Chapter 9: Differential Equations

CONTENTS — 6 exercises · 98 questions

9.1	Exercise 9.1	12 Q
9.2	Exercise 9.2	12 Q
9.3	Exercise 9.3	23 Q
9.4	Exercise 9.4	17 Q
9.5	Exercise 9.5	19 Q
Misc	Miscellaneous Exercise	15 Q

Exercise 9.1 — Exercise 9.1

Q1 $d^4y/dx^4 + \sin(y''') = 0$

SOLUTION

The highest order derivative present is d^4y/dx^4, so the order is 4.
Since the equation contains $\sin(y''')$, it is NOT a polynomial equation in its derivatives.
Therefore, the degree is not defined.

Answer: Order: 4, Degree: not defined

Q2 $y' + 5y = 0$

SOLUTION

The highest order derivative is $y' = dy/dx$, so the order is 1.
The equation is a polynomial in y' with the highest power of y' being 1.
Therefore, the degree is 1.

Answer: Order: 1, Degree: 1

Q3 $(ds/dt)^4 + 3s * d^2s/dt^2 = 0$

SOLUTION

The highest order derivative present is d^2s/dt^2, so the order is 2.
The equation is a polynomial in d^2s/dt^2 and ds/dt.
The highest power of the highest order derivative d^2s/dt^2 is 1.
Therefore, the degree is 1.

Answer: Order: 2, Degree: 1

Q4 $(d^2y/dx^2)^2 + \cos(dy/dx) = 0$

SOLUTION

The highest order derivative is d^2y/dx^2, so the order is 2.
The term $\cos(dy/dx)$ means the equation is NOT a polynomial in dy/dx.
Therefore, the degree is not defined.

Answer: Order: 2, Degree: not defined

Q5 $d^2y/dx^2 = \cos 3x + \sin 3x$

SOLUTION

The highest order derivative is d^2y/dx^2, so the order is 2.
The equation is a polynomial in d^2y/dx^2 with the highest power equal to 1.
Therefore, the degree is 1.

Answer: Order: 2, Degree: 1

Q6 $(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$

SOLUTION

The highest order derivative is y''' (third order), so the order is 3.
The equation is a polynomial in y''', y'', y'.
The highest power of the highest order derivative y''' is 2.
Therefore, the degree is 2.

Answer: Order: 3, Degree: 2

Q7 $y''' + 2y'' + y' = 0$

SOLUTION

The highest order derivative is y''' (third order), so the order is 3.
The equation is a polynomial in y''', y'', y', with highest power of y''' equal to 1.
Therefore, the degree is 1.

Answer: Order: 3, Degree: 1

Q8 $y' + y = e^x$

SOLUTION

The highest order derivative is y', so the order is 1.
The equation is a polynomial in y' with highest power 1.
Therefore, the degree is 1.

Answer: Order: 1, Degree: 1

Q9 $y'' + (y')^2 + 2y = 0$

SOLUTION

The highest order derivative is y'', so the order is 2.
The equation is a polynomial in y'' and y', with highest power of y'' equal to 1.
Therefore, the degree is 1.

Answer: Order: 2, Degree: 1

Q10 $y'' + 2y' + \sin y = 0$

SOLUTION

The highest order derivative is y'', so the order is 2.
The term $\sin y$ involves y (the dependent variable), not a derivative, so the equation is still a polynomial in y'' and y' with highest power of y'' equal to 1.
Therefore, the degree is 1.

Answer: Order: 2, Degree: 1

Q11 The degree of the differential equation $(d^2y/dx^2)^3 + (dy/dx)^2 + \sin(dy/dx) + 1 = 0$ is:

(A) 3 (B) 2 (C) 1 (D) not defined

SOLUTION

The term sin(dy/dx) makes this equation NOT a polynomial in the derivatives.

Degree is defined only when the differential equation is a polynomial in its derivatives.

Here the equation is not a polynomial in dy/dx.

Therefore, the degree is not defined.

Answer: (D) not defined

Q12 The order of the differential equation $2x^2 * d^2y/dx^2 - 3 * dy/dx + y = 0$ is:

(A) 2 (B) 1 (C) 0 (D) not defined

SOLUTION

The highest order derivative present is d^2y/dx^2 (second order).

Therefore, the order is 2.

Answer: (A) 2

Exercise 9.2 — Exercise 9.2

Q1 y = e^x + 1 : y'' - y' = 0

SOLUTION

Given y = e^x + 1.
Differentiate: y' = e^x.
Differentiate again: y'' = e^x.
Now y'' - y' = e^x - e^x = 0 = R.H.S.
Hence y = e^x + 1 is a solution of y'' - y' = 0.

Answer: Verified: y = e^x + 1 satisfies y'' - y' = 0.

Q2 $y = x^2 + 2x + C : y' - 2x - 2 = 0$

SOLUTION

Given $y = x^2 + 2x + C$.
Differentiate: $y' = 2x + 2$.
Now $y' - 2x - 2 = (2x + 2) - 2x - 2 = 0$ = R.H.S.
Hence $y = x^2 + 2x + C$ is a solution of $y' - 2x - 2 = 0$.

Answer: Verified: $y = x^2 + 2x + C$ satisfies $y' - 2x - 2 = 0$.

Q3 y = cos x + C : y' + sin x = 0

SOLUTION

Given y = cos x + C.
Differentiate: y' = -sin x.
Now y' + sin x = -sin x + sin x = 0 = R.H.S.
Hence y = cos x + C is a solution of y' + sin x = 0.

Answer: Verified: y = cos x + C satisfies y' + sin x = 0.

Q4 $y = \sqrt{1 + x^2} : y' = xy/(1 + x^2)$

SOLUTION

Given $y = \sqrt{1 + x^2}$.
Differentiate: $y' = x / \sqrt{1 + x^2}$.
Now $xy/(1 + x^2) = x * \sqrt{1 + x^2} / (1 + x^2) = x / \sqrt{1 + x^2} = y'$.

Hence y = sqrt(1 + x²) is a solution of y' = xy/(1 + x²).

Answer: Verified: y = sqrt(1 + x²) satisfies y' = xy/(1 + x²).

Q5 y = Ax : xy' = y (x ≠ 0)

SOLUTION

Given y = Ax.
Differentiate: y' = A.
Now xy' = x * A = Ax = y.
Hence y = Ax is a solution of xy' = y.

Answer: Verified: y = Ax satisfies xy' = y.

Q6 y = x sin x : xy' = y + x*sqrt(x² - y²) (x ≠ 0, x > y or x < -y)

SOLUTION

Given y = x sin x.
Differentiate: y' = sin x + x cos x.
So xy' = x(sin x + x cos x) = x sin x + x² cos x.
Now y + x*sqrt(x² - y²) = x sin x + x*sqrt(x² - x² sin² x)
= x sin x + x * |x| * |cos x|.
For x > 0 and 0 < x < pi/2 (so cos x > 0): = x sin x + x² cos x = xy'.
Hence y = x sin x is a solution of xy' = y + x*sqrt(x² - y²).

Answer: Verified: y = x sin x satisfies xy' = y + x*sqrt(x² - y²).

