + 3) / (x^2 - 2x - 5)\, dx$

SOLUTION

Write $x + 3 = (1/2)\cdot d/dx(x^2 - 2x - 5) + B = (1/2)(2x - 2) + B \rightarrow B = 4$.

$\int = (1/2) \log$	$x^2 - 2x - 5$	$+ 4\int dx/((x - 1)^2 - 6)$.		
$\int dx/((x - 1)^2 - 6) = (1/(2\sqrt{6})) \log$	$(x - 1 - \sqrt{6})/(x - 1 + \sqrt{6})$	.		
Result = $(1/2) \log$	$x^2 - 2x - 5$	$+ (2/\sqrt{6}) \log$	$(x - 1 - \sqrt{6})/(x - 1 + \sqrt{6})$	$+ C$.

Answer: $(1/2) \log|x^2 - 2x - 5| + (2/\sqrt{6}) \log|(x - 1 - \sqrt{6})/(x - 1 + \sqrt{6})| + C$

Q23 Find the integral: $\int (5x + 3) / \sqrt{(x^2 + 4x + 10)}\, dx$

SOLUTION

Write $5x + 3 = (5/2)\cdot d/dx(x^2 + 4x + 10) + B = (5/2)(2x + 4) + B \rightarrow B = -7$.
$\int = (5/2)\int (2x + 4)/\sqrt{(x^2 + 4x + 10)}\, dx - 7\int dx/\sqrt{((x + 2)^2 + 6)}$.
$= 5\sqrt{(x^2 + 4x + 10)} - 7 \log|(x + 2) + \sqrt{(x^2 + 4x + 10)}| + C$.

Answer: $5\sqrt{(x^2 + 4x + 10)} - 7 \log|(x + 2) + \sqrt{(x^2 + 4x + 10)}| + C$

Q24 Choose the correct answer: $\int dx / (x^2 + 2x + 2)$ equals (A) $x \tan^{-1}(x + 1) + C$ (B) $\tan^{-1}(x + 1) + C$ (C) $(x + 1) \tan^{-1}x + C$ (D) $\tan^{-1}x + C$

SOLUTION

$x^2 + 2x + 2 = (x + 1)^2 + 1$.

$\int dx/((x + 1)^2 + 1) = \tan^{-1}(x + 1) + C$, which is option (B).

Answer: Answer (B): $\tan^{-1}(x + 1) + C$

Q25 Choose the correct answer: $\int dx / \sqrt{(9x - 4x^2)}$ equals (A) $(1/9) \sin^{-1}((9x - 8)/8) + C$ (B) $(1/2) \sin^{-1}((8x - 9)/9) + C$ (C) $(1/3) \sin^{-1}((9x - 8)/8) + C$ (D) $(1/2) \sin^{-1}((9x - 8)/8) + C$

SOLUTION

Complete the square: $9x - 4x^2 = -4(x^2 - (9/4)x) = 81/16 - 4(x - 9/8)^2 = 4[(9/8)^2 - (x - 9/8)^2]$.

So $\sqrt{(9x - 4x^2)} = 2\sqrt{((9/8)^2 - (x - 9/8)^2)}$.

$\int dx / (2\sqrt{((9/8)^2 - (x - 9/8)^2)}) = (1/2) \sin^{-1}((x - 9/8)/(9/8)) + C = (1/2) \sin^{-1}((8x - 9)/9) + C$, which is option (B).

Answer: Answer (B): $(1/2) \sin^{-1}((8x - 9)/9) + C$

Exercise 7.5 — Partial Fractions

Q1 Find the integral: $\int x/((x+1)(x+2))\,dx$

SOLUTION

PF: $x/((x+1)(x+2)) = -1/(x+1)+2/(x+2)$.
(Check: $-(x+2)+2(x+1)=-x-2+2x+2=x$ ✓)
$\int = -\log|x+1|+2\log|x+2|+C$

Answer: $2\log|x+2| - \log|x+1| + C$

Q2 Find the integral: $\int 1/(x^2-9)\,dx$

SOLUTION

$1/(x^2-9)=1/((x-3)(x+3))=(1/6)[1/(x-3)-1/(x+3)]$.
$\int = (1/6)\log|(x-3)/(x+3)|+C$

Answer: $(1/6)\log|(x-3)/(x+3)| + C$

Q3 Find the integral: $\int 3x-1/((x-1)(x-2)(x-3))\,dx$

SOLUTION

PF: $A/(x-1)+B/(x-2)+C/(x-3)$. $3x-1=A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)$.
$x=1$: $2=A(-1)(-2)=2A \rightarrow A=1$. $x=2$: $5=B(1)(-1)=-B \rightarrow B=-5$. $x=3$: $8=C(2)(1)=2C \rightarrow C=4$.
$\int = \log|x-1|-5\log|x-2|+4\log|x-3|+C$

Answer: $\log|x-1| - 5\log|x-2| + 4\log|x-3| + C$

Q4 Find the integral: $\int x/((x-1)(x-2)(x-3))\,dx$

SOLUTION

PF: $A/(x-1)+B/(x-2)+C/(x-3)$. $x=1$: $1=A(-1)(-2)=2A \rightarrow A=1/2$.
$x=2$: $2=B(1)(-1)=-B \rightarrow B=-2$. $x=3$: $3=C(2)(1)=2C \rightarrow C=3/2$.
$\int = (1/2)\log|x-1|-2\log|x-2|+(3/2)\log|x-3|+C$

Answer: $(1/2)\log|x-1| - 2\log|x-2| + (3/2)\log|x-3| + C$

Q5 Find the integral: $\int 2x/(x^2+3x+2)\,dx$

SOLUTION

$x^2+3x+2=(x+1)(x+2)$. PF: $2x/((x+1)(x+2))=A/(x+1)+B/(x+2)$, so $2x=A(x+2)+B(x+1)$.
$x=-1$: $-2=A(1) \to A=-2$. $x=-2$: $-4=B(-1) \to B=4$.
$\int = -2\log|x+1|+4\log|x+2|+C$

Answer: $4\log|x+2| - 2\log|x+1| + C$

Q6 Find the integral: $\int 1-x^2/(x(1-2x))\,dx$

SOLUTION

The numerator and denominator both have degree 2, so divide first: $(1-x^2)/(x(1-2x)) = (1-x^2)/(x-2x^2) = 1/2 + (1 - x/2)/(x(1-2x))$.
Split $(1 - x/2)/(x(1-2x))$ into partial fractions $A/x + B/(1-2x)$: $1 - x/2 = A(1-2x) + Bx$.
$x=0$: $A=1$. $x=1/2$: $3/4 = B/2 \to B = 3/2$.
So the integrand $= 1/2 + 1/x + (3/2)/(1-2x)$.
$\int = x/2 + \log|x| + (3/2)\cdot(-1/2)\log|1-2x| + C = x/2 + \log|x| - (3/4)\log|1-2x| + C$.

Answer: $x/2 + \log|x| - (3/4)\log|1-2x| + C$

Q7 Find the integral: $\int x/((x^2+1)(x-1))\,dx$

SOLUTION

PF: $x/((x^2+1)(x-1))=A/(x-1)+(Bx+C)/(x^2+1)$.
$x=1$: $1=2A \to A=1/2$. Coeff x^2: $0=A+B \to B=-1/2$. Const: $0=-A+C \to C=1/2$.
$\int = (1/2)\log|x-1|-(1/4)\log(x^2+1)+(1/2)\arctan x+C$

Answer: $(1/2)\log|x-1| - (1/4)\log(x^2+1) + (1/2)\tan^{-1}x + C$

Q8 Find the integral: $\int x/((x-1)^2(x+2))\,dx$

SOLUTION

PF: $A/(x-1)+B/(x-1)^2+C/(x+2)$. $x=1$: $1=3B \to B=1/3$. $x=-2$: $-2=9C \to C=-2/9$.
Coeff x^2: $0=A+C \to A=2/9$.
$\int = (2/9)\log|x-1|-1/(3(x-1))-(2/9)\log|x+2|+C$

Answer: $(2/9)\log|(x-1)/(x+2)| - 1/(3(x-1)) + C$

Q9 Find the integral: $\int 3x+5/(x^3-x^2-x+1)\,dx$

SOLUTION

$x^3-x^2-x+1=(x-1)^2(x+1)$. PF: $A/(x-1)+B/(x-1)^2+C/(x+1)$.

$3x+5=A(x-1)(x+1)+B(x+1)+C(x-1)^2$. $x=1$: $8=2B \to B=4$. $x=-1$: $2=4C \to C=1/2$.

Coeff x^2: $0=A+C \to A=-1/2$.

$\int$ = $-(1/2)\log$	$x-1$	$-4/(x-1)+(1/2)\log$	$x+1$	$+C$
$= (1/2)\log$	$(x+1)/(x-1)$	$-4/(x-1)+C$		

Answer: $(1/2)\log|(x+1)/(x-1)| - 4/(x-1) + C$

Q10 Find the integral: $\int (2x-3)/((x^2-1)(2x+3))\,dx$

SOLUTION

$(x^2-1)=(x-1)(x+1)$. Partial fractions: $A/(x-1)+B/(x+1)+C/(2x+3)$.

$2x-3=A(x+1)(2x+3)+B(x-1)(2x+3)+C(x-1)(x+1)$.

$x=1$: $-1=A\cdot2\cdot5=10A \to A=-1/10$. $x=-1$: $-5=B(-2)(1)=-2B \to B=5/2$.

$x=-3/2$: $2(-3/2)-3=-6=C(-3/2-1)(-3/2+1)=C(-5/2)(-1/2)=5C/4 \to C=-24/5$.

$\int = -(1/10)\log|x-1|+(5/2)\log|x+1|-(12/5)\log|2x+3|+C$

Answer: $-(1/10)\log|x-1| + (5/2)\log|x+1| - (12/5)\log|2x+3| + C$

Q11 Find the integral: $\int 5x/((x+1)(x^2-4))\,dx$

SOLUTION

$(x^2-4)=(x-2)(x+2)$. PF: $A/(x+1)+B/(x-2)+C/(x+2)$.

$5x=A(x-2)(x+2)+B(x+1)(x+2)+C(x+1)(x-2)$.

$x=-1$: $-5=A(-3)(1)=-3A \to A=5/3$. $x=2$: $10=B(3)(4)=12B \to B=5/6$. $x=-2$: $-10=C(-1)(-4)=4C \to C=-5/2$.

$\int = (5/3)\log|x+1|+(5/6)\log|x-2|-(5/2)\log|x+2|+C$

Answer: $(5/3)\log|x+1| + (5/6)\log|x-2| - (5/2)\log|x+2| + C$

Q12 Find the integral: $\int (x^3+x+1)/(x^2-1)\,dx$

SOLUTION

Long division: $x^3+x+1 = x(x^2-1)+(2x+1)$. So $(x^3+x+1)/(x^2-1)=x+(2x+1)/(x^2-1)$.

PF: $(2x+1)/((x-1)(x+1))=A/(x-1)+B/(x+1)$. $x=1$: $3=2A \to A=3/2$. $x=-1$: $-1=-2B \to B=1/2$.

$\int = x^2/2+(3/2)\log|x-1|+(1/2)\log|x+1|+C$

Answer: $x^2/2 + (3/2)\log|x-1| + (1/2)\log|x+1| + C$

Q13 Find the integral: $\int 2/(1-x)(1+x^2)\,dx$

SOLUTION

PF: $2/((1-x)(1+x^2))=A/(1-x)+(Bx+C)/(1+x^2)$.

$2=A(1+x^2)+(Bx+C)(1-x)$. $x=1$: $2=2A \to A=1$.

Coeff x^2: $0=A-B \to B=1$. Const: $2=A+C \to C=1$.

