Chapter 8: Application of Integrals

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Exercise 8.1 — Exercise 8.1

Q1 Find the area of the region bounded by the ellipse $x^2/16 + y^2/9 = 1$.

SOLUTION

The ellipse is $x^2/16 + y^2/9 = 1$, giving $a = 4$ and $b = 3$.

From the equation: $y = (3/4)\sqrt{(16 - x^2)}$ (taking positive value in the first quadrant).

Since the ellipse is symmetrical about both axes, the total area is 4 times the area in the first quadrant:
Area $= 4 \int_0^4 y\, dx = 4 \int_0^4 (3/4)\sqrt{(16 - x^2)}\, dx = 3 \int_0^4 \sqrt{(16 - x^2)}\, dx$

Using the standard formula $\int\sqrt{(a^2 - x^2)}\, dx = (x/2)\sqrt{(a^2 - x^2)} + (a^2/2)\sin^{-1}(x/a) + C$, with $a = 4$:
Area $= 3\,[x/2 \cdot \sqrt{(16 - x^2)} + 8\sin^{-1}(x/4)]_0^4$

At $x = 4$: $(4/2)\cdot\sqrt{(16-16)} + 8\sin^{-1}(1) = 0 + 8\cdot(\pi/2) = 4\pi$
At $x = 0$: $(0/2)\cdot\sqrt{16} + 8\sin^{-1}(0) = 0$

Area $= 3\,[4\pi - 0] = 12\pi$

Answer: 12π square units

Q2 Find the area of the region bounded by the ellipse $x^2/4 + y^2/9 = 1$.

SOLUTION

The ellipse is $x^2/4 + y^2/9 = 1$, giving $a = 2$ and $b = 3$.

From the equation: $y = (3/2)\sqrt{(4 - x^2)}$ (taking positive value in the first quadrant).

Since the ellipse is symmetrical about both axes, the total area is 4 times the area in the first quadrant:
Area $= 4 \int_0^2 y\, dx = 4 \int_0^2 (3/2)\sqrt{(4 - x^2)}\, dx = 6 \int_0^2 \sqrt{(4 - x^2)}\, dx$

Using the standard formula $\int\sqrt{(a^2 - x^2)}\, dx = (x/2)\sqrt{(a^2 - x^2)} + (a^2/2)\sin^{-1}(x/a) + C$, with $a = 2$:

Area = $6\,[x/2 \cdot \sqrt{(4 - x^2)} + 2\sin^{-1}(x/2)]_0^2$

At x = 2: $(2/2)\cdot\sqrt{(4-4)} + 2\sin^{-1}(1) = 0 + 2\cdot(\pi/2) = \pi$
At x = 0: $(0/2)\cdot\sqrt{4} + 2\sin^{-1}(0) = 0$

Area = $6\,[\pi - 0] = 6\pi$

Answer: 6π square units

Q3 Area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines x = 0 and x = 2 is
(A) π
(B) $\pi/2$
(C) $\pi/3$
(D) $\pi/4$

SOLUTION

The circle is $x^2 + y^2 = 4$ (radius 2). In the first quadrant, $y = \sqrt{(4 - x^2)} \geq 0$.

Area = $\int_0^2 \sqrt{(4 - x^2)}\,dx$

Using the formula $\int\sqrt{(a^2 - x^2)}\,dx = (x/2)\sqrt{(a^2 - x^2)} + (a^2/2)\sin^{-1}(x/a) + C$, with a = 2:
Area = $[x/2 \cdot \sqrt{(4 - x^2)} + 2\sin^{-1}(x/2)]_0^2$

At x = 2: $(2/2)\cdot\sqrt{(4-4)} + 2\sin^{-1}(1) = 0 + 2\cdot(\pi/2) = \pi$
At x = 0: $0 + 2\sin^{-1}(0) = 0$

Area = $\pi - 0 = \pi$

Answer: (A) π

Q4 Area of the region bounded by the curve $y^2 = 4x$, y-axis and the line y = 3 is
(A) 2
(B) 9/4
(C) 9/3
(D) 9/2

SOLUTION

The curve is $y^2 = 4x$, so $x = y^2/4$.

The region is bounded by the curve $y^2 = 4x$, the y-axis (x = 0), and the line y = 3. Using horizontal strips, x ranges from 0 to $y^2/4$ for y in [0, 3].

Area = $\int_0^3 x\,dy = \int_0^3 (y^2/4)\,dy$

$= [y^3/12]_0^3$

$= 27/12 - 0$

$= 9/4$

Answer: (B) 9/4

Misc — Miscellaneous Exercise

Q1 Find the area under the given curves and given lines:
(i) $y = x^2$, $x = 1$, $x = 2$ and x-axis
(ii) $y = x^4$, $x = 1$, $x = 5$ and x-axis

SOLUTION

(i) $y = x^2$, bounded by $x = 1$, $x = 2$, and the x-axis.

Since $y = x^2 \geq 0$ for all x, the curve lies above the x-axis on [1, 2].
Area $= \int_1^2 x^2\,dx = [x^3/3]_1^2$
$= 8/3 - 1/3$
$= 7/3$

(ii) $y = x^4$, bounded by $x = 1$, $x = 5$, and the x-axis.