Q7 xy = log y + C : y' = y²/(1 - xy) (xy ≠ 1)

SOLUTION

Given xy = log y + C.
Differentiate implicitly with respect to x:
y + x*y' = y'/y.
Multiply through by y: y² + xy*y' = y'.
y² = y' - xy*y' = y'(1 - xy).
Therefore y' = y²/(1 - xy).
Hence xy = log y + C is a solution of y' = y²/(1 - xy).

Answer: Verified: xy = log y + C satisfies y' = y²/(1 - xy).

Q8 $y - \cos y = x : (y \sin y + \cos y + x)\, y' = y$

SOLUTION

Given $y - \cos y = x$.
Differentiate implicitly: $y' + \sin y \cdot y' = 1$, so $y'(1 + \sin y) = 1$.
Therefore $y' = 1/(1 + \sin y)$.
Now check $(y \sin y + \cos y + x)\, y' = y$:
LHS $= (y \sin y + \cos y + x) / (1 + \sin y)$.
From the given relation: $x = y - \cos y$.
So LHS $= (y \sin y + \cos y + y - \cos y)/(1 + \sin y) = y(1 + \sin y)/(1 + \sin y) = y =$ RHS.
Hence $y - \cos y = x$ is a solution of the given differential equation.

Answer: Verified: $y - \cos y = x$ satisfies $(y \sin y + \cos y + x)\, y' = y$.

Q9 $x + y = \tan^{-1}(y) : y^2 y' + y^2 + 1 = 0$

SOLUTION

Given $x + y = \tan^{-1}(y)$.
Differentiate both sides with respect to x:
$1 + y' = y'/(1 + y^2)$.
$y'[1 - 1/(1 + y^2)] = -1$.
$y' \cdot [y^2/(1 + y^2)] = -1$.
$y' = -(1 + y^2)/y^2$.
Now $y^2 y' + y^2 + 1 = y^2 \cdot [-(1+y^2)/y^2] + y^2 + 1$
$= -(1 + y^2) + y^2 + 1 = 0 =$ RHS.
Hence $x + y = \tan^{-1}(y)$ is a solution of $y^2 y' + y^2 + 1 = 0$.

Answer: Verified: $x + y = \tan^{-1}(y)$ satisfies $y^2 y' + y^2 + 1 = 0$.

Q10 $y = \sqrt{a^2 - x^2},\ x \in (-a, a) : x + y \cdot dy/dx = 0 \quad (y \neq 0)$

SOLUTION

Given $y = \sqrt{a^2 - x^2}$.
Differentiate: $dy/dx = -x / \sqrt{a^2 - x^2} = -x/y$.
Now $x + y \cdot dy/dx = x + y \cdot (-x/y) = x - x = 0 =$ RHS.
Hence $y = \sqrt{a^2 - x^2}$ is a solution of $x + y \cdot dy/dx = 0$.

Answer: Verified: $y = \sqrt{a^2 - x^2}$ satisfies $x + y\, dy/dx = 0$.

Q11 The number of arbitrary constants in the general solution of a differential equation of fourth order are:
(A) 0 (B) 2 (C) 3 (D) 4

SOLUTION

The general solution of a differential equation of order n contains exactly n arbitrary constants.

For a fourth order differential equation, the general solution contains 4 arbitrary constants.

Answer: (D) 4

Q12 The number of arbitrary constants in the particular solution of a differential equation of third order are:
(A) 3 (B) 2 (C) 1 (D) 0

SOLUTION

A particular solution is obtained from the general solution by assigning specific values to all arbitrary constants.

Therefore, a particular solution contains no arbitrary constants, regardless of the order.

Answer: (D) 0

Exercise 9.3 — Exercise 9.3

Q1 dy/dx = (1 - cos x)/(1 + cos x)

SOLUTION

Use the identities: $1 - \cos x = 2\sin^2(x/2)$ and $1 + \cos x = 2\cos^2(x/2)$.
So $dy/dx = 2\sin^2(x/2) / (2\cos^2(x/2)) = \tan^2(x/2) = \sec^2(x/2) - 1$.
Separate variables: $dy = (\sec^2(x/2) - 1)\,dx$.
Integrate both sides:
$y = 2\tan(x/2) - x + C$.

Answer: $y = 2\tan(x/2) - x + C$

Q2 dy/dx = sqrt(4 - y²) (-2 < y < 2)

SOLUTION

Separate variables: $dy/\sqrt{4 - y^2} = dx$.
Integrate both sides: $\sin^{-1}(y/2) = x + C$.
Therefore $y = 2\sin(x + C)$.

Answer: $\sin^{-1}(y/2) = x + C$, or equivalently $y = 2\sin(x + C)$

Q3 dy/dx + y = 1 (y ≠ 1)

SOLUTION

Rewrite: $dy/dx = 1 - y$.
Separate variables: $dy/(1 - y) = dx$.

Integrate both sides: -log	1 - y	= x + C_1.
So log	1 - y	= -x - C_1.

$1 - y = e^{(-x - C_1)} = A\,e^{(-x)}$ where $A = e^{(-C_1)}$.
$y = 1 - A\,e^{(-x)}$, which is written as $y = 1 - C\,e^{(-x)}$ (absorbing the constant).

Answer: $y = 1 - C\,e^{(-x)}$ (or equivalently $\log|1 - y| = -x + C$)

Q4 $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

SOLUTION

Divide throughout by tan x tan y:
sec²(x)/tan(x) dx + sec²(y)/tan(y) dy = 0.
Integrate both sides:
∫ sec²(x)/tan(x) dx + ∫ sec²(y)/tan(y) dy = 0.
log|tan x| + log|tan y| = log C.
tan x * tan y = C.

Answer: tan x * tan y = C

Q5 (e^x + e⁻x) dy - (e^x - e⁻x) dx = 0

SOLUTION

Separate variables: dy = (e^x - e⁻x)/(e^x + e⁻x) dx.
Integrate both sides: y = ∫ (e^x - e⁻x)/(e^x + e⁻x) dx.
Let t = e^x + e⁻x, then dt = (e^x - e⁻x) dx.
y = ∫ dt/t = log|t| + C = log(e^x + e⁻x) + C.

Answer: y = log(e^x + e⁻x) + C

Q6 $dy/dx = (1 + x^2)(1 + y^2)$

SOLUTION

Separate variables: dy/(1 + y²) = (1 + x²) dx.
Integrate both sides:
$\tan^{-1}(y) = x + x^3/3 + C$.

Answer: $\tan^{-1}(y) = x + x^3/3 + C$

Q7 y log y dx - x dy = 0

SOLUTION

Separate variables: dy/(y log y) = dx/x.
Integrate both sides: ∫ dy/(y log y) = ∫ dx/x.

Let t = log y, dt = dy/y: ∫ dt/t = log	t	= log	log y	.
So log	log y	= log	x	+ log C.

log y = Cx (where C is an arbitrary constant).
y = e^(Cx).

Answer: log y = Cx (or y = e^(Cx))

Q8 $x^5 \, dy/dx = -y^5$

SOLUTION

Separate variables: $dy/y^5 = -dx/x^5$.
Integrate both sides: $\int y^{-5} \, dy = -\int x^{-5} \, dx$.
$-1/(4y^4) = 1/(4x^4) + C_1$.
Multiply by -4: $1/y^4 = -1/x^4 - 4C_1$.
Or equivalently: $x^{-4} + y^{-4} = C$ (absorbing constant).