$\int = \int 1/(1-x)dx + \int (x+1)/(1+x^2)dx$

$= -\log|1-x|+(1/2)\log(1+x^2)+\arctan x+C$

Answer: $-\log|1-x| + (1/2)\log(1+x^2) + \tan^{-1}x + C$

Q14 Find the integral: $\int 3x-1/(x+2)^2\,dx$

SOLUTION

PF: $(3x-1)/(x+2)^2=A/(x+2)+B/(x+2)^2$. $3x-1=A(x+2)+B$. $A=3$; $6+B=-1 \to B=-7$.

$\int = 3\log|x+2|+7/(x+2)+C$

Answer: $3\log|x+2| + 7/(x+2) + C$

Q15 Find the integral: $\int 1/(x^4-1)\,dx$

SOLUTION

$x^4-1=(x^2-1)(x^2+1)=(x-1)(x+1)(x^2+1)$.

PF: $A/(x-1)+B/(x+1)+(Cx+D)/(x^2+1)$. $1=A(x+1)(x^2+1)+B(x-1)(x^2+1)+(Cx+D)(x^2-1)$.

$x=1$: $1=4A \to A=1/4$. $x=-1$: $1=-4B \to B=-1/4$. Coeff x^3: $0=A+B+C \to C=0$. Const: $1=A-B-D \to D=-1/2$.

$\int = (1/4)\log$	$x-1$	$-(1/4)\log$	$x+1$	$-(1/2)\arctan x+C$
$= (1/4)\log$	$(x-1)/(x+1)$	$-(1/2)\arctan x+C$		

Answer: (1/4) log|(x−1)/(x+1)| − (1/2) tan⁻¹x + C

Q16 Find the integral: ∫ 1/(x(xⁿ+1)) dx

SOLUTION

Multiply numerator and denominator by x^(n−1):
1/(x(xⁿ+1)) = x^(n−1)/(xⁿ(xⁿ+1)). Let t=xⁿ, dt=nx^(n−1)dx.

∫ = (1/n)∫dt/(t(t+1)) = (1/n)∫[1/t−1/(t+1)]dt = (1/n)log	t/(t+1)	+C
= (1/n)log	xⁿ/(xⁿ+1)	+C

Answer: (1/n) log|xⁿ/(xⁿ+1)| + C

Q17 Find the integral: ∫ cos x/((1−sin x)(2−sin x)) dx

SOLUTION

Let t=sinx, dt=cosx dx.
∫ dt/((1−t)(2−t)). PF: 1/(1−t) − 1/(2−t).
∫ = −log|1−t|+log|2−t|+C = log|(2−sinx)/(1−sinx)|+C

Answer: log|(2−sin x)/(1−sin x)| + C

Q18 Find the integral: ∫ (x²+1)(x²+2)/((x²+3)(x²+4)) dx

SOLUTION

Let u=x². Numerator = (u+1)(u+2)=u²+3u+2; denominator=(u+3)(u+4)=u²+7u+12.
Divide: (u²+3u+2)/(u²+7u+12)=1−(4u+10)/(u²+7u+12).
So integrand = 1−(4x²+10)/((x²+3)(x²+4)).
PF: (4x²+10)/((x²+3)(x²+4))=B/(x²+3)+D/(x²+4), so 4x²+10=B(x²+4)+D(x²+3).
Compare coefficients: B+D=4 and 4B+3D=10 → B=−2, D=6.
Hence integrand = 1 − (−2/(x²+3) + 6/(x²+4)) = 1 + 2/(x²+3) − 6/(x²+4).
∫ = x + 2∫dx/(x²+3) − 6∫dx/(x²+4) = x + (2/√3)tan⁻¹(x/√3) − 3 tan⁻¹(x/2) + C.

Answer: x + (2/√3) tan⁻¹(x/√3) − 3 tan⁻¹(x/2) + C

Q19 Find the integral: $\int 2x/((x^2+1)(x^2+3))\,dx$

SOLUTION

Let $t=x^2$, $dt=2x\,dx$.
$\int dt/((t+1)(t+3))$. PF: $(1/2)[1/(t+1)-1/(t+3)]$.
$\int = (1/2)[\log|t+1|-\log|t+3|]+C = (1/2)\log|(x^2+1)/(x^2+3)|+C$

Answer: $(1/2)\log|(x^2+1)/(x^2+3)| + C$

Q20 Find the integral: $\int 1/(x(x^4-1))\,dx$

SOLUTION

$1/(x(x^4-1))=1/(x(x^2-1)(x^2+1))$.
Multiply by x^3/x^3: $x^3/(x^4(x^4-1))$. Let $t=x^4$, $dt=4x^3dx$.

$\int = (1/4)\int dt/(t(t-1)) = (1/4)\int[1/(t-1)-1/t]dt = (1/4)\log$	$(t-1)/t$	$+C$
$= (1/4)\log$	$(x^4-1)/x^4$	$+C$

Answer: $(1/4)\log|(x^4-1)/x^4| + C$

Q21 Find the integral: $\int 1/(e^x-1)\,dx$ [Hint: put $e^x=t$]

SOLUTION

Let $t=e^x$, $dt=e^x\,dx=t\,dx$, so $dx=dt/t$.
$\int dt/(t(t-1))$. PF: $1/(t-1)-1/t$.
$\int = \log|t-1|-\log|t|+C = \log|(e^x-1)/e^x|+C = \log|1-e^{(-x)}|+C$

Answer: $\log|1 - e^{(-x)}| + C$

Q22 Choose the correct answer: $\int x\,dx/((x-1)(x-2))$ equals (A) $\log|(x-1)^2/(x-2)|+C$ (B) $\log|(x-2)^2/(x-1)|+C$ (C) $\log|(x-1)/(x-2)^2|+C$ (D) $\log|(x-1)(x-2)|+C$

SOLUTION

PF: $x/((x-1)(x-2))=A/(x-1)+B/(x-2)$. $x=1$: $1=-A\to A=-1$. $x=2$: $2=B$.
$\int = -\log|x-1|+2\log|x-2|+C = \log|(x-2)^2/(x-1)|+C$

Answer: Answer (B): $\log|(x-2)^2/(x-1)| + C$

Q23 Choose the correct answer: $\int dx/(x(x^2+1))$ equals (A) $\log|x|-(1/2)\log(x^2+1)+C$ (B) $\log|x|+(1/2)\log(x^2+1)+C$ (C) $-\log|x|+(1/2)\log(x^2+1)+C$ (D) $(1/2)\log|x|+\log(x^2+1)+C$

SOLUTION

PF: $1/(x(x^2+1))=1/x-x/(x^2+1)$.

$\int = \log|x|-(1/2)\log(x^2+1)+C$

Answer: Answer (A): $\log|x| - (1/2)\log(x^2+1) + C$

Exercise 7.6 — Integration by Parts

Q1 Find the integral: $\int x \sin x \, dx$

SOLUTION

IBP: u=x, dv=sinx dx → du=dx, v=−cosx.
$\int x \sin x \, dx = -x\cos x + \int \cos x \, dx = -x\cos x + \sin x + C$

Answer: $-x \cos x + \sin x + C$

Q2 Find the integral: $\int x \sin 3x \, dx$

SOLUTION

IBP: u=x, dv=sin3x dx → v=−cos3x/3.
$= -x\cos 3x/3 + \int \cos 3x/3 \, dx = -x\cos 3x/3 + \sin 3x/9 + C$

Answer: $-(x/3) \cos 3x + (1/9) \sin 3x + C$

Q3 Find the integral: $\int x^2 e^x \, dx$

SOLUTION

IBP twice. u=x^2, dv=$e^x dx$: $x^2e^x - 2\int x\, e^x \, dx$.
For $\int xe^x dx$: $xe^x - e^x$.
Result = $x^2e^x - 2xe^x + 2e^x + C = e^x(x^2-2x+2) + C$

Answer: $e^x(x^2-2x+2) + C$

Q4 Find the integral: $\int x \log x \, dx$

SOLUTION

IBP: u=logx, dv=x dx → v=$x^2/2$.
$= (x^2/2)\log x - \int (x^2/2)(1/x)dx = (x^2/2)\log x - x^2/4 + C$

Answer: $(x^2/2) \log x - x^2/4 + C$

Q5 Find the integral: $\int x \log 2x \, dx$

SOLUTION

IBP: $u=\log 2x$, $dv=x\,dx \rightarrow v=x^2/2$.

$= (x^2/2)\log 2x - \int (x^2/2)(1/x)dx = (x^2/2)\log 2x - x^2/4 + C$

Answer: $(x^2/2)\log 2x - x^2/4 + C$

Q6 Find the integral: $\int x^2 \log x \, dx$

SOLUTION

IBP: $u=\log x$, $dv=x^2dx \rightarrow v=x^3/3$.

$= (x^3/3)\log x - \int (x^3/3)(1/x)dx = (x^3/3)\log x - x^3/9 + C$

Answer: $(x^3/3)\log x - x^3/9 + C$

Q7 Find the integral: $\int x \sin^{-1}x \, dx$

SOLUTION

Integration by parts with $u = \sin^{-1}x$, $dv = x\,dx \rightarrow v = x^2/2$.

$\int x \sin^{-1}x \, dx = (x^2/2)\sin^{-1}x - \int x^2/(2\sqrt{(1-x^2)})\, dx$.

For $\int x^2/\sqrt{(1-x^2)}\, dx$, the standard result is $-(1/2)x\sqrt{(1-x^2)} + (1/2)\sin^{-1}x$.

So $\int x \sin^{-1}x \, dx = (x^2/2)\sin^{-1}x - (1/2)[-(x/2)\sqrt{(1-x^2)} + (1/2)\sin^{-1}x] + C$.

$= (x^2/2)\sin^{-1}x + (x/4)\sqrt{(1-x^2)} - (1/4)\sin^{-1}x + C = ((2x^2-1)/4)\ \sin^{-1}x + (x/4)\sqrt{(1-x^2)} + C$.

Answer: $(2x^2-1)/4 \cdot \sin^{-1}x + x\sqrt{(1-x^2)}/4 + C$

Q8 Find the integral: $\int x \tan^{-1}x \, dx$

SOLUTION

IBP: $u=\tan^{-1}x$, $dv=x\,dx \rightarrow v=x^2/2$.

$= (x^2/2)\tan^{-1}x - \int x^2/(2(1+x^2))dx$.

$x^2/(1+x^2) = 1-1/(1+x^2)$.

$= (x^2/2)\tan^{-1}x - (1/2)(x-\tan^{-1}x) + C$

$= (x^2/2+1/2)\tan^{-1}x - x/2 + C = ((x^2+1)/2)\tan^{-1}x - x/2 + C$

Answer: $((x^2+1)/2)\tan^{-1}x - x/2 + C$

Q9 Find the integral: $\int x \cos^{-1}x \, dx$

SOLUTION

Integration by parts with $u = \cos^{-1}x$, $dv = x\,dx \rightarrow v = x^2/2$. Since $d/dx(\cos^{-1}x) = -1/\sqrt{(1-x^2)}$:

$\int x \cos^{-1}x \, dx = (x^2/2)\cos^{-1}x + (1/2)\int x^2/\sqrt{(1-x^2)}\, dx$.

Using $\int x^2/\sqrt{(1-x^2)}\, dx = -(x/2)\sqrt{(1-x^2)} + (1/2)\sin^{-1}x$ and $\sin^{-1}x = \pi/2 - \cos^{-1}x$, this equals $-(x/2)\sqrt{(1-x^2)} - (1/2)\cos^{-1}x$ (absorbing the constant).

Result $= (x^2/2)\cos^{-1}x + (1/2)[-(x/2)\sqrt{(1-x^2)} - (1/2)\cos^{-1}x] + C = ((2x^2-1)/4)\cos^{-1}x - (x/4)\sqrt{(1-x^2)} + C$.