Since $y = x^4 \geq 0$ for all x, the curve lies above the x-axis on [1, 5].
Area $= \int_1^5 x^4\,dx = [x^5/5]_1^5$
$= 5^5/5 - 1^5/5$
$= 3125/5 - 1/5$
$= 3124/5$

Answer: (i) 7/3 square units (ii) 3124/5 square units

Q2 Sketch the graph of $y = |x + 3|$ and evaluate $\int_{-6}^{0} |x + 3|\,dx$.

SOLUTION

The function $y = |x + 3|$ equals $-(x+3)$ when $x < -3$ and equals $(x+3)$ when $x \geq -3$.

The graph: for $x < -3$, $y = -(x+3)$ is a line with slope -1; for $x \geq -3$, $y = x+3$ is a line with slope +1. The vertex (minimum point) is at (-3, 0). The graph is V-shaped with vertex at (-3, 0).

To evaluate $\int_{-6}^{0}$	$x + 3$	dx, split at x = -3:
$\int_{-6}^{0}$	$x + 3$	$dx = \int_{-6}^{-3} [-(x + 3)]\,dx + \int_{-3}^{0} (x + 3)\,dx$

First integral: $\int_{-6}^{-3} -(x+3)\,dx = [-x^2/2 - 3x]_{-6}^{-3}$
At x = -3: $-9/2 - 3(-3) = -9/2 + 9 = 9/2$
At x = -6: $-36/2 - 3(-6) = -18 + 18 = 0$
First integral $= 9/2 - 0 = 9/2$

Second integral: $\int_{-3}^{0} (x+3)\,dx = [x^2/2 + 3x]_{-3}^{0}$

At x = 0: 0 + 0 = 0

At x = -3: 9/2 + 3(-3) = 9/2 - 9 = -9/2

Second integral = 0 - (-9/2) = 9/2

Total = 9/2 + 9/2 = 9

Answer: $\int_{-6}^{0} |x + 3| \, dx = 9$

Q3 Find the area bounded by the curve y = sin x between x = 0 and x = 2π.

SOLUTION

The curve y = sin x is positive on (0, π) and negative on (π, 2π).

Area = $\int_0^{\pi} \sin x \, dx + |\int_{\pi}^{2\pi} \sin x \, dx|$

First part ($x \in [0, \pi]$, $\sin x \geq 0$):

$\int_0^{\pi} \sin x \, dx = [-\cos x]_0^{\pi} = (-\cos \pi) - (-\cos 0) = 1 - (-1) = 2$

Second part ($x \in [\pi, 2\pi]$, $\sin x \leq 0$):

$\int_{\pi}^{2\pi} \sin x \, dx = [-\cos x]_{\pi}^{2\pi} = (-\cos 2\pi) - (-\cos \pi) = -1 - 1 = -2$

Area of this part = |-2| = 2

Total area = 2 + 2 = 4

Answer: 4 square units

Q4 Area bounded by the curve $y = x^3$, the x-axis and the ordinates x = -2 and x = 1 is

(A) -9

(B) -15/4

(C) 15/4

(D) 17/4

SOLUTION

$y = x^3$ is negative for x in [-2, 0] and positive for x in [0, 1].

For $x \in [-2, 0]$: $x^3 \leq 0$, so the curve is below the x-axis.

$\text{Area}_1 = |\int_{-2}^{0} x^3 \, dx| = |[x^4/4]_{-2}^{0}| = |0 - 16/4| = |-4| = 4$

For $x \in [0, 1]$: $x^3 \geq 0$, so the curve is above the x-axis.

$\text{Area}_2 = \int_0^1 x^3 \, dx = [x^4/4]_0^1 = 1/4 - 0 = 1/4$

Total area = $\text{Area}_1 + \text{Area}_2 = 4 + 1/4 = 17/4$

Answer: (D) 17/4

Q5 The area bounded by the curve $y = x|x|$, x-axis and the ordinates $x = -1$ and $x = 1$ is given by

(A) 0

(B) 1/3

(C) 2/3

(D) 4/3

[Hint: $y = x^2$ if $x > 0$ and $y = -x^2$ if $x < 0$]

SOLUTION

The function $y = x|x|$ is defined as:

$y = x^2$ if $x > 0$ (positive, above x-axis)

$y = -x^2$ if $x < 0$ (negative, below x-axis)

For $x \in [-1, 0]$: $y = -x^2 \leq 0$, curve is below the x-axis.

$\text{Area}_1 = |\int_{-1}^{0} (-x^2)\, dx| = |[-x^3/3]_{-1}^{0}| = |0 - (1/3)| = 1/3$

For $x \in [0, 1]$: $y = x^2 \geq 0$, curve is above the x-axis.

$\text{Area}_2 = \int_{0}^{1} x^2\, dx = [x^3/3]_0^1 = 1/3 - 0 = 1/3$

Total area = $\text{Area}_1 + \text{Area}_2 = 1/3 + 1/3 = 2/3$

Answer: (C) 2/3