Answer: $x^{-4} + y^{-4} = C$

Q9 $dy/dx = \sin^{-1} x$

SOLUTION

Integrate both sides: $y = \int \sin^{-1}(x) \, dx$.
Use integration by parts: $u = \sin^{-1}(x)$, $dv = dx$.
$du = 1/\sqrt{1-x^2} \, dx$, $v = x$.
$y = x \sin^{-1}(x) - \int x/\sqrt{1-x^2} \, dx$.
For $\int x/\sqrt{1-x^2} \, dx$, let $t = 1 - x^2$, $dt = -2x \, dx$: integral = $\sqrt{1-x^2}$.
$y = x \sin^{-1}(x) + \sqrt{1 - x^2} + C$.

Answer: $y = x \sin^{-1}(x) + \sqrt{1 - x^2} + C$

Q10 $e^x \tan y \, dx + (1 - e^x) \sec^2 y \, dy = 0$

SOLUTION

Separate variables: $e^x/(1 - e^x) \, dx = -\sec^2(y)/\tan(y) \, dy$.
Equivalently: $e^x/(e^x - 1) \, dx = \sec^2(y)/\tan(y) \, dy$.
Integrate both sides:
$\log|e^x - 1| = \log|\tan y| + \log C$.
$\tan y = C(e^x - 1)$.

Answer: tan y = C(e^x - 1)

Q11 $(x^3 + x^2 + x + 1)\, dy/dx = 2x^2 + x$; $y = 1$ when $x = 0$

SOLUTION

Factor denominator: $x^3 + x^2 + x + 1 = (x^2 + 1)(x + 1)$.
Separate and apply partial fractions to $(2x^2 + x)/((x^2+1)(x+1))$:
$= 1/(2(x+1)) + (3x-1)/(2(x^2+1))$.
Integrate both sides:
$y = (1/2) \log|x+1| + (3/4) \log(x^2+1) - (1/2) \tan^{-1}(x) + C$.
Apply initial condition $y = 1$ at $x = 0$:
$1 = 0 + 0 - 0 + C \Rightarrow C = 1$.
Particular solution: $y = (1/2) \log|x+1| + (3/4) \log(x^2+1) - (1/2) \tan^{-1}(x) + 1$.

Answer: $y = (1/2) \log|x+1| + (3/4) \log(x^2+1) - (1/2) \tan^{-1}(x) + 1$

Q12 $x(x^2 - 1)\, dy/dx = 1$; $y = 0$ when $x = 2$

SOLUTION

Separate variables: $dy = dx / (x(x^2-1))$.
Apply partial fractions: $1/(x(x-1)(x+1)) = -1/x + 1/(2(x-1)) + 1/(2(x+1))$.

Integrate: $y = -\log$	x	$+ (1/2) \log$	$x-1$	$+ (1/2) \log$	$x+1$	$+ C$.
$= (1/2) \log(x^2-1) - \log$	x	$+ C$.				

Apply $y = 0$ at $x = 2$: $0 = (1/2) \log(3) - \log(2) + C$.
$C = \log(2) - (1/2) \log(3) = \log(2/\sqrt{3})$.
Particular solution: $y = (1/2) \log|x^2-1| - \log|x| + \log(2/\sqrt{3}) = \log(2\sqrt{x^2-1}/(\sqrt{3}x))$.

Answer: $y = \log(2 \sqrt{x^2-1} / (\sqrt{3}\, |x|))$

Q13 $\cos(dy/dx) = a$ $(a \in R)$; $y = 1$ when $x = 0$

SOLUTION

From $\cos(dy/dx) = a$, we get $dy/dx = \cos^{-1}(a)$, which is a constant.
Integrate: $y = x \cos^{-1}(a) + C$.
Apply $y = 1$ at $x = 0$: $1 = 0 + C \Rightarrow C = 1$.
Particular solution: $y = x \cos^{-1}(a) + 1$.

Answer: $y = x \cos^{-1}(a) + 1$

Q14 dy/dx = y tan x; y = 1 when x = 0

SOLUTION

Separate variables: dy/y = tan x dx.
Integrate: log|y| = log|sec x| + log C = log|C sec x|.
y = C sec x.
Apply y = 1 at x = 0: 1 = C sec(0) = C*1 => C = 1.
Particular solution: y = sec x.

Answer: y = sec x

Q15 Find the equation of a curve passing through the point (0, 0) and whose differential equation is y' = e^x sin x.

SOLUTION

We need to solve dy/dx = e^x sin x with y(0) = 0.
Integrate: y = $\int$ e^x sin x dx.
Use integration by parts twice (or the standard result):
Let I = $\int$ e^x sin x dx. Integrating by parts: I = e^x sin x - $\int$ e^x cos x dx.
= e^x sin x - [e^x cos x + $\int$ e^x sin x dx] = e^x sin x - e^x cos x - I.
2I = e^x(sin x - cos x), so I = e^x(sin x - cos x)/2.
y = e^x(sin x - cos x)/2 + C.
Apply y(0) = 0: 0 = e⁰(0 - 1)/2 + C = -1/2 + C => C = 1/2.
Particular solution: y = e^x(sin x - cos x)/2 + 1/2 = (e^x(sin x - cos x) + 1)/2.

Answer: y = (e^x(sin x - cos x) + 1)/2

Q16 For the differential equation xy dy/dx = (x + 2)(y + 2), find the solution curve passing through the point (1, -1).

SOLUTION

Separate variables: y/(y+2) dy = (x+2)/x dx.
Rewrite left side: y/(y+2) = 1 - 2/(y+2).
Right side: (x+2)/x = 1 + 2/x.
Integrate both sides:

y - 2 log	y+2	= x + 2 log	x	+ C.
Apply condition (1, -1): -1 - 2 log	1	= 1 + 2 log(1) + C.		

-1 - 0 = 1 + 0 + C => C = -2.

Solution curve: y - 2 log	y+2	= x + 2 log	x	- 2.
Or: y - x + 2 = 2 log	x(y+2)	.		

Answer: y - x + 2 = 2 log|x(y+2)|

Q17 Find the equation of a curve passing through the point (0, -2) given that at any point (x, y) on the curve, the product of the slope of its tangent and the y coordinate of the point is equal to the x coordinate of the point.

SOLUTION

Slope of tangent = dy/dx. Product of slope and y equals x: y * dy/dx = x.
Separate variables: y dy = x dx.
Integrate: $y^2/2 = x^2/2 + C$, so $y^2 - x^2 = K$ (where K = 2C).
Apply condition (0, -2): $(-2)^2 - 0 = K \Rightarrow K = 4$.
Equation of the curve: $y^2 - x^2 = 4$.

Answer: $y^2 - x^2 = 4$

Q18 At any point (x, y) of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point (-4, -3). Find the equation of the curve given that it passes through (-2, 1).

SOLUTION

Slope of line joining (x, y) to (-4, -3) is (y+3)/(x+4).
Given: dy/dx = 2(y+3)/(x+4).
Separate variables: dy/(y+3) = 2dx/(x+4).
Integrate: log|y+3| = 2 log|x+4| + log C.
$y + 3 = C(x+4)^2$.
Apply (-2, 1): $1 + 3 = C(-2+4)^2 = 4C \Rightarrow C = 1$.
Equation of the curve: $y + 3 = (x+4)^2$.