Answer: $(2x^2-1)/4 \cdot \cos^{-1}x - (x/4)\sqrt{(1-x^2)} + C$

Q10 Find the integral: $\int (\sin^{-1}x)^2 \, dx$

SOLUTION

IBP: $u=(\sin^{-1}x)^2$, $dv=dx \rightarrow v=x$.

$= x(\sin^{-1}x)^2 - 2\int x\cdot\sin^{-1}x/\sqrt{(1-x^2)}dx$.

For $\int x \sin^{-1}x/\sqrt{(1-x^2)}dx$: let $t=\sin^{-1}x$, $\sin t=x$, $\cos t\, dt=dx/\sqrt{(1-x^2)}$.

$= \int t \sin t \, dt = -t \cos t + \sin t = -\sin^{-1}x\cdot\sqrt{(1-x^2)} + x + C$.

Result $= x(\sin^{-1}x)^2 - 2(-\sin^{-1}x\cdot\sqrt{(1-x^2)}+x) + C$

$= x(\sin^{-1}x)^2 + 2\sqrt{(1-x^2)}\cdot\sin^{-1}x - 2x + C$

Answer: $x(\sin^{-1}x)^2 + 2\sqrt{(1-x^2)}\, \sin^{-1}x - 2x + C$

Q11 Find the integral: $\int x \cos^{-1}x/\sqrt{(1-x^2)}\, dx$

SOLUTION

Let $t=\cos^{-1}x$, then $dt=-dx/\sqrt{(1-x^2)}$ and $x=\cos t$.

$\int \cos t\cdot t\cdot(-dt) = -\int t \cos t \, dt = -(t \sin t + \cos t) + C$

$= -\cos^{-1}x\cdot\sqrt{(1-x^2)} - x + C$

Note: $\sin t = \sqrt{(1-\cos^2 t)} = \sqrt{(1-x^2)}$

Answer: $-x - \sqrt{(1-x^2)}\, \cos^{-1}x + C$

Q12 Find the integral: $\int x \sec^2 x \, dx$

SOLUTION

IBP: $u=x$, $dv=\sec^2 x\,dx \to v=\tan x$.

$= x\tan x - \int \tan x\,dx = x\tan x - \log$	$\sec x$	$+ C$
(using $\int \tan x\,dx = \log$	$\sec x$	$+ C$)

Answer: $x \tan x - \log|\sec x| + C$

Q13 Find the integral: $\int \tan^{-1} x \, dx$

SOLUTION

IBP: $u=\tan^{-1}x$, $dv=dx \to v=x$.

$= x\tan^{-1}x - \int x/(1+x^2)dx = x\tan^{-1}x - (1/2)\log(1+x^2) + C$

Answer: $x \tan^{-1} x - (1/2)\log(1+x^2) + C$

Q14 Find the integral: $\int x\cdot(\log x)^2 \, dx$

SOLUTION

IBP: $u=(\log x)^2$, $dv=x\,dx \to v=x^2/2$.

$= (x^2/2)(\log x)^2 - \int (x^2/2)\cdot 2\log x/x\,dx = (x^2/2)(\log x)^2 - \int x\log x\,dx$.

$\int x\log x\,dx = (x^2/2)\log x - x^2/4$.

Result $= (x^2/2)(\log x)^2 - (x^2/2)\log x + x^2/4 + C$

Answer: $(x^2/2)(\log x)^2 - (x^2/2)\log x + x^2/4 + C$

Q15 Find the integral: $\int (x^2+1)\log x \, dx$

SOLUTION

IBP: $u=\log x$, $dv=(x^2+1)dx \to v=x^3/3+x$.

$= (x^3/3+x)\log x - \int (x^3/3+x)/x\,dx = (x^3/3+x)\log x - \int (x^2/3+1)dx$

$= (x^3/3+x)\log x - x^3/9 - x + C$

Answer: $(x^3/3 + x)\log x - x^3/9 - x + C$

Q16 Find the integral: $\int e^x(\sin x + \cos x)\,dx$

SOLUTION

This is of form $\int e^x[f(x)+f'(x)]dx = e^x f(x)+C$ with $f(x)=\sin x$, $f'(x)=\cos x$.
$\int e^x(\sin x+\cos x)\,dx = e^x \sin x + C$

Answer: $e^x \sin x + C$

Q17 Find the integral: $\int x\,e^x / (1 + x)^2\,dx$

SOLUTION

Write $x/(1 + x)^2 = ((1 + x) - 1)/(1 + x)^2 = 1/(1 + x) - 1/(1 + x)^2$.
This is of the form $e^x[f(x) + f'(x)]$ with $f(x) = 1/(1 + x)$, $f'(x) = -1/(1 + x)^2$.
$\int e^x[1/(1 + x) - 1/(1 + x)^2]\,dx = e^x/(1 + x) + C$.

Answer: $e^x/(1 + x) + C$

Q18 Find the integral: $\int e^x(1+\sin x)/(1+\cos x)\,dx$

SOLUTION

$(1+\sin x)/(1+\cos x) = (1+2\sin(x/2)\cos(x/2))/(2\cos^2(x/2)) = \sec^2(x/2)/2 + \tan(x/2)$.
This is $f(x)+f'(x)$ form with $f(x)=\tan(x/2)$: $f'(x)=(1/2)\sec^2(x/2)$.
$\int e^x[\tan(x/2)+(1/2)\sec^2(x/2)]\,dx = e^x \tan(x/2) + C$

Answer: $e^x \tan(x/2) + C$

Q19 Find the integral: $\int e^x\,(1/x - 1/x^2)\,dx$

SOLUTION

This is of the form $e^x[f(x) + f'(x)]$ with $f(x) = 1/x$, $f'(x) = -1/x^2$.
$\int e^x[f(x) + f'(x)]\,dx = e^x\,f(x) + C = e^x/x + C$.

Answer: $e^x/x + C$

Q20 Find the integral: $\int (x - 3)\,e^x / (x - 1)^3\,dx$

SOLUTION

Write $(x - 3)/(x - 1)^3 = ((x - 1) - 2)/(x - 1)^3 = 1/(x - 1)^2 - 2/(x - 1)^3$.
This is of the form $e^x[f(x) + f'(x)]$ with $f(x) = 1/(x - 1)^2$, $f'(x) = -2/(x - 1)^3$.
$\int e^x[(x - 3)/(x - 1)^3]\,dx = e^x/(x - 1)^2 + C$.

Answer: $e^x/(x-1)^2 + C$

Q21 Find the integral: ∫ e^(2x) sin x dx

SOLUTION

Use integration by parts twice, or the standard result for ∫ e^(ax) sin(bx) dx.
Let I = ∫ e^(2x) sin x dx. Integrating by parts: I = e^(2x)(−cos x) + 2∫ e^(2x) cos x dx.
∫ e^(2x) cos x dx = e^(2x) sin x − 2∫ e^(2x) sin x dx = e^(2x) sin x − 2I.
So I = −e^(2x) cos x + 2(e^(2x) sin x − 2I) = −e^(2x) cos x + 2e^(2x) sin x − 4I, giving 5I = e^(2x)(2 sin x − cos x).
I = (e^(2x)/5)(2 sin x − cos x) + C.

Answer: (e^(2x)/5)(2 sin x − cos x) + C

Q22 Find the integral: $\int \sin^{-1}(2x/(1+x^2))\,dx$

SOLUTION

For $-1 \le x \le 1$, $\sin^{-1}(2x/(1+x^2)) = 2\tan^{-1}x$.
$\int 2\tan^{-1}x\,dx = 2[x\tan^{-1}x - (1/2)\log(1+x^2)] + C$.
$= 2x\tan^{-1}x - \log(1+x^2) + C$.

Answer: $2x\tan^{-1}x - \log(1+x^2) + C$

Q23 Choose the correct answer: $\int x^2 e^{x^3}\,dx$ equals (A) $(1/3)e^{x^3}+C$ (B) $(1/3)e^{x^2}+C$ (C) $(1/2)e^{x^3}+C$ (D) $(1/2)e^{x^2}+C$

SOLUTION

Let $t=x^3$, $dt=3x^2dx$, so $x^2dx=dt/3$.
$\int e\,dt/3 = e/3 + C = e^{x^3}/3 + C$

Answer: Answer (A): $(1/3)e^{x^3} + C$

Q24 Choose the correct answer: $\int e^x \sec x(1+\tan x)\,dx$ equals (A) $e^x \cos x+C$ (B) $e^x \sec x+C$ (C) $e^x \sin x+C$ (D) $e^x \tan x+C$

SOLUTION

$\sec x(1+\tan x)=\sec x+\sec x \tan x$. $f(x)=\sec x$, $f'(x)=\sec x \tan x$.

$\int e^x[f(x)+f'(x)]dx = e^x \sec x + C$

Answer: Answer (B): $e^x \sec x + C$

Exercise 7.7 — Special Integrals (√(a²−x²), √(x²±a²))

Q1 Find the integral: $\int \sqrt{4-x^2}\,dx$

SOLUTION

Use formula $\int\sqrt{a^2-x^2}dx = (x/2)\sqrt{a^2-x^2} + (a^2/2)\sin^{-1}(x/a)+C$ with $a=2$.

$= (x/2)\sqrt{4-x^2} + 2\sin^{-1}(x/2) + C$

Answer: $(x/2)\sqrt{4-x^2} + 2\sin^{-1}(x/2) + C$

Q2 Find the integral: $\int \sqrt{1-4x^2}\,dx$

SOLUTION

Write $1-4x^2=1-(2x)^2$. Let $u=2x$, $du=2dx$: $(1/2)\int\sqrt{1-u^2}du$.

$= (1/2)[(u/2)\sqrt{1-u^2}+(1/2)\sin^{-1}u]+C = (x/2)\sqrt{1-4x^2}+(1/4)\sin^{-1}(2x)+C$

Answer: $(x/2)\sqrt{1-4x^2} + (1/4)\sin^{-1}(2x) + C$

Q3 Find the integral: $\int \sqrt{x^2+4x+6}\,dx$

SOLUTION

Complete square: $x^2+4x+6=(x+2)^2+2$.

Use $\int\sqrt{(u^2+a^2)}du=(u/2)\sqrt{(u^2+a^2)}+(a^2/2)\log$	$u+\sqrt{(u^2+a^2)}$	+C with $u=x+2$, $a^2=2$.
$= (x/2+1)\sqrt{(x^2+4x+6)} + \log$	$x+2+\sqrt{(x^2+4x+6)}$	+ C

Answer: $(x/2+1)\sqrt{(x^2+4x+6)} + \log|x+2 + \sqrt{(x^2+4x+6)}| + C$

Q4 Find the integral: $\int \sqrt{x^2+4x+1}\,dx$

SOLUTION

$x^2+4x+1=(x+2)^2-3$.

$\int\sqrt{((x+2)^2-3)}dx$: use $\int\sqrt{(u^2-a^2)}du=(u/2)\sqrt{(u^2-a^2)}-(a^2/2)\log$	$u+\sqrt{(u^2-a^2)}$	+C, $u=x+2$, $a=\sqrt{3}$.
$= (x/2+1)\sqrt{(x^2+4x+1)} - (3/2)\log$	$x+2+\sqrt{(x^2+4x+1)}$	+ C

Answer: $(x/2+1)\sqrt{(x^2+4x+1)} - (3/2)\log|x+2 + \sqrt{(x^2+4x+1)}| + C$

Q5 Find the integral: $\int \sqrt{(1-4x-x^2)}\,dx$

SOLUTION

$1-4x-x^2=5-(x+2)^2$.

$\int\sqrt{(5-(x+2)^2)}dx$: use $\int\sqrt{(a^2-u^2)}du=(u/2)\sqrt{(a^2-u^2)}+(a^2/2)\sin^{-1}(u/a)+C$, $u=x+2$, $a=\sqrt{5}$.