Answer: $y + 3 = (x + 4)^2$

Q19 The volume of a spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units, find the radius of the balloon after t seconds.

SOLUTION

Volume of sphere: $V = (4/3)\pi r^3$.
dV/dt = constant => $d/dt[(4/3)\pi r^3] = k$.
$4\pi r^2 * dr/dt = k$.
Integrate: $(4/3)\pi r^3 = kt + C_1$.
At t = 0, r = 3: $(4/3)\pi 27 = 0 + C_1 \Rightarrow C_1 = 36\pi$.
At t = 3, r = 6: $(4/3)\pi 216 = 3k + 36\pi \Rightarrow 288\pi = 3k + 36\pi \Rightarrow 3k = 252\pi \Rightarrow k = 84\pi$.
So $(4/3)\pi r^3 = 84\pi t + 36\pi$.
$r^3 = 63t + 27$.
$r = (63t + 27)^{(1/3)}$.

Answer: $r = (63t + 27)^{(1/3)}$ units

Q20 In a bank, principal increases continuously at the rate of r% per year. Find the value of r if Rs 100 doubles itself in 10 years ($\log_e 2 = 0.6931$).

SOLUTION

$dP/dt = (r/100)P$.
Separate: $dP/P = (r/100)\,dt$.
Integrate: $\log P = rt/100 + C_1 \Rightarrow P = P_0 * e^{(rt/100)}$.
At t = 0: $P_0 = 100$. At t = 10: P = 200.
$200 = 100 * e^{(r*10/100)} = 100 * e^{(r/10)}$.
$e^{(r/10)} = 2 \Rightarrow r/10 = \log_e 2 = 0.6931$.
r = 6.931.

Answer: r = 6.931

Q21 In a bank, principal increases continuously at the rate of 5% per year. An amount of Rs 1000 is deposited with this bank, how much will it worth after 10 years ($e^{0.5} = 1.648$).

SOLUTION

dP/dt = (5/100)*P => P = 1000 * e^(t/20).

At t = 10: P = 1000 * e^(10/20) = 1000 * $e^{0.5}$ = 1000 * 1.648 = Rs 1648.

Answer: Rs 1648

Q22 In a culture, the bacteria count is 1,00,000. The number is increased by 10% in 2 hours. In how many hours will the count reach 2,00,000, if the rate of growth of bacteria is proportional to the number present?

SOLUTION

Let N be the count at time t. dN/dt = kN => N = N_0 * e^(kt).

At t = 0: N_0 = 1,00,000.

At t = 2: N = 1,10,000 => 1,10,000 = 1,00,000 * e^(2k).

e^(2k) = 1.1 => 2k = log_e(1.1).

Find t when N = 2,00,000: 2,00,000 = 1,00,000 * e^(kt).

e^(kt) = 2 => kt = log_e(2).

t = log_e(2)/k = 2 log_e(2)/log_e(1.1).

Answer: t = 2 log_e(2) / log_e(1.1) hours

Q23 The general solution of the differential equation dy/dx = e^(x+y) is:
(A) $e^x + e^{-y} = C$ (B) $e^x + e^y = C$ (C) $e^{-x} + e^y = C$ (D) $e^{-x} + e^{-y} = C$

SOLUTION

dy/dx = e^x * e^y.

Separate variables: e^{-y} dy = e^x dx.

Integrate both sides: $-e^{-y} = e^x + C_1$.

$e^x + e^{-y} = C$ (where $C = -C_1$).

Answer: (A) $e^x + e^{-y} = C$

Exercise 9.4 — Exercise 9.4

Q1 $(x^2 + xy)\,dy = (x^2 + y^2)\,dx$

SOLUTION

Rewrite: $dy/dx = (x^2 + y^2)/(x^2 + xy) = (1 + (y/x)^2)/(1 + y/x)$.
This is of the form $g(y/x)$, so it is homogeneous.
Substitute $y = vx$: $dy/dx = v + x\,dv/dx$.
$v + x\,dv/dx = (1 + v^2)/(1 + v)$.
$x\,dv/dx = (1 + v^2)/(1 + v) - v = (1 - v)/(1 + v)$.
Separate: $(1+v)/(1-v)\,dv = dx/x$.
Write $(1+v)/(1-v) = -1 + 2/(1-v)$. Integrate:
$-v - 2\log|1-v| = \log|x| + C$.
Substitute back $v = y/x$:

-y/x - 2 log	1 - y/x	= log	x	+ C.		
-y/x - 2 log	(x-y)/x	= log	x	+ C.		
-y/x - 2 log	x-y	+ 2 log	x	= log	x	+ C.
-y/x + log	x	= 2 log	x-y	+ C.		
Or: y/x + 2 log	x-y	= log	x	+ C'.		

Answer: $y/x + 2\log|x - y| = \log|x| + C$

Q2 $y' = (x + y)/x$

SOLUTION

$dy/dx = 1 + y/x$. This is homogeneous of degree 0.
Substitute $y = vx$: $v + x\,dv/dx = 1 + v$.
$x\,dv/dx = 1$.
$dv = dx/x$.

Integrate: v = log	x	+ C.
Substitute back v = y/x: y/x = log	x	+ C.
y = x(log	x	+ C).

Answer: $y = x(\log|x| + C)$

Q3 $(x - y)\,dy - (x + y)\,dx = 0$

SOLUTION

Rewrite: $dy/dx = (x+y)/(x-y) = (1 + y/x)/(1 - y/x)$. Homogeneous.
Substitute $y = vx$: $v + x\cdot dv/dx = (1+v)/(1-v)$.
$x\cdot dv/dx = (1+v)/(1-v) - v = (1+v^2)/(1-v)$.
Separate: $(1-v)/(1+v^2)\,dv = dx/x$.
Integrate: $\tan^{-1}(v) - (1/2)\log(1+v^2) = \log|x| + C$.
Substitute $v = y/x$:

$\tan^{-1}(y/x) - (1/2)\log(1 + y^2/x^2) = \log$	x	$+ C.$		
$\tan^{-1}(y/x) - (1/2)[\log(x^2+y^2) - \log(x^2)] = \log$	x	$+ C.$		
$\tan^{-1}(y/x) - (1/2)\log(x^2+y^2) + \log$	x	$= \log$	x	$+ C.$

$\tan^{-1}(y/x) = (1/2)\log(x^2+y^2) + C.$

Answer: $\tan^{-1}(y/x) = (1/2)\log(x^2 + y^2) + C$

Q4 $(x^2 - y^2)\,dx + 2xy\,dy = 0$

SOLUTION

Rewrite: $dy/dx = -(x^2 - y^2)/(2xy) = (y^2 - x^2)/(2xy) = (v^2 - 1)/(2v)$ where $v = y/x$. Homogeneous.
Substitute $y = vx$: $v + x\cdot dv/dx = (v^2-1)/(2v)$.
$x\cdot dv/dx = (v^2-1)/(2v) - v = -(1+v^2)/(2v)$.
Separate: $2v/(1+v^2)\,dv = -dx/x$.
Integrate: $\log(1+v^2) = -\log|x| + \log C$.
$(1 + v^2)\cdot x = C$.
Substitute $v = y/x$: $(1 + y^2/x^2)\cdot x = C \Rightarrow (x^2+y^2)/x = C$.
$x^2 + y^2 = Cx$.

Answer: $x^2 + y^2 = Cx$

Q5 $x^2\,dy/dx = x^2 - 2y^2 + xy$

SOLUTION

$dy/dx = 1 - 2(y/x)^2 + y/x$. Homogeneous.
Substitute $y = vx$: $v + x\cdot dv/dx = 1 + v - 2v^2$.
$x\cdot dv/dx = 1 - 2v^2$.

Separate: dv/(1-2v²) = dx/x.
Partial fractions: 1/(1-2v²) = 1/((1-sqrt(2)v)(1+sqrt(2)v)) = (1/(2sqrt(2)))*[1/(1-sqrt(2)v) + 1/(1+sqrt(2)v)] ... using A=1/2, B=1/2.
Actually: 1/(2sqrt(2)) * log|(1+sqrt(2)v)/(1-sqrt(2)v)| = log|x| + C.
Substitute v = y/x:
log|(x + sqrt(2)y)/(x - sqrt(2)y)| = 2*sqrt(2)*log|x| + C.

Answer: log|(x + sqrt(2) y)/(x - sqrt(2) y)| = 2*sqrt(2) log|x| + C

Q6 x dy - y dx = sqrt(x² + y²) dx

SOLUTION

Rewrite: dy/dx = (y + sqrt(x²+y²))/x = y/x + sqrt(1 + (y/x)²). Homogeneous.
Substitute y = vx: v + x*dv/dx = v + sqrt(1+v²).
x*dv/dx = sqrt(1+v²).
Separate: dv/sqrt(1+v²) = dx/x.
Integrate: log|v + sqrt(1+v²)| = log|x| + log C.
v + sqrt(1+v²) = Cx.
Substitute v = y/x: y/x + sqrt(1 + y²/x²) = Cx.
(y + sqrt(x²+y²))/x = Cx.
y + sqrt(x²+y²) = Cx².

Answer: y + sqrt(x² + y²) = Cx²

Q7 {x cos(y/x) + y sin(y/x)} y dx = {y sin(y/x) - x cos(y/x)} x dy

SOLUTION

Rearrange: dy/dx = [x cos(y/x) + y sin(y/x)] * y / [(y sin(y/x) - x cos(y/x)) * x].
Let v = y/x (so y = vx), simplify numerator and denominator:
dy/dx = v(cos v + v sin v)/(v sin v - cos v). Homogeneous.
Substitute y = vx: v + x*dv/dx = v(cos v + v sin v)/(v sin v - cos v).
x*dv/dx = [v(cos v + v sin v) - v(v sin v - cos v)]/(v sin v - cos v) = 2v cos v/(v sin v - cos v).
Separate: (v sin v - cos v)/(2v cos v) dv = dx/x.
= (1/2)(tan v - 1/v) dv = dx/x.

Integrate: (1/2)[-log	cos v	- log	v	] = log	x	+ log C.
log	cos v	+ log	v	= -2 log	x	+ C.

v cos v = C/x².
Substitute v = y/x: (y/x) cos(y/x) = C/x².
xy cos(y/x) = C.

Answer: xy cos(y/x) = C

Q8 x dy/dx - y + x sin(y/x) = 0

SOLUTION

Rewrite: dy/dx = y/x - sin(y/x). Homogeneous.
Substitute y = vx: v + x*dv/dx = v - sin v.
x*dv/dx = -sin v.
Separate: dv/sin v = -dx/x, i.e., cosec v dv = -dx/x.
Integrate: log|tan(v/2)| = -log|x| + log C.
tan(v/2) = C/x.
Substitute v = y/x: tan(y/(2x)) = C/x.
x tan(y/(2x)) = C.

Answer: x tan(y/(2x)) = C

Q9 y dx + x log(y/x) dy - 2x dy = 0

SOLUTION

Rewrite: y dx = (2x - x log(y/x)) dy => dx/dy = x(2 - log(y/x))/y.
Let w = x/y (so x = wy): dx/dy = w + y*dw/dy.
w + y*dw/dy = wy(2 - log(1/w))/y = w(2 + log w).
y*dw/dy = w(2 + log w) - w = w(1 + log w).
Separate: dw/(w(1 + log w)) = dy/y.

Let t = 1 + log w, dt = dw/w: ∫ dt/t = log	t	= log	1 + log w	.
log	1 + log w	= log	y	+ log C.

1 + log w = Cy.
Substitute w = x/y: 1 + log(x/y) = Cy.
log(x/y) = Cy - 1.

Answer: log(x/y) = Cy - 1 (or 1 + log(x/y) = Cy)

Q10 (1 + e^(x/y)) dx + e^(x/y)(1 - x/y) dy = 0

SOLUTION

Let v = x/y (so x = vy, dx = v dy + y dv):
(1 + e^v)(v dy + y dv) + e^v(1 - v) dy = 0.
Expand: v(1+e^v) dy + y(1+e^v) dv + e^v(1-v) dy = 0.
dy[v + ve^v + e^v - ve^v] + y(1+e^v) dv = 0.
dy(v + e^v) + y(1 + e^v) dv = 0.
dy/y = -(1 + e^v)/(v + e^v) dv.
Integrate: log|y| = -log|v + e^v| + log C.
y(v + e^v) = C.
Substitute v = x/y: y(x/y + e^(x/y)) = C.
x + y e^(x/y) = C.

Answer: x + y e^(x/y) = C

Q11 (x + y) dy + (x - y) dx = 0; y = 1 when x = 1

SOLUTION

dy/dx = (y-x)/(y+x). Homogeneous.
Substitute y = vx: v + x*dv/dx = (v-1)/(v+1).
x*dv/dx = -(1+v²)/(1+v).
Separate: (1+v)/(1+v²) dv = -dx/x.

Integrate: (1/2) log(1+v²) + tan⁻¹(v) = -log	x	+ C.
(1/2) log(1+v²) + tan⁻¹(v) + log	x	= C.

Substitute v = y/x:
(1/2)[log(x²+y²) - log x²] + tan⁻¹(y/x) + log|x| = C.
(1/2) log(x²+y²) + tan⁻¹(y/x) = C.
Apply x = 1, y = 1: (1/2) log(2) + pi/4 = C.
Particular solution: (1/2) log(x²+y²) + tan⁻¹(y/x) = (1/2) log 2 + pi/4.

Answer: (1/2) log(x² + y²) + tan⁻¹(y/x) = (1/2) log 2 + pi/4

Q12 $x^2\,dy + (xy + y^2)\,dx = 0$; $y = 1$ when $x = 1$

SOLUTION

$dy/dx = -(xy + y^2)/x^2 = -(y/x) - (y/x)^2$. Homogeneous.

Substitute $y = vx$: $v + x\,dv/dx = -v - v^2$.

$x\,dv/dx = -2v - v^2 = -v(2+v)$.

Separate: $dv/(v(v+2)) = -dx/x$.

Partial fractions: $1/(v(v+2)) = (1/2)/v - (1/2)/(v+2)$.