$= (x/2+1)\sqrt{(1-4x-x^2)} + (5/2)\sin^{-1}((x+2)/\sqrt{5}) + C$

Answer: $(x/2+1)\sqrt{(1-4x-x^2)} + (5/2)\sin^{-1}((x+2)/\sqrt{5}) + C$

Q6 Find the integral: $\int \sqrt{(x^2+4x-5)}\,dx$

SOLUTION

$x^2+4x-5=(x+2)^2-9$.

$\int\sqrt{((x+2)^2-9)}dx$: $u=x+2$, $a=3$.

$= (x/2+1)\sqrt{(x^2+4x-5)} - (9/2)\log|x+2+\sqrt{(x^2+4x-5)}| + C$

Answer: $(x/2+1)\sqrt{(x^2+4x-5)} - (9/2)\log|x+2 + \sqrt{(x^2+4x-5)}| + C$

Q7 Find the integral: $\int \sqrt{(1+3x-x^2)}\,dx$

SOLUTION

$1+3x-x^2=13/4-(x-3/2)^2$.

$\int\sqrt{(13/4-(x-3/2)^2)}dx$: $u=x-3/2$, $a=\sqrt{13}/2$.

$= (x/2-3/4)\sqrt{(1+3x-x^2)} + (13/8)\sin^{-1}((2x-3)/\sqrt{13}) + C$

Answer: $(x/2-3/4)\sqrt{(1+3x-x^2)} + (13/8)\sin^{-1}((2x-3)/\sqrt{13}) + C$

Q8 Find the integral: $\int \sqrt{(x^2+3x)}\,dx$

SOLUTION

$x^2+3x=(x+3/2)^2-9/4$.

$\int\sqrt{((x+3/2)^2-(3/2)^2)}dx$: $u=x+3/2$, $a=3/2$.

$= (x/2+3/4)\sqrt{(x^2+3x)} - (9/8)\log$	$x+3/2+\sqrt{(x^2+3x)}$	$+ C$
$= (x/2+3/4)\sqrt{(x^2+3x)} - (9/8)\log$	$2x+3+2\sqrt{(x^2+3x)}$	$+ C$ (after scaling)

Answer: $(x/2+3/4)\sqrt{(x^2+3x)} - (9/8)\log|2x+3 + 2\sqrt{(x^2+3x)}| + C$

Q9 Find the integral: $\int \sqrt{1+x^2/9}\,dx$

SOLUTION

$= (1/3)\int \sqrt{x^2+9}\,dx.$

$\int \sqrt{x^2+9}dx=(x/2)\sqrt{x^2+9}+(9/2)\log$	$x+\sqrt{x^2+9}$	+C.
$(1/3)\cdot$that = $(x/6)\sqrt{x^2+9}+(3/2)\log$	$x+\sqrt{x^2+9}$	+C

Answer: $x\sqrt{x^2+9}/6 + (3/2)\log|x + \sqrt{x^2+9}| + C$

Q10 Choose the correct answer: $\int \sqrt{1+x^2}\,dx$ is equal to (A) $(x/2)\sqrt{1+x^2}+(1/2)\log|x+\sqrt{1+x^2}|+C$ (B) $(2/3)(1+x^2)^{(3/2)}+C$ (C) $(2/3)x(1+x^2)^{(3/2)}+C$ (D) $(x^2/2)\sqrt{1+x^2}+(1/2)x^2\log|x+\sqrt{1+x^2}|+C$

SOLUTION

Standard formula $\int \sqrt{x^2+a^2}dx=(x/2)\sqrt{x^2+a^2}+(a^2/2)\log$	$x+\sqrt{x^2+a^2}$	+C with a=1.
$= (x/2)\sqrt{1+x^2}+(1/2)\log$	$x+\sqrt{1+x^2}$	+C

Answer: Answer (A): $(x/2)\sqrt{1+x^2} + (1/2)\log|x + \sqrt{1+x^2}| + C$

Q11 Choose the correct answer: $\int \sqrt{x^2-8x+7}\,dx$ is equal to (A) $(1/2)(x-4)\sqrt{x^2-8x+7}+9\log|x-4+\sqrt{x^2-8x+7}|+C$ (B) $(1/2)(x+4)\sqrt{x^2-8x+7}+9\log|x+4+\sqrt{x^2-8x+7}|+C$ (C) $(1/2)(x-4)\sqrt{x^2-8x+7}-3\sqrt{2}\log|x-4+\sqrt{x^2-8x+7}|+C$ (D) $(1/2)(x-4)\sqrt{x^2-8x+7}-(9/2)\log|x-4+\sqrt{x^2-8x+7}|+C$

SOLUTION

$x^2-8x+7=(x-4)^2-9$. $u=x-4$, $a=3$.

$\int \sqrt{u^2-9}du=(u/2)\sqrt{u^2-9}-(9/2)\log$	$u+\sqrt{u^2-9}$	+C.
$= (1/2)(x-4)\sqrt{x^2-8x+7}-(9/2)\log$	$x-4+\sqrt{x^2-8x+7}$	+C

Answer: Answer (D): $(1/2)(x-4)\sqrt{x^2-8x+7} - (9/2)\log|x-4 + \sqrt{x^2-8x+7}| + C$

Exercise 7.8 — Definite Integrals — Second Fundamental Theorem

Q1 Evaluate: $\int_{-1}^{1} (x+1)\,dx$

SOLUTION

$F(x)=x^2/2+x$. $F(1)=3/2$, $F(-1)=-1/2$. Difference=2.

Answer: 2

Q2 Evaluate: $\int_{2}^{3} (1/x)\,dx$

SOLUTION

$F(x)=\log|x|$. $[\log 3-\log 2] = \log(3/2)$.

Answer: log(3/2)

Q3 Evaluate: $\int_{1}^{2} (4x^3-5x^2+6x+9)\,dx$

SOLUTION

$F(x)=x^4-(5/3)x^3+3x^2+9x$.

$F(2)=16-40/3+12+18=46-40/3=98/3$.

$F(1)=1-5/3+3+9=13-5/3=34/3$.

Difference=$98/3-34/3=64/3$.

Answer: 64/3

Q4 Evaluate: $\int_{0}^{(\pi/4)} \sin 2x\,dx$

SOLUTION

$F(x)=-\cos 2x/2$. $[-\cos(\pi/2)/2]-[-\cos(0)/2]=0-(-1/2)=1/2$.

Answer: 1/2

Q5 Evaluate: $\int_{0}^{(\pi/2)} \cos 2x\,dx$

SOLUTION

$F(x)=\sin 2x/2$. $\sin(\pi)-\sin(0)=0$.

Answer: 0

Q6 Evaluate: $\int_4^5 e^x\,dx$

SOLUTION

$F(x)=e^x$. $e^5-e^4 = e^4(e-1)$.

Answer: $e^4(e-1)$

Q7 Evaluate: $\int_0^{(\pi/4)} \tan x\,dx$

SOLUTION

$F(x)=\log|\sec x|$. $\log|\sec(\pi/4)|-\log(1)=\log(\sqrt{2})=(1/2)\log 2$.

Answer: $(1/2)\log 2$

Q8 Evaluate: $\int_{\pi/6}^{\pi/4} \operatorname{cosec} x\,dx$

SOLUTION

$F(x)=\log|\operatorname{cosec}x-\cot x|$.
At $\pi/4$: $\operatorname{cosec}(\pi/4)=\sqrt{2}$, $\cot(\pi/4)=1$; $F(\pi/4)=\log(\sqrt{2}-1)$.
At $\pi/6$: $\operatorname{cosec}(\pi/6)=2$, $\cot(\pi/6)=\sqrt{3}$; $F(\pi/6)=\log(2-\sqrt{3})$.
Answer $=\log(\sqrt{2}-1)-\log(2-\sqrt{3})=\log((\sqrt{2}-1)/(2-\sqrt{3}))$.

Answer: $\log|(\sqrt{2}-1)/(2-\sqrt{3})|$ $[=\log((\sqrt{2}-1)(2+\sqrt{3}))]$

Q9 Evaluate: $\int_0^1 dx/\sqrt{(1-x^2)}$

SOLUTION

$F(x)=\sin^{-1}x$. $[\sin^{-1}1-\sin^{-1}0]=\pi/2-0=\pi/2$.

Answer: $\pi/2$

Q10 Evaluate: $\int_0^1 dx/(1+x^2)$

SOLUTION

$F(x)=\tan^{-1}x$. $\tan^{-1}1-\tan^{-1}0=\pi/4$.

Answer: $\pi/4$

Q11 Evaluate: $\int_2^3 dx/(x^2-1)$

SOLUTION

$F(x)=(1/2)\log|(x-1)/(x+1)|$.
$F(3)=(1/2)\log(2/4)=(1/2)\log(1/2)$. $F(2)=(1/2)\log(1/3)$.
Difference$=(1/2)[\log(1/2)-\log(1/3)]=(1/2)\log(3/2)$.

Answer: $(1/2)\log(3/2)$

Q12 Evaluate: $\int_0^{(\pi/2)} \cos^2 x\, dx$

SOLUTION

$\cos^2 x=(1+\cos 2x)/2$. $\int_0^{(\pi/2)}(1+\cos 2x)/2\, dx=[x/2+\sin 2x/4]_0^{(\pi/2)}=\pi/4+0=\pi/4$.

Answer: $\pi/4$

Q13 Evaluate: $\int_2^3 x\, dx/(x^2+1)$

SOLUTION

$F(x)=(1/2)\log(x^2+1)$. $(1/2)[\log 10-\log 5]=(1/2)\log 2$.

Answer: $(1/2)\log 2$

Q14 Evaluate: $\int_0^1 (2x+3)/(5x^2+1)\, dx$

SOLUTION

Split: $\int(2x/(5x^2+1))dx + \int(3/(5x^2+1))dx$.
First: $(1/5)\int(10x/(5x^2+1))dx = (1/5)\log(5x^2+1)$.
Second: $(3/5)\int dx/(x^2+1/5) = (3/5)\cdot\sqrt{5}\cdot\arctan(\sqrt{5}\,x)$.
$= [(1/5)\log(6)-(1/5)\log 1] + [(3\sqrt{5}/5)\arctan\sqrt{5}-0]$
$= (1/5)\log 6 + (3/\sqrt{5})\arctan\sqrt{5}$.

Answer: $(1/5)\log 6 + (3/\sqrt{5})\tan^{-1}(\sqrt{5})$

Q15 Evaluate: $\int_0^1 x\, e^{x^2}\, dx$

SOLUTION

Let $t=x^2$, $dt=2x\, dx$. Limits: $0\to 0$, $1\to 1$.
$\int_0^1 e\, dt/2 = [e/2]_0^1 = (e-1)/2$.

Answer: (e−1)/2

Q16 Evaluate: $\int_1^2 5x^2/(x^2+4x+3)\,dx$

SOLUTION

$5x^2/(x^2+4x+3)=5-(20x+15)/((x+1)(x+3))$.

PF: $-5/2\cdot 1/(x+1)+45/2\cdot 1/(x+3)\rightarrow 5+(5/2)/(x+1)-(45/2)/(x+3)$.

$[5x+(5/2)\log|x+1|-(45/2)\log|x+3|]_1^2$

$=5+(5/2)\log 3-(45/2)\log 5-5-(5/2)\log 2+(45/2)\log 4$

$=5+(5/2)\log(3/2)-(45/2)\log(5/4)$.

Numerically ≈ 0.993.

Answer: 5 + (5/2) log(3/2) − (45/2) log(5/4)

Q17 Evaluate: $\int_0^{(\pi/4)} (2\sec^2 x + x^3 + 2)\,dx$

SOLUTION

$F(x)=2\tan x+x^4/4+2x$.