Integrate: $(1/2)\log$	v	$-(1/2)\log$	$v+2$	$= -\log$	x	$+ C.$
$(1/2)\log$	$v/(v+2)$	$= -\log$	x	$+ C.$		

$v/(v+2) = A/x^2$.

Substitute $v = y/x$: $(y/x)/((y/x)+2) = A/x^2 \Rightarrow y/(y+2x) = A/x^2$.

$x^2 y = A(y+2x)$.

Apply $x=1$, $y=1$: $1 = A*3 \Rightarrow A = 1/3$.

Particular solution: $x^2 y = (y+2x)/3$, i.e., $3x^2 y = y + 2x$.

Answer: $3x^2 y = y + 2x$

Q13 $[x \sin^2(y/x) - y]\,dx + x\,dy = 0$; $y = \pi/4$ when $x = 1$

SOLUTION

$dy/dx = (y - x \sin^2(y/x))/x = y/x - \sin^2(y/x)$. Homogeneous.

Substitute $y = vx$: $v + x\,dv/dx = v - \sin^2(v)$.

$x\,dv/dx = -\sin^2(v)$.

Separate: $-dv/\sin^2(v) = dx/x$, i.e., $\operatorname{cosec}^2(v)\,dv = -dx/x$.

Integrate: $\cot(v) = \log$	x	$+ C.$
Substitute $v = y/x$: $\cot(y/x) = \log$	x	$+ C.$

Apply $x=1$, $y=\pi/4$: $\cot(\pi/4) = \log(1) + C \Rightarrow 1 = 0 + C \Rightarrow C = 1$.

Particular solution: $\cot(y/x) = \log|x| + 1$.

Answer: $\cot(y/x) = \log|x| + 1$

Q14 dy/dx - y/x + cosec(y/x) = 0; y = 0 when x = 1

SOLUTION

dy/dx = y/x - cosec(y/x). Homogeneous.
Substitute y = vx: v + x*dv/dx = v - cosec(v).
x*dv/dx = -cosec(v).
Separate: sin(v) dv = -dx/x.

Integrate: -cos(v) = -log	x	+ C, i.e., cos(v) = log	x	+ C.
Substitute v = y/x: cos(y/x) = log	x	+ C.		

Apply x=1, y=0: cos(0) = log(1) + C => 1 = 0 + C => C = 1.
Particular solution: cos(y/x) = log|x| + 1.

Answer: cos(y/x) = log|x| + 1

Q15 2xy + y² - 2x² dy/dx = 0; y = 2 when x = 1

SOLUTION

2x² dy/dx = 2xy + y² => dy/dx = y/x + (1/2)(y/x)². Homogeneous.
Substitute y = vx: v + x*dv/dx = v + v²/2.
x*dv/dx = v²/2.
Separate: 2dv/v² = dx/x.

Integrate: -2/v = log	x	+ C.
Substitute v = y/x: -2x/y = log	x	+ C.

Apply x=1, y=2: -1 = 0 + C => C = -1.
Particular solution: -2x/y = log|x| - 1 => y = 2x/(1 - log|x|).

Answer: y = 2x/(1 - log|x|)

Q16 A homogeneous differential equation of the form dx/dy = h(x/y) can be solved by making the substitution:
(A) y = vx (B) v = yx (C) x = vy (D) x = v

SOLUTION

For a homogeneous DE of the form dx/dy = h(x/y), we substitute x = vy (where v = x/y is a function of y).
Then dx/dy = v + y*dv/dy, and x/y = v.

The equation becomes separable in v and y.

Answer: (C) x = vy

Q17 Which of the following is a homogeneous differential equation?

(A) (4x + 6y + 5) dy - (3y + 2x + 4) dx = 0

(B) (xy) dx - (x^3 + y^3) dy = 0

(C) (x^3 + 2y^2) dx + 2xy dy = 0

(D) y^2 dx + (x^2 - xy - y^2) dy = 0

SOLUTION

A homogeneous DE has F(x,y) of degree zero, i.e., every term has the same total degree.

(A) Contains constant terms 5 and 4 — not homogeneous.

(B) xy is degree 2, x^3+y^3 is degree 3 — not homogeneous.

(C) x^3+2y^2 has terms of degree 3 and 2 — not homogeneous.

(D) y^2 is degree 2, x^2−xy-y^2 is degree 2 — all terms degree 2. Homogeneous.

The DE dy/dx = -y^2/(x^2−xy-y^2) is F(x,y)/G(x,y) with both homogeneous of degree 2, giving degree 0. Homogeneous.

Answer: (D) y^2 dx + (x^2 - xy - y^2) dy = 0

Exercise 9.5 — Exercise 9.5

Q1 dy/dx + 2y = sin x

SOLUTION

Linear DE: P = 2, Q = sin x.
I.F. = e^(∫2 dx) = e^(2x).
Multiply: d/dx(y e^(2x)) = e^(2x) sin x.
Integrate: y e^(2x) = ∫ e^(2x) sin x dx.
Using the formula ∫ e^(ax) sin(bx) dx = e^(ax)(a sin(bx) - b cos(bx))/(a²+b²):
∫ e^(2x) sin x dx = e^(2x)(2 sin x - cos x)/5.
y e^(2x) = e^(2x)(2 sin x - cos x)/5 + C.
y = (2 sin x - cos x)/5 + C e^(-2x).

Answer: y = (2 sin x - cos x)/5 + C e^(-2x)

Q2 dy/dx + 3y = e^(-2x)

SOLUTION

Linear DE: P = 3, Q = e^(-2x).
I.F. = e^(3x).
y e^(3x) = ∫ e^(-2x) * e^(3x) dx = ∫ e^x dx = e^x + C.
y = e^(-2x) + C e^(-3x).

Answer: y = e^(-2x) + C e^(-3x)

Q3 dy/dx + y/x = x²

SOLUTION

Linear DE: P = 1/x, Q = x².
I.F. = e^(∫ 1/x dx) = e^(log x) = x.
y*x = ∫ x² * x dx = ∫ x³ dx = x⁴/4 + C.
y = x³/4 + C/x.

Answer: y = x³/4 + C/x

Q4 dy/dx + (sec x) y = tan x (0 ≤ x < pi/2)

SOLUTION

Linear DE: P = sec x, Q = tan x.
I.F. = e^(∫ sec x dx) = e^(log|sec x + tan x|) = sec x + tan x.
y(sec x + tan x) = ∫ tan x (sec x + tan x) dx.
= ∫ (sec x tan x + tan² x) dx = ∫ (sec x tan x + sec² x - 1) dx.
= sec x + tan x - x + C.
y(sec x + tan x) = sec x + tan x - x + C.

Answer: y(sec x + tan x) = sec x + tan x - x + C

Q5 cos² x * dy/dx + y = tan x (0 ≤ x < pi/2)

SOLUTION

Divide by cos² x: dy/dx + y sec² x = tan x sec² x.
Linear DE: P = sec² x, Q = sec² x tan x.
I.F. = e^(∫ sec² x dx) = e^(tan x).
y e^(tan x) = ∫ sec²(x) tan(x) e^(tan x) dx.
Let t = tan x, dt = sec² x dx:
= ∫ t e^t dt = e^t(t - 1) = e^(tan x)(tan x - 1) + C.
y e^(tan x) = e^(tan x)(tan x - 1) + C.
y = (tan x - 1) + C e^(-tan x).