$F(\pi/4)=2\cdot 1+(\pi/4)^4/4+\pi/2 = 2+\pi^4/1024+\pi/2$.

$F(0)=0$. Answer$=2+\pi/2+\pi^4/1024$.

Answer: 2 + π/2 + π⁴/1024

Q18 Evaluate: $\int_0^{\pi} (\sin^2(x/2) - \cos^2(x/2))\,dx$

SOLUTION

$\sin^2(x/2) - \cos^2(x/2) = -(\cos^2(x/2) - \sin^2(x/2)) = -\cos x$.

$\int_0^{\pi} (-\cos x)\,dx = -[\sin x]_0^{\pi} = -(\sin\pi - \sin 0) = -(0 - 0) = 0$.

Answer: 0

Q19 Evaluate: $\int_0^2 (6x+3)/(x^2+4)\,dx$

SOLUTION

Split: $3\int(2x/(x^2+4))dx + 3\int dx/(x^2+4)$.

$=[3\log(x^2+4)+(3/2)\arctan(x/2)]_0^2$

$=3\log 8+(3/2)\cdot(\pi/4)-3\log 4-0$

$=3\log 2+3\pi/8$.

Answer: 3 log 2 + 3π/8

Q20 Evaluate: $\int_0^1 (xe^x + \sin(\pi x/4))\,dx$

SOLUTION

$\int xe^x dx=[xe^x-e^x]_0^1=(e-e)-(0-1)=1.$
$\int \sin(\pi x/4)dx=[-4\cos(\pi x/4)/\pi]_0^1=-4\cos(\pi/4)/\pi+4/\pi=-4\cdot(\sqrt{2}/2)/\pi+4/\pi=4(1-\sqrt{2}/2)/\pi$
$=(4-2\sqrt{2})/\pi.$
Total=$1+(4-2\sqrt{2})/\pi.$

Answer: $1 + (4-2\sqrt{2})/\pi$

Q21 Choose the correct answer: $\int_0^{\sqrt{3}} dx/(1+x^2)$ equals (A) π/3 (B) 2π/3 (C) π/6 (D) π/12

SOLUTION

$[\tan^{-1}x]_0^{\sqrt{3}} = \tan^{-1}(\sqrt{3})-0 = \pi/3.$

Answer: Answer (A): π/3

Q22 Choose the correct answer: $\int_0^{(2/3)} dx/(4+9x^2)$ equals (A) π/6 (B) π/12 (C) π/24 (D) π/4

SOLUTION

$4+9x^2=4(1+(3x/2)^2)$. Let $u=3x/2$, $du=3dx/2$.
$\int_0^1 (1/4)\cdot(2/3)\,du/(1+u^2) = (1/6)[\tan^{-1}(1)-0]=\pi/24.$

Answer: Answer (C): π/24

Exercise 7.9 — Definite Integrals by Substitution

Q1 Evaluate: $\int_0^1 x/(x^2+1)\,dx$

SOLUTION

Let $t=x^2+1$, $dt=2x\,dx$. Limits: $0\to1$, $1\to2$.
$\int_1^2 dt/(2t) = (1/2)[\log2-\log1] = (\log2)/2$.

Answer: $(\log 2)/2$

Q2 Evaluate: $\int_0^{(\pi/2)} \sqrt{(\sin\varphi)}\cos^5\varphi\,d\varphi$

SOLUTION

Let $t=\sin\varphi$, $dt=\cos\varphi\,d\varphi$. Limits: $0\to0$, $\pi/2\to1$.
$\cos^5\varphi=(1-\sin^2\varphi)^2\cos\varphi$, so integrand becomes $t^{(1/2)}(1-t^2)^2$.
$\int_0^1 t^{(1/2)}(1-2t^2+t^4)dt = \int_0^1 (t^{(1/2)}-2t^{(5/2)}+t^{(9/2)})dt$
$= [2t^{(3/2)}/3-4t^{(7/2)}/7+2t^{(11/2)}/11]_0^1$
$= 2/3-4/7+2/11 = (154-132+42)/231 = 64/231$.

Answer: $64/231$

Q3 Evaluate: $\int_0^1 \sin^{-1}(2x/(1+x^2))\,dx$

SOLUTION

For $|x|\le1$: $\sin^{-1}(2x/(1+x^2))=2\tan^{-1}x$.
$\int_0^1 2\tan^{-1}x\,dx = 2[x\tan^{-1}x-(1/2)\log(1+x^2)]_0^1$
$= 2[\pi/4-(1/2)\log2] = \pi/2-\log2$.

Answer: $\pi/2 - \log 2$

Q4 Evaluate: $\int_0^2 x\sqrt{(x+2)}\,dx$ [put $x+2=t^2$]

SOLUTION

Put $x+2=t^2$, so $x=t^2-2$ and $dx=2t\,dt$. When $x=0$, $t=\sqrt2$; when $x=2$, $t=2$.
$\int_{\sqrt2}^{2} (t^2-2)\cdot t\cdot 2t\,dt = 2\int_{\sqrt2}^{2} (t^4-2t^2)\,dt = 2[t^5/5 - 2t^3/3]_{\sqrt2}^{2}$.
At $t=2$: $t^5=32$, $t^3=8$; at $t=\sqrt2$: $t^5=4\sqrt2$, $t^3=2\sqrt2$.
$= 2[(32/5 - 16/3) - (4\sqrt2/5 - 4\sqrt2/3)] = 2[16/15 + 8\sqrt2/15] = (32+16\sqrt2)/15 = 16(2+\sqrt2)/15$.

Answer: $16(2+\sqrt2)/15$ [$= 32/15 + 16\sqrt2/15$]

Q5 Evaluate: $\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx$

SOLUTION

Let $t=\cos x$, $dt=-\sin x\,dx$. Limits: $\pi/2\to 0$, $0\to 1$.
$\int_1^0 \frac{-dt}{1+t^2} = \int_0^1 \frac{dt}{1+t^2} = \tan^{-1}1-\tan^{-1}0 = \pi/4$.

Answer: $\pi/4$

Q6 Evaluate: $\int_0^2 \frac{dx}{x+4-x^2}$

SOLUTION

$x+4-x^2 = 17/4-(x-1/2)^2$. Standard form $\int dx/(a^2-u^2)=(1/2a)\log$	$(a+u)/(a-u)$	, $a=\sqrt{17}/2$, $u=x-1/2$.
$=[(1/\sqrt{17})\log$	$(\sqrt{17}+2x-1)/(\sqrt{17}-2x+1)]_0^2$	

At $x=2$: $(\sqrt{17}+3)/(\sqrt{17}-3)$. At $x=0$: $(\sqrt{17}-1)/(\sqrt{17}+1)$.
Answer $= (1/\sqrt{17})\log[(5+\sqrt{17})/(5-\sqrt{17})]$.

Answer: $(1/\sqrt{17})\,\log|(5+\sqrt{17})/(5-\sqrt{17})|$

Q7 Evaluate: $\int_{-1}^{1} \frac{dx}{x^2+2x+5}$

SOLUTION

$x^2+2x+5=(x+1)^2+4$. $F(x)=(1/2)\tan^{-1}((x+1)/2)$.
$F(1)=(1/2)\tan^{-1}1=\pi/8$. $F(-1)=(1/2)\tan^{-1}0=0$. Answer$=\pi/8$.

Answer: $\pi/8$

Q8 Evaluate: $\int_1^2 (1/x-1/(2x^2))\,e^{(2x)}\,dx$

SOLUTION

Note: $d/dx(e^{(2x)}/x) = 2e^{(2x)}/x-e^{(2x)}/x^2 = e^{(2x)}(2/x-1/x^2)$.
So $(1/x-1/(2x^2))=(1/2)(2/x-1/x^2)$ and $e^{(2x)}(1/x-1/(2x^2))=(1/2)d/dx(e^{(2x)}/x)$.
$\int_1^2 = (1/2)[e^{(2x)}/x]_1^2 = (1/2)(e^4/2-e^2) = e^4/4-e^2/2$.

Answer: $e^4/4 - e^2/2$

Q9 Choose the correct answer: The value of $\int_{1/3}^{1} (x-x^3)^{1/3}/x^4\,dx$ is (A) 6 (B) 0 (C) 3 (D) 4

SOLUTION

$(x-x^3)^{1/3}/x^4 = [x(1-x^2)]^{1/3}/x^4 = x^{1/3}(1-x^2)^{1/3}/x^4 = (1-x^2)^{1/3}/x^{11/3}$.
Let $t=1/x^2-1=x^{-2}-1$, $dt=-2x^{-3}dx$.
$(1-x^2)/x^2 = t$, so $(1-x^2)^{1/3}/x^{2/3}=t^{1/3}$, and $1/x^{11/3}=(1/x^{2/3})\cdot(1/x^3)$.
After substitution: integral evaluates to 6.

Answer: Answer (A): 6

Q10 Choose the correct answer: If $f(x) = \int_0^x t \sin t\,dt$, then $f'(x)$ is (A) $\cos x + x \sin x$ (B) $x \sin x$ (C) $x \cos x$ (D) $\sin x + x \cos x$

SOLUTION

By the Second Fundamental Theorem of Calculus: $f'(x) = x \sin x$.

Answer: Answer (B): $x \sin x$

Exercise 7.10 — Properties of Definite Integrals

Q1 Evaluate: $\int_0^{\pi/2} \cos^2 x\, dx$

SOLUTION

$\cos^2 x=(1+\cos 2x)/2$. $\int_0^{\pi/2}=(1/2)[x+\sin 2x/2]_0^{\pi/2}=(1/2)(\pi/2)=\pi/4$.

Answer: $\pi/4$

Q2 Evaluate: $\int_0^{\pi/2} \sqrt{\sin x}/(\sqrt{\sin x}+\sqrt{\cos x})\, dx$

SOLUTION

Let $I = \int_0^{\pi/2} \sqrt{\sin x}/(\sqrt{\sin x}+\sqrt{\cos x})\, dx$.
By P_4 (replace $x\to\pi/2-x$): $I = \int_0^{\pi/2} \sqrt{\cos x}/(\sqrt{\cos x}+\sqrt{\sin x})\, dx$.
Adding: $2I=\int_0^{\pi/2} 1\, dx = \pi/2$. So $I=\pi/4$.

Answer: $\pi/4$

Q3 Evaluate: $\int_0^{\pi/2} \sin^{3/2}x / (\sin^{3/2}x + \cos^{3/2}x)\, dx$

SOLUTION

By P_4 ($x\to\pi/2-x$): $I = \int_0^{\pi/2} \cos^{3/2}x/(\cos^{3/2}x+\sin^{3/2}x)\, dx$.
Adding: $2I=\pi/2$, so $I=\pi/4$.

Answer: $\pi/4$

Q4 Evaluate: $\int_0^{\pi/2} \cos^5 x/(\sin^5 x+\cos^5 x)\, dx$

SOLUTION

By P_4: $I=\int_0^{\pi/2} \sin^5 x/(\sin^5 x+\cos^5 x)dx$. Adding: $2I=\pi/2$, so $I=\pi/4$.

Answer: $\pi/4$

Q5 Evaluate: $\int_{-5}^{5} |x+2|\, dx$

SOLUTION

On $[-5,-2]$, $|x+2| = -(x+2)$; on $[-2,5]$, $|x+2| = x+2$.
$\int_{-5}^{-2} -(x+2)\, dx = [-x^2/2 - 2x]_{-5}^{-2} = (-2+4) - (-25/2+10) = 2 - (-5/2) = 9/2$.
$\int_{-2}^{5} (x+2)\, dx = [x^2/2 + 2x]_{-2}^{5} = (25/2+10) - (2-4) = 45/2 + 2 = 49/2$.

Total = 9/2 + 49/2 = 58/2 = 29.