Answer: y = (tan x - 1) + C e^(-tan x)

Q6 x dy/dx + 2y = x² log x

SOLUTION

Divide by x: dy/dx + 2y/x = x log x.
Linear DE: P = 2/x, Q = x log x.
I.F. = e^(∫ 2/x dx) = e^(2 log x) = x².
y x² = ∫ x log x * x² dx = ∫ x³ log x dx.
Integrate by parts: u = log x, dv = x³ dx => v = x⁴/4.
∫ x³ log x dx = (x⁴/4) log x - ∫ (x⁴/4)(1/x) dx = (x⁴/4) log x - x⁴/16 + C.
y x² = (x⁴/4) log x - x⁴/16 + C.
y = (x²/4) log x - x²/16 + C/x².
y = x²(4 log x - 1)/16 + C/x².

Answer: $y = x^2(4 \log x - 1)/16 + C/x^2$

Q7 $x \log x \cdot dy/dx + y = 2/x \cdot \log x$

SOLUTION

Divide by x log x: $dy/dx + y/(x \log x) = 2/(x^2)$.
Linear DE: $P = 1/(x \log x)$, $Q = 2/x^2$.
I.F. = $e^{\int 1/(x \log x)\, dx} = e^{\log|\log x|} = \log x$.
$y \log x = \int (2/x^2) \log x\, dx$.
Integrate by parts: $u = \log x$, $dv = 2/x^2\, dx \Rightarrow v = -2/x$.
$= -(2 \log x)/x + \int (2/x)(1/x)\, dx = -(2 \log x)/x - 2/x + C$.
$y \log x = -2(\log x + 1)/x + C$.

Answer: $y \log x = -2(\log x + 1)/x + C$

Q8 $(1 + x^2)\, dy + 2xy\, dx = \cot x\, dx \quad (x \neq 0)$

SOLUTION

Rewrite: $dy/dx + 2xy/(1+x^2) = \cot x/(1+x^2)$.
Linear DE: $P = 2x/(1+x^2)$, $Q = \cot x/(1+x^2)$.
I.F. = $e^{\int 2x/(1+x^2)\, dx} = e^{\log(1+x^2)} = 1 + x^2$.

$y(1+x^2) = \int \cot x/(1+x^2) \cdot (1+x^2)\, dx = \int \cot x\, dx = \log$	sin x	+ C.
y = (log	sin x	+ C)/(1 + x^2).

Answer: $y(1 + x^2) = \log|\sin x| + C$

Q9 $x\, dy/dx + y - x + xy \cot x = 0 \quad (x \neq 0)$

SOLUTION

Rewrite: $x\, dy/dx + y(1 + x \cot x) = x$.
Divide by x: $dy/dx + y(1/x + \cot x) = 1$.
Linear DE: $P = 1/x + \cot x$, $Q = 1$.
I.F. = $e^{\int (1/x + \cot x)\, dx} = e^{\log x + \log \sin x} = x \sin x$.
$y \cdot x \sin x = \int 1 \cdot x \sin x\, dx$.
Integrate by parts: $\int x \sin x\, dx = -x \cos x + \sin x + C$.
$y \cdot x \sin x = \sin x - x \cos x + C$.
$y = (\sin x - x \cos x)/(x \sin x) + C/(x \sin x)$.

y = 1/x - cot x + C/(x sin x).

Answer: xy sin x = sin x - x cos x + C

Q10 (x + y) dy/dx = 1

SOLUTION

Rewrite treating x as a function of y: dx/dy = x + y.
dx/dy - x = y.
Linear DE in x: P_1 = -1, Q_1 = y.
I.F. = e^(∫ -1 dy) = e^(-y).
x e^(-y) = ∫ y e^(-y) dy.
Integrate by parts: ∫ y e^(-y) dy = -y e^(-y) + ∫ e^(-y) dy = -y e^(-y) - e^(-y) = -e^(-y)(y+1).
x e^(-y) = -e^(-y)(y+1) + C.
x = -(y+1) + C e^y.
x = C e^y - y - 1.

Answer: x = C e^y - y - 1

Q11 y dx + (x - y^2) dy = 0

SOLUTION

Rewrite: dx/dy = (y^2 - x)/y = y - x/y.
dx/dy + x/y = y.
Linear DE in x: P_1 = 1/y, Q_1 = y.
I.F. = e^(∫ 1/y dy) = e^(log y) = y.
x*y = ∫ y * y dy = ∫ y^2 dy = y^3/3 + C.
x = y^2/3 + C/y.

Answer: xy = y^3/3 + C

Q12 (x + 3y^2) dy/dx = y

SOLUTION

Rewrite: dx/dy = (x + 3y^2)/y = x/y + 3y.
dx/dy - x/y = 3y.
Linear DE in x: P_1 = -1/y, Q_1 = 3y.

I.F. = e^(∫ -1/y dy) = e^(-log y) = 1/y.
x/y = ∫ 3y * (1/y) dy = ∫ 3 dy = 3y + C.
x = 3y² + Cy.

Answer: x = 3y² + Cy

Q13 dy/dx + 2y tan x = sin x; y = 0 when x = pi/3

SOLUTION

Linear DE: P = 2 tan x, Q = sin x.
I.F. = e^(∫ 2 tan x dx) = e^(2 log sec x) = sec² x.
y sec² x = ∫ sin x sec² x dx = ∫ (sin x/cos² x) dx = ∫ sec x tan x dx = sec x + C.
Apply y = 0, x = pi/3: 0 = sec(pi/3) + C = 2 + C => C = -2.
Particular solution: y sec² x = sec x - 2.
y = cos x - 2 cos² x.

Answer: y = cos x - 2 cos² x

Q14 (1 + x²) dy/dx + 2xy = 1/(1 + x²); y = 0 when x = 1

SOLUTION

Divide by (1+x²): dy/dx + 2xy/(1+x²) = 1/(1+x²)².
I.F. = 1 + x² (same as Q8).
y(1+x²) = ∫ 1/(1+x²) dx = tan⁻¹(x) + C.
Apply y = 0, x = 1: 0 = tan⁻¹(1) + C = pi/4 + C => C = -pi/4.
Particular solution: y(1+x²) = tan⁻¹(x) - pi/4.

Answer: y(1 + x²) = tan⁻¹(x) - pi/4

Q15 dy/dx - 3y cot x = sin 2x; y = 2 when x = pi/2

SOLUTION

Linear DE: P = -3 cot x, Q = sin 2x = 2 sin x cos x.
I.F. = e^(∫ -3 cot x dx) = e^(-3 log|sin x|) = sin^(-3) x = cosec³ x.
y cosec³ x = ∫ 2 sin x cos x * cosec³ x dx = ∫ 2 cos x/sin² x dx.
Let t = sin x: = ∫ 2/t² dt = -2/t = -2/sin x = -2 cosec x.
y cosec³ x = -2 cosec x + C.
y = -2 sin² x + C sin³ x.

Apply y = 2, x = pi/2: 2 = -2(1) + C(1) = -2 + C => C = 4.
Particular solution: $y = 4 \sin^3 x - 2 \sin^2 x$.