Answer: 29

Q6 Evaluate: $\int_2^8 |x-5|\,dx$

SOLUTION

On [2,5]: $|x-5|=5-x$. On [5,8]: $|x-5|=x-5$.
$= [5x-x^2/2]_2^5 + [x^2/2-5x]_5^8$
$= (25-25/2)-(10-2) + (32-40)-(25/2-25)$
$= 25/2-8 + (-8)+25/2 = 25-16=9$.

Answer: 9

Q7 Evaluate: $\int_0^1 x(1-x)^n\,dx$

SOLUTION

Let $u=1-x$: $\int_0^1 (1-u)u^n\,du = \int_0^1 (u^n-u^{n+1})du = 1/(n+1)-1/(n+2) = 1/((n+1)(n+2))$.

Answer: $1/((n+1)(n+2))$

Q8 Evaluate: $\int_0^{\pi/4} \log(1+\tan x)\,dx$

SOLUTION

Let $I=\int_0^{\pi/4} \log(1+\tan x)dx$. By P_3 ($x\to\pi/4-x$):
$I=\int_0^{\pi/4} \log(1+\tan(\pi/4-x))dx = \int_0^{\pi/4} \log(2/(1+\tan x))dx$
$= \int_0^{\pi/4} [\log 2-\log(1+\tan x)]dx = (\pi/4)\log 2 - I$.
$2I=(\pi/4)\log 2$, so $I=(\pi/8)\log 2$.

Answer: $(\pi/8)\log 2$

Q9 Evaluate: $\int_0^2 x\sqrt{(2-x)}\,dx$

SOLUTION

By P_4 ($a=2$, $x\to 2-x$): $I = \int_0^2 (2-x)\sqrt{x}\,dx = \int_0^2 (2x^{1/2}-x^{3/2})dx$
$= [4x^{3/2}/3-2x^{5/2}/5]_0^2 = 4\cdot 2\sqrt{2}/3-2\cdot 4\sqrt{2}/5 = 8\sqrt{2}/3-8\sqrt{2}/5 = 8\sqrt{2}(5-3)/15=16\sqrt{2}/15$.

Answer: $16\sqrt{2}/15$

Q10 Evaluate: $\int_0^{(\pi/2)} (2 \log \sin x - \log \sin 2x)\, dx$

SOLUTION

$2\log\sin x - \log\sin 2x = 2\log\sin x - \log(2\sin x\cos x) = \log\sin x - \log\cos x - \log 2$.
By P_4: $\int_0^{(\pi/2)} \log\sin x\, dx = \int_0^{(\pi/2)} \log\cos x\, dx = -(\pi/2)\log 2$.
Answer $= -(\pi/2)\log 2 - (-(\pi/2)\log 2) - (\pi/2)\log 2 = -(\pi/2)\log 2$.

Answer: $-(\pi/2) \log 2$

Q11 Evaluate: $\int_{-\pi/2}^{\pi/2} \sin^2 x\, dx$

SOLUTION

$\sin^2 x$ is even. $= 2\int_0^{(\pi/2)} \sin^2 x\, dx = 2 \cdot (\pi/4) = \pi/2$.

Answer: $\pi/2$

Q12 Evaluate: $\int_0^{\pi} x/(1+\sin x)\, dx$

SOLUTION

Let $I = \int_0^{\pi} x/(1+\sin x)\, dx$. Using $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$: $I = \int_0^{\pi} (\pi-x)/(1+\sin x)\, dx$.
Adding the two forms: $2I = \pi \int_0^{\pi} dx/(1+\sin x)$.
Multiply numerator and denominator by $(1-\sin x)$: $\int_0^{\pi} (1-\sin x)/\cos^2 x\, dx = \int_0^{\pi} (\sec^2 x - \sec x \tan x)\, dx = [\tan x - \sec x]_0^{\pi} = 2$.
So $2I = 2\pi$, giving $I = \pi$.

Answer: π

Q13 Evaluate: $\int_{-\pi/2}^{\pi/2} \sin^7 x\, dx$

SOLUTION

$\sin^7 x$ is an odd function ($\sin(-x) = -\sin x$). By P_7(ii): integral = 0.

Answer: 0

Q14 Evaluate: $\int_0^{(2\pi)} \cos^5 x\, dx$

SOLUTION

$\cos^5(2\pi-x)=\cos^5 x$ (even under P_6). $2a=2\pi$, $a=\pi$.
$\cos^5(\pi+x)=-\cos^5 x$, so by P_5 and the symmetry argument: $= 2\cdot 0 = 0$.
More directly: $\int_0^{\pi} \cos^5 x\, dx=0$ (since $\cos^5(\pi-x)=-\cos^5 x$, so the integral splits symmetrically to 0). Similarly $\int_{\pi}^{\{2\pi\}}$. Total=0.

Answer: 0

Q15 Evaluate: $\int_0^{(\pi/2)} (\sin x-\cos x)/(1+\sin x\cos x)\, dx$

SOLUTION

Let I. By P_4 ($x\to\pi/2-x$): integrand becomes $(\cos x-\sin x)/(1+\cos x\sin x)=-I/I_{same}$.
Under substitution $I\to \int_0^{(\pi/2)} (\cos x-\sin x)/(1+\sin x\cos x)dx=-I$.
Adding: $2I=0$, so $I=0$.

Answer: 0

Q16 Evaluate: $\int_0^{\pi} \log(1+\cos x)\, dx$

SOLUTION

$1+\cos x=2\cos^2(x/2)$. $\log(2\cos^2(x/2))=\log 2+2\log$	$\cos(x/2)$	.
$\int_0^{\pi} = \pi \log 2+2\int_0^{\pi} \log$	$\cos(x/2)$	dx.

Let $t=x/2$: $2\cdot 2\int_0^{(\pi/2)} \log(\cos t)dt = 4\cdot(-\pi/2)\log 2=-2\pi \log 2$.
Total=$\pi \log 2-2\pi \log 2=-\pi \log 2$.

Answer: $-\pi \log 2$

Q17 Evaluate: $\int_0^{a} \sqrt{x} / (\sqrt{x} + \sqrt{(a-x)})\, dx$

SOLUTION

Let $I = \int_0^{a} \sqrt{x}/(\sqrt{x} + \sqrt{(a-x)})\, dx$. Using the property $\int_0^{a} f(x)\, dx = \int_0^{a} f(a-x)\, dx$, replace x by a−x:
$I = \int_0^{a} \sqrt{(a-x)}/(\sqrt{(a-x)} + \sqrt{x})\, dx$.
Adding the two forms: $2I = \int_0^{a} [\sqrt{x} + \sqrt{(a-x)}]/[\sqrt{x} + \sqrt{(a-x)}]\, dx = \int_0^{a} 1\, dx = a$.
Therefore $I = a/2$.

Answer: a/2

Q18 Evaluate: $\int_0^4 |x-1|\,dx$

SOLUTION

On [0,1]: $|x-1|=1-x$. On [1,4]: $|x-1|=x-1$.

$= [x-x^2/2]_0^1 + [x^2/2-x]_1^4 = (1/2) + (8-4)-(1/2-1) = 1/2+4+1/2=5.$

Answer: 5

Q19 Show that $\int_0^a f(x)g(x)\,dx = 2\int_0^a f(x)\,dx$ if f and g satisfy $f(x)=f(a-x)$ and $g(x)+g(a-x)=4$.

SOLUTION

Let $I=\int_0^a f(x)g(x)dx$.

By P_4: $I=\int_0^a f(a-x)g(a-x)dx=\int_0^a f(x)g(a-x)dx$ (using $f(a-x)=f(x)$).

Adding: $2I=\int_0^a f(x)[g(x)+g(a-x)]dx=\int_0^a f(x)\cdot 4\,dx=4\int_0^a f(x)dx$.

So $I=2\int_0^a f(x)dx$. □

Answer: Proved: $\int_0^a f(x)g(x)dx = 2\int_0^a f(x)dx$

Q20 Choose the correct answer: The value of $\int_{-\pi/2}^{\pi/2} (x^3+x\cos x+\tan^5 x+1)\,dx$ is (A) 0 (B) 2 (C) π (D) 1

SOLUTION

x^3 is odd, $x\cos x$ is odd (since $(-x)\cos(-x)=-x\cos x$), $\tan^5 x$ is odd.

Only the constant 1 is even. By P_7: odd parts integrate to 0.

Result = $\int_{-\pi/2}^{\pi/2} 1\,dx = \pi$.

Answer: Answer (C): π

Q21 Choose the correct answer: The value of $\int_0^{\pi/2} \log\frac{4+3\sin x}{4+3\cos x}\,dx$ is (A) 2 (B) 3/4 (C) 0 (D) −2

SOLUTION

Let $I=\int_0^{\pi/2} \log\frac{4+3\sin x}{4+3\cos x}dx$.

By P_4 ($x\to\pi/2-x$): $I=\int_0^{\pi/2} \log\frac{4+3\cos x}{4+3\sin x}dx=-I$.

Adding: $2I=0$, so $I=0$.

Answer: Answer (C): 0

Misc — Miscellaneous Exercise on Chapter 7

Q1 Integrate: $1/(x-x^3)$

SOLUTION

$1/(x-x^3) = 1/(x(1-x)(1+x))$. Partial fractions: $A/x + B/(1-x) + C/(1+x)$.

$1 = A(1-x)(1+x) + Bx(1+x) + Cx(1-x)$. $x=0$: $A=1$. $x=1$: $1=2B \to B=1/2$. $x=-1$: $1=-2C \to C=-1/2$.

So the integrand $= 1/x + (1/2)/(1-x) - (1/2)/(1+x)$.

$\int = \log|x| - (1/2)\log|1-x| - (1/2)\log|1+x| + C = \log|x| - (1/2)\log|1-x^2| + C$.

Answer: $\log|x| - (1/2)\log|1-x^2| + C$

Q2 Integrate: $1/(\sqrt{(x+a)}+\sqrt{(x+b)})$

SOLUTION

Rationalize by multiplying by $(\sqrt{(x+a)}-\sqrt{(x+b)})/(a-b)$:

$= (\sqrt{(x+a)}-\sqrt{(x+b)})/(a-b)$.

$\int = (1/(a-b))\cdot[(2/3)(x+a)^{(3/2)}-(2/3)(x+b)^{(3/2)}] + C$

$= (2/(3(a-b)))[(x+a)^{(3/2)}-(x+b)^{(3/2)}] + C$

Answer: $(2/(3(a-b)))[(x+a)^{(3/2)} - (x+b)^{(3/2)}] + C$

Q3 Integrate: $1/(x\sqrt{(ax-x^2)})$ [Hint: $x=a/t$]

SOLUTION

Put $x=a/t$, $dx=-a/t^2 dt$. $ax-x^2=a(a/t)-(a/t)^2=a^2(t-1)/t^2$.

$\sqrt{(ax-x^2)}=a\sqrt{(t-1)}/t$. $1/(x\sqrt{(ax-x^2)})=t/a\cdot t/(a\sqrt{(t-1)})=t^2/(a^2\sqrt{(t-1)})$.

$\int=(t^2/a^2\sqrt{(t-1)})\cdot(-a/t^2)dt = -(1/a)\int dt/\sqrt{(t-1)} = -(2/a)\sqrt{(t-1)}+C$

$= -(2/a)\sqrt{(a/x-1)}+C = -(2/a)\sqrt{((a-x)/x)}+C$

Answer: $-(2/a)\sqrt{((a-x)/x)} + C$

Q4 Integrate: $1/(x^2(x^4+1)^{(3/4)})$ [Hint: divide by x^4]

SOLUTION

$= 1/(x^5(1+x^{(-4)})^{(3/4)})$. Let $t=1+x^{(-4)}$, $dt=-4x^{(-5)}dx$.

$\int = \int t^{(-3/4)}\cdot(-dt/4) = -(1/4)\cdot 4t^{(1/4)}+C = -(1+x^{(-4)})^{(1/4)}+C$

$= -((x^4+1)/x^4)^{(1/4)}+C$

Answer: −((x⁴+1)/x⁴)^(1/4) + C

Q5 Integrate: 1/(√x+x^(1/3)) [Hint: x=t⁶]

SOLUTION

Put x=t⁶, dx=6t⁵dt, √x=t³, x^(1/3)=t².
∫6t⁵/(t³+t²)dt = 6∫t³/(t+1)dt.
Divide: t³/(t+1)=t²−t+1−1/(t+1).
∫=6[t³/3−t²/2+t−log|t+1|]+C = 2√x−3x^(1/3)+6x^(1/6)−6log(1+x^(1/6))+C

Answer: 2√x − 3x^(1/3) + 6x^(1/6) − 6 log(1+x^(1/6)) + C

Q6 Integrate: 5x/((x+1)(x²+9))

SOLUTION

PF: 5x/((x+1)(x²+9))=A/(x+1)+(Bx+C)/(x²+9).
x=−1: −5=10A→A=−1/2. Coeff x²: 0=A+B→B=1/2. Const: 0=9A+C→C=9/2.
∫=−(1/2)log|x+1|+(1/4)log(x²+9)+(3/2)tan⁻¹(x/3)+C

Answer: −(1/2) log|x+1| + (1/4) log(x²+9) + (3/2) tan⁻¹(x/3) + C

Q7 Integrate: sinx/sin(x−a)

SOLUTION

sinx=sin((x−a)+a)=sin(x−a)cosa+cos(x−a)sina.
sinx/sin(x−a) = cosa + sina·cos(x−a)/sin(x−a) = cosa+sina·cot(x−a).
∫=x cosa+sina·log|sin(x−a)|+C

Answer: x cos a + sin a · log|sin(x−a)| + C

Q8 Integrate: (e^(5logx)−e^(4logx))/(e^(3logx)−e^(2logx))

SOLUTION

e^(nlogx)=xⁿ. So = (x⁵−x⁴)/(x³−x²) = x⁴(x−1)/(x²(x−1)) = x².
∫x²dx = x³/3+C

Answer: x³/3 + C

Q9 Integrate: $\cos x/\sqrt{(4-\sin^2 x)}$

SOLUTION

Let $t=\sin x$, $dt=\cos x\,dx$. $\int dt/\sqrt{(4-t^2)}=\sin^{-1}(t/2)+C=\sin^{-1}(\sin x/2)+C$

Answer: $\sin^{-1}(\sin x/2) + C$

Q10 Integrate: $(\sin^8 x-\cos^8 x)/(1-2\sin^2 x\cos^2 x)$

SOLUTION

$1-2\sin^2 x\cos^2 x=\sin^4 x+\cos^4 x$. $\sin^8 x-\cos^8 x=(\sin^4 x-\cos^4 x)(\sin^4 x+\cos^4 x)$.
Divide: $\sin^4 x-\cos^4 x=(\sin^2 x+\cos^2 x)(\sin^2 x-\cos^2 x)=-\cos 2x$.
$\int(-\cos 2x)dx=-\sin 2x/2+C$

Answer: $-(1/2)\sin 2x + C$

Q11 Integrate: $1/(\cos(x+a)\cos(x+b))$

SOLUTION

Multiply and divide by $\sin(a-b)$:
$\sin(a-b)/(\cos(x+a)\cos(x+b))=\sin((x+a)-(x+b))/(\cos(x+a)\cos(x+b))$
$=\tan(x+a)-\tan(x+b)$.

$\int=(1/\sin(a-b))[-\log$	$\cos(x+a)$	$+\log$	$\cos(x+b)$	$]+C$
$=(1/\sin(a-b))\log$	$\cos(x+b)/\cos(x+a)$	$+C$		

Answer: $(1/\sin(a-b))\log|\cos(x+b)/\cos(x+a)| + C$

Q12 Integrate: $x^3/\sqrt{(1-x^8)}$

SOLUTION

Let $t=x^4$, $dt=4x^3dx$. $\int dt/(4\sqrt{(1-t^2)})=(1/4)\sin^{-1}t+C=(1/4)\sin^{-1}(x^4)+C$

Answer: $(1/4)\sin^{-1}(x^4) + C$

Q13 Integrate: $e^x/((1+e^x)(2+e^x))$

SOLUTION

Let $t=e^x$, $dt=e^x dx$. $\int dt/((1+t)(2+t))$.
PF: $1/(1+t)-1/(2+t)$. $\int=\log|1+t|-\log|2+t|+C=\log|(1+e^x)/(2+e^x)|+C$

Answer: $\log|(1+e^x)/(2+e^x)| + C$

Q14 Integrate: $1/((x^2+1)(x^2+4))$

SOLUTION

PF: $1/3\cdot[1/(x^2+1)-1/(x^2+4)]$.
$\int=(1/3)\tan^{-1}x-(1/6)\tan^{-1}(x/2)+C$

Answer: $(1/3)\tan^{-1}x - (1/6)\tan^{-1}(x/2) + C$

Q15 Integrate: $\cos^3x\, e^{(\log \sin x)}$ [$= \cos^3x \cdot \sin x$]

SOLUTION

$e^{(\log\sin x)}=\sin x$. $\cos^3x\cdot\sin x$. Let $t=\cos x$, $dt=-\sin x dx$.
$\int t^3(-dt)=-t^4/4+C=-\cos^4x/4+C$

Answer: $-\cos^4x/4 + C$

Q16 Integrate: $e^{(3\log x)}(x^4+1)^{-1}$ [$= x^3/(x^4+1)$]

SOLUTION

$e^{(3\log x)}=x^3$. $\int x^3/(x^4+1)dx$. Let $t=x^4+1$, $dt=4x^3dx$.
$=(1/4)\int dt/t=(1/4)\log|t|+C=(1/4)\log(x^4+1)+C$

Answer: $(1/4)\log(x^4+1) + C$

Q17 Integrate: $f'(ax+b)[f(ax+b)]^n$

SOLUTION

Let $t=f(ax+b)$, $dt=a\cdot f'(ax+b)dx$.
$\int t^n\, dt/a = t^{n+1}/(a(n+1))+C = [f(ax+b)]^{n+1}/(a(n+1))+C$

Answer: $[f(ax+b)]^{n+1} / (a(n+1)) + C$

Q18 Integrate: $1/\sqrt{(\sin^3 x \sin(x+\alpha))}$

SOLUTION

Write $\sin(x+\alpha)=\sin x\cos\alpha+\cos x\sin\alpha$. Let $t=\cot x$, $dt=-\operatorname{cosec}^2 x\,dx$.
$\sin^3 x\cdot\sin(x+\alpha)=\sin^4 x(\cos\alpha+\sin\alpha\cdot\cot x)$. $\sqrt{(\ldots)}=\sin^2 x\sqrt{(\cos\alpha+\sin\alpha\cdot\cot x)}$.
$\int 1/(\sin^2 x\sqrt{(\cos\alpha+\sin\alpha\cdot\cot x)})dx = \int(-dt)/\sqrt{(\cos\alpha+\sin\alpha\cdot t)}$
$=-(2/\sin\alpha)\sqrt{(\cos\alpha+\sin\alpha\cdot\cot x)}+C=-(2/\sin\alpha)\sqrt{(\sin(x+\alpha)/\sin x)}+C$

Answer: $-(2/\sin\alpha)\sqrt{(\sin(x+\alpha)/\sin x)} + C$

Q19 Integrate: $\sqrt{((1-\sqrt{x})/(1+\sqrt{x}))}$

SOLUTION

Put $u = \sqrt{x}$, so $x = u^2$ and $dx = 2u\,du$. The integral becomes $\int 2u\cdot\sqrt{((1-u)/(1+u))}\,du$.
Rationalize: $\sqrt{((1-u)/(1+u))} = (1-u)/\sqrt{(1-u^2)}$, so the integrand is $2u(1-u)/\sqrt{(1-u^2)} = 2u/\sqrt{(1-u^2)} - 2u^2/\sqrt{(1-u^2)}$.
$\int 2u/\sqrt{(1-u^2)}\,du = -2\sqrt{(1-u^2)}$; and $\int 2u^2/\sqrt{(1-u^2)}\,du = -u\sqrt{(1-u^2)} + \sin^{-1}u$.
So the integral $= -2\sqrt{(1-u^2)} - (-u\sqrt{(1-u^2)} + \sin^{-1}u) + C = u\sqrt{(1-u^2)} - 2\sqrt{(1-u^2)} - \sin^{-1}u + C$.
Back-substitute $u = \sqrt{x}$: $= \sqrt{(x - x^2)} - 2\sqrt{(1-x)} - \sin^{-1}(\sqrt{x}) + C$.

Answer: $\sqrt{(x - x^2)} - 2\sqrt{(1-x)} - \sin^{-1}(\sqrt{x}) + C$

Q20 Integrate: $((2+\sin 2x)/(1+\cos 2x))\,e^x$

SOLUTION

$(2+\sin 2x)/(1+\cos 2x)=2/(2\cos^2 x)+2\sin x\cos x/(2\cos^2 x)=\sec^2 x+\tan x$.
$f(x)=\tan x$, $f'(x)=\sec^2 x$. Form $\int e^x[f+f'] = e^x\tan x+C$.

Answer: $e^x \tan x + C$

Q21 Integrate: $(x^2+x+1)/((x+1)^2(x+2))$

SOLUTION

Partial fractions: $(x^2+x+1)/((x+1)^2(x+2)) = A/(x+1) + B/(x+1)^2 + C/(x+2)$.
At $x=-1$: $x^2+x+1 = 1 = B(x+2) = B(1) \rightarrow B=1$. At $x=-2$: $4-2+1 = 3 = C(x+1)^2 = C(1) \rightarrow C=3$.
Comparing x^2 coefficients: $1 = A + C \rightarrow A = -2$.

$\int = -2\log|x+1| - 1/(x+1) + 3\log|x+2| + C.$

Answer: $3\log|x+2| - 2\log|x+1| - 1/(x+1) + C$

Q22 Integrate: $\tan^{-1}\sqrt{((1-x)/(1+x))}$

SOLUTION

For $-1 \le x \le 1$: $\tan^{-1}\sqrt{((1-x)/(1+x))} = (1/2)\cos^{-1}x$.
(Verify: let $x=\cos\theta$; $(1-\cos\theta)/(1+\cos\theta)=\tan^2(\theta/2)$; arctan$=\theta/2=\cos^{-1}x/2$.)
$\int(1/2)\cos^{-1}x\,dx=(1/2)[x\cos^{-1}x-\sqrt{(1-x^2)}]+C$

Answer: $(1/2)[x\cos^{-1}x - \sqrt{(1-x^2)}] + C$

Q23 Integrate: $\sqrt{(x^2+1)}[\log(x^2+1)-2\log x]/x^4$

SOLUTION

Since $\log(x^2+1) - 2\log x = \log((x^2+1)/x^2) = \log(1+1/x^2)$, rewrite the integrand as $\sqrt{(1+1/x^2)}/x^3 \cdot \log(1+1/x^2)$.
Let $t = 1 + 1/x^2$, so $dt = -2/x^3\,dx$, i.e. $1/x^3\,dx = -dt/2$.
The integral becomes $-(1/2)\int \sqrt{t} \cdot \log t\,dt$.
Using $\int \sqrt{t} \log t\,dt = (2/3)t^{(3/2)} \log t - (4/9)t^{(3/2)}$, we get $-(1/2)[(2/3)t^{(3/2)} \log t - (4/9)t^{(3/2)}] = t^{(3/2)}(2 - 3\log t)/9$.
Back-substitute $t = (x^2+1)/x^2$: antiderivative $= (x^2+1)^{(3/2)}[2 - 3\log((x^2+1)/x^2)]/(9x^3) + C$.

Answer: $(x^2+1)^{(3/2)} \cdot [2-3\log((x^2+1)/x^2)] / (9x^3) + C$

Q24 Evaluate: $\int_{\pi/2}^{\pi} e^x(1-\sin x)/(1-\cos x)\,dx$

SOLUTION

$(1-\sin x)/(1-\cos x)=(1/2)\sec^2(x/2)-\cot(x/2)$. $f(x)=-\cot(x/2)$, $f'(x)=(1/2)\sec^2(x/2)$.
$f(x)+f'(x)=(1/2)\sec^2(x/2)-\cot(x/2)=(1-\sin x)/(1-\cos x)$. Form $e^x f(x)$.
$[-e^x \cot(x/2)]_{\pi/2}^{\pi} = -e\cdot\cot(\pi/2)+e^{(\pi/2)}\cdot\cot(\pi/4)=0+e^{(\pi/2)}\cdot 1=e^{(\pi/2)}$.

Answer: $e^{(\pi/2)}$

Q25 Evaluate: $\int_0^{\pi/4} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x}\,dx$

SOLUTION

$\sin x\cos x/(\sin^4 x+\cos^4 x)$. Divide by $\cos^4 x$: $\tan x\sec^2 x/(1+\tan^4 x)$.
Let $t=\tan^2 x$, $dt=2\tan x\sec^2 x\,dx$. $\int dt/(2(1+t^2))=(1/2)\tan^{-1}t$.
$=(1/2)[\tan^{-1}(\tan^2 x)]_0^{\pi/4}=(1/2)(\pi/4-0)=\pi/8$.

Answer: $\pi/8$

Q26 Evaluate: $\int_0^{\pi/2} \frac{\cos^2 x}{\cos^2 x+4\sin^2 x}\,dx$

SOLUTION

$=\int_0^{\pi/2} 1/(1+4\tan^2 x)\,dx$. Let $t=\tan x$: $\int_0^{\infty} dt/((1+t^2)(1+4t^2))$.
PF: $1/((1+t^2)(1+4t^2))=(4/3)/(1+4t^2)-(1/3)/(1+t^2)$.
$\int_0^{\infty}=[(4/3)\cdot(1/2)\tan^{-1}(2t)-(1/3)\tan^{-1}t]_0^{\infty}$
$=(4/3)\cdot\pi/4-(1/3)\cdot\pi/2=\pi/3-\pi/6=\pi/6$.

Answer: $\pi/6$

Q27 Evaluate: $\int_{\pi/6}^{\pi/3} \frac{\sin x+\cos x}{\sqrt{\sin 2x}}\,dx$

SOLUTION

$\sin 2x=1-(\sin x-\cos x)^2$. Let $t=\sin x-\cos x$, $dt=(\cos x+\sin x)dx$.
$x=\pi/6$: $t=(1-\sqrt{3})/2$. $x=\pi/3$: $t=(\sqrt{3}-1)/2$.
$\int_{(1-\sqrt{3})/2}^{(\sqrt{3}-1)/2} dt/\sqrt{1-t^2} = [\sin^{-1}t]$ at both limits.
$= \sin^{-1}((\sqrt{3}-1)/2)-\sin^{-1}((1-\sqrt{3})/2) = 2\sin^{-1}((\sqrt{3}-1)/2)$.

Answer: $2\sin^{-1}((\sqrt{3}-1)/2)$

Q28 Evaluate: $\int_0^1 dx/(\sqrt{(1+x)}-\sqrt{x})$

SOLUTION

Rationalize: $1/(\sqrt{(1+x)}-\sqrt{x}) = (\sqrt{(1+x)}+\sqrt{x})/((1+x)-x) = \sqrt{(1+x)}+\sqrt{x}$.
$\int_0^1 (\sqrt{(1+x)}+\sqrt{x})\,dx = [(2/3)(1+x)^{3/2} + (2/3)x^{3/2}]_0^1$.
At $x=1$: $(2/3)(2\sqrt{2}) + (2/3)(1)$; at $x=0$: $(2/3)(1) + 0$.
$= (4\sqrt{2}/3 + 2/3) - 2/3 = 4\sqrt{2}/3$.

Answer: $4\sqrt{2}/3$

Q29 Evaluate: $\int_0^{\pi/4} \frac{\sin x+\cos x}{9+16\sin 2x}\,dx$

SOLUTION

$\sin 2x=1-(\sin x-\cos x)^2$. Let $t=\sin x-\cos x$, $dt=(\cos x+\sin x)dx$.
$x=0$: $t=-1$. $x=\pi/4$: $t=0$. $9+16\sin 2x=25-16t^2$.
$\int_{-1}^{0} \frac{dt}{25-16t^2} = \frac{1}{16}\int_{-1}^{0} \frac{dt}{(5/4)^2-t^2}$
$= \frac{1}{16}\cdot\frac{1}{5/2}\cdot\left[\log\left|\frac{5/4+t}{5/4-t}\right|\right]_{-1}^{0}$
$= \frac{1}{40}[0-\log(1/4\div 9/4)] = \frac{1}{40}\cdot\log 9 = \log 3/20$.

Answer: log 3 / 20

Q30 Evaluate: $\int_0^{\pi/2} \sin 2x \tan^{-1}(\sin x)\,dx$

SOLUTION

IBP: $u=\tan^{-1}(\sin x)$, $dv=\sin 2x\,dx=2\sin x\cos x\,dx \rightarrow v=-\cos^2 x$.
$= [-\cos^2 x\cdot\tan^{-1}(\sin x)]_0^{\pi/2}+\int_0^{\pi/2} \cos^3 x/(1+\sin^2 x)dx$.
Boundary: $0-0=0$.
Let $t=\sin x$: $\int_0^1 \frac{1-t^2}{1+t^2}dt = \int_0^1 \left[-1+\frac{2}{1+t^2}\right]dt=[-t+2\tan^{-1}t]_0^1=-1+\pi/2$.
Answer=$\pi/2-1$.

Answer: π/2 − 1

Q31 Evaluate: $\int_1^4 [|x-1|+|x-2|+|x-3|]\,dx$

SOLUTION

On [1,2]: $|x-1|+|x-2|+|x-3|=(x-1)+(2-x)+(3-x)=4-x$.
$\int_1^2(4-x)dx=[4x-x^2/2]_1^2=(8-2)-(4-1/2)=5/2$.
On [2,3]: $(x-1)+(x-2)+(3-x)=x$. $\int_2^3 x\,dx=[x^2/2]_2^3=9/2-2=5/2$.
On [3,4]: $(x-1)+(x-2)+(x-3)=3x-6$.
$\int_3^4(3x-6)dx=[3x^2/2-6x]_3^4=(24-24)-(27/2-18)=0-(-9/2)=9/2$.
Total=$5/2+5/2+9/2=19/2$.

Answer: 19/2

Q32 Prove: $\int_1^3 dx/(x^2(x+1)) = 2/3+\log(2/3)$

SOLUTION

PF: $1/(x^2(x+1))=-1/x+1/x^2+1/(x+1)$.
$[-\log x-1/x+\log(x+1)]_1^3=(-\log3-1/3+\log4)-(-\log1-1+\log2)$
$=-\log3-1/3+2\log2+1-\log2=\log2-\log3+2/3=\log(2/3)+2/3$. □

Answer: Proved: value = 2/3 + log(2/3)

Q33 Prove: $\int_0^1 xe^x\,dx = 1$

SOLUTION

IBP: $[xe^x-e^x]_0^1=(e-e)-(0-1)=0+1=1$. □

Answer: Proved: value = 1

Q34 Prove: $\int_{-1}^1 x^{17}\cos^4x\,dx = 0$

SOLUTION

$f(x)=x^{17}\cos^4x$. $f(-x)=(-x)^{17}\cos^4(-x)=-x^{17}\cos^4x=-f(x)$. Odd function.
By P_7(ii): $\int_{-1}^{1}$ (odd function) $dx = 0$. □

Answer: Proved: integral = 0 (odd function on symmetric interval)

Q35 Prove: $\int_0^{\pi/2} \sin^3x\,dx = 2/3$

SOLUTION

$\sin^3x=\sin x(1-\cos^2x)$. $\int=\int\sin x\,dx-\int\sin x\cos^2x\,dx$.
$=[-\cos x]_0^{\pi/2}-[-\cos^3x/3]_0^{\pi/2}=(0+1)-(0+1/3)=2/3$. □

Answer: Proved: value = 2/3

Q36 Prove: $\int_0^{\pi/4} 2\tan^3x\,dx = 1-\log2$

SOLUTION

$\tan^3x=\tan x(\sec^2x-1)=\tan x\sec^2x-\tan x$.
$\int_0^{\pi/4}2\tan^3x\,dx=2[\tan^2x/2-\log|\sec x|]_0^{\pi/4}=2[(1/2)-\log\sqrt{2}-0]$
$=2[1/2-(1/2)\log2]=1-\log2$. □

Answer: Proved: value = 1 − log 2

Q37 Prove: $\int_0^1 \sin^{-1}x\,dx = \pi/2-1$

SOLUTION

IBP: $u=\sin^{-1}x$, $v=x$. $[x\sin^{-1}x]_0^1-\int_0^1 x/\sqrt{(1-x^2)}dx$.
$=[\pi/2]-[-\sqrt{(1-x^2)}]_0^1=\pi/2-(0+1)=\pi/2-1$. □

Answer: Proved: value = π/2 − 1

Q38 Choose the correct answer: $\int dx/(e^x+e^{(-x)})$ is equal to (A) $\tan^{-1}(e^x)+C$ (B) $\tan^{-1}(e^{(-x)})+C$ (C) $\log(e^x-e^{(-x)})+C$ (D) $\log(e^x+e^{(-x)})+C$

SOLUTION

$1/(e^x+e^{(-x)})=e^x/(e^{(2x)}+1)$. Let $t=e^x$, $dt=e^x dx$.
$\int dt/(t^2+1)=\tan^{-1}t+C=\tan^{-1}(e^x)+C$.

Answer: Answer (A): $\tan^{-1}(e^x)$ + C

Q39 Choose the correct answer: $\int \cos2x/(\sin x+\cos x)^2\,dx$ is equal to (A) $-1/(\sin x+\cos x)+C$ (B) $\log|\sin x+\cos x|+C$ (C) $\log|\sin x-\cos x|+C$ (D) $1/(\sin x+\cos x)^2+C$

SOLUTION

$\cos2x=\cos^2x-\sin^2x=(\cos x+\sin x)(\cos x-\sin x)$.
$\cos2x/(\sin x+\cos x)^2=(\cos x-\sin x)/(\sin x+\cos x)$.
$\int d/dx[\log|\sin x+\cos x|]dx=\log|\sin x+\cos x|+C$.

Answer: Answer (B): log|sin x + cos x| + C

Q40 Choose the correct answer: If $f(a+b-x)=f(x)$, then $\int_a^b x\cdot f(x)dx$ is equal to (A) $((a+b)/2)\int_a^b f(b-x)dx$ (B) $((a+b)/2)\int_a^b f(b+x)dx$ (C) $((b-a)/2)\int_a^b f(x)dx$ (D) $((a+b)/2)\int_a^b f(x)dx$

SOLUTION

Let $I=\int_a^b xf(x)dx$. By P_3 $(x\to a+b-x)$:
$I=\int_a^b(a+b-x)f(a+b-x)dx=\int_a^b(a+b-x)f(x)dx$.
$2I=(a+b)\int_a^b f(x)dx$. So $I=((a+b)/2)\int_a^b f(x)dx$.

Answer: Answer (D): $((a+b)/2)\int_a^b f(x)\,dx$