Answer: $y = 4 \sin^3 x - 2 \sin^2 x$

Q16 Find the equation of a curve passing through the origin given that the slope of the tangent to the curve at any point (x, y) is equal to the sum of the coordinates of the point.

SOLUTION

Slope = dy/dx = x + y.
dy/dx - y = x.
Linear DE: P = -1, Q = x.
I.F. = e^(-x).
y e^(-x) = $\int$ x e^(-x) dx.
Integrate by parts: $\int$ x e^(-x) dx = -x e^(-x) - e^(-x) = -e^(-x)(x+1).
y e^(-x) = -e^(-x)(x+1) + C.
y = -(x+1) + C e^x.
Apply (0, 0): 0 = -1 + C => C = 1.
Equation of curve: y = e^x - x - 1.

Answer: y = e^x - x - 1

Q17 Find the equation of a curve passing through the point (0, 2) given that the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent to the curve at that point by 5.

SOLUTION

Condition: x + y - |dy/dx| = 5, equivalently dy/dx = x + y - 5.
dy/dx - y = x - 5.
Linear DE: P = -1, Q = x - 5.
I.F. = e^(-x).
y e^(-x) = $\int$ (x-5) e^(-x) dx.
Integrate by parts (u = x-5, dv = e^(-x) dx): -(x-5)e^(-x) - e^(-x) = -(x-4)e^(-x).
y e^(-x) = -(x-4) e^(-x) + C.
y = -(x-4) + C e^x = 4 - x + C e^x.
Apply (0, 2): 2 = 4 + C => C = -2.
Equation of curve: y = 4 - x - 2 e^x.

Answer: y = (4 - x) - 2 e^x

Q18 The Integrating Factor of the differential equation x dy/dx - y = 2x² is:
(A) e^(-x) (B) e^(-y) (C) 1/x (D) x

SOLUTION

Rewrite in standard form: dy/dx - y/x = 2x.
P = -1/x.
I.F. = e^(∫ -1/x dx) = e^(-log x) = 1/x.

Answer: (C) 1/x

Q19 The Integrating Factor of the differential equation (1 - y²) dx/dy + yx = ay (-1 < y < 1) is:
(A) 1/(y²-1) (B) 1/sqrt(y²-1) (C) 1/(1-y²) (D) 1/sqrt(1-y²)

SOLUTION

Rewrite: dx/dy + xy/(1-y²) = ay/(1-y²).
P_1 = y/(1-y²).
I.F. = e^(∫ y/(1-y²) dy).
Let t = 1-y², dt = -2y dy: ∫ y/(1-y²) dy = -1/2 * ∫ dt/t = -1/2 log|1-y²|.
I.F. = e^(-1/2 * log(1-y²)) = (1-y²)^(-1/2) = 1/sqrt(1-y²).

Answer: (D) 1/sqrt(1 - y²)

Misc — Miscellaneous Exercise

Q1 For each of the differential equations given below, indicate its order and degree (if defined).
(i) $d^2y/dx^2 + 5x(dy/dx)^2 - 6y = \log x$
(ii) $(dy/dx)^3 - 4(dy/dx)^2 + 7y = \sin x$
(iii) $d^4y/dx^4 - \sin(d^3y/dx^3) = 0$

SOLUTION

(i) Highest order derivative: d^2y/dx^2 => order 2. Polynomial in d^2y/dx^2 with highest power 1 => degree 1.

(ii) Highest order derivative: dy/dx => order 1. Polynomial in dy/dx with highest power 3 => degree 3.

(iii) Highest order derivative: d^4y/dx^4 => order 4. Contains $\sin(d^3y/dx^3)$, so NOT a polynomial in derivatives => degree not defined.

Answer: (i) Order 2, Degree 1; (ii) Order 1, Degree 3; (iii) Order 4, Degree not defined

---

Q2 For each of the exercises given below, verify that the given function (implicit or explicit) is a solution of the corresponding differential equation.
(i) $xy = a\,e^x + b\,e^{-x} + x^2$: $x\,d^2y/dx^2 + 2\,dy/dx - xy + x^2 - 2 = 0$
(ii) $y = e^x(a\cos x + b\sin x)$: $d^2y/dx^2 - 2\,dy/dx + 2y = 0$
(iii) $y = x\sin 3x$: $d^2y/dx^2 + 9y - 6\cos 3x = 0$
(iv) $x^2 = 2y^2 \log y$: $(x^2 + y^2)\,dy/dx - xy = 0$

SOLUTION

(i) Let $F = xy = ae^x + be^{(-x)} + x^2$. Then $F' = ae^x - be^{(-x)} + 2x$ and $F'' = ae^x + be^{(-x)} + 2$.
Note: $d^2(xy)/dx^2 = xy'' + 2y'$. So $x\,y'' + 2y' = F'' = ae^x + be^{(-x)} + 2$.
LHS $= (xy'' + 2y') - xy + x^2 - 2 = (ae^x + be^{(-x)} + 2) - xy + x^2 - 2$.
Since $xy = ae^x + be^{(-x)} + x^2$, we get $ae^x + be^{(-x)} = xy - x^2$.
LHS $= (xy - x^2 + 2) - xy + x^2 - 2 = 0 =$ RHS. Verified.

(ii) $y = e^x(a\cos x + b\sin x)$. Differentiate:
$y' = e^x(a\cos x + b\sin x) + e^x(-a\sin x + b\cos x) = e^x[(a+b)\cos x + (b-a)\sin x]$.
$y'' = e^x[(a+b)\cos x + (b-a)\sin x] + e^x[-(a+b)\sin x + (b-a)\cos x] = e^x[2b\cos x - 2a\sin x]$.

Now: y'' - 2y' + 2y = 2e^x(b cos x - a sin x) - 2e^x[(a+b)cos x + (b-a)sin x] + 2e^x(a cos x + b sin x).
Coefficients of e^x cos x: 2b - 2(a+b) + 2a = 2b - 2a - 2b + 2a = 0.
Coefficients of e^x sin x: -2a - 2(b-a) + 2b = -2a - 2b + 2a + 2b = 0.
LHS = 0 = RHS. Verified.

(iii) y = x sin 3x. y' = sin 3x + 3x cos 3x. y'' = 6 cos 3x - 9x sin 3x.
y'' + 9y = 6 cos 3x - 9x sin 3x + 9x sin 3x = 6 cos 3x.
y'' + 9y - 6 cos 3x = 0. Verified.

(iv) $x^2 = 2y^2 \log y$. Differentiate: $2x = 4y \log y * y' + 2y * y' = 2y\,y'(2 \log y + 1)$.
$y' = x / [y(2 \log y + 1)]$.
Check $(x^2+y^2)y' - xy = (x^2+y^2) * x/[y(2 \log y+1)] - xy$.
$= x[(x^2+y^2) - y^2(2 \log y+1)] / [y(2 \log y+1)]$.
$= x[x^2 - 2y^2 \log y] / [y(2 \log y+1)]$.
Since $x^2 = 2y^2 \log y$, the numerator $x^2 - 2y^2 \log y = 0$.
LHS = 0 = RHS. Verified.

Answer: All four functions verified as solutions of their respective differential equations.

Q3 Prove that $x^2 - y^2 = c(x^2 + y^2)^2$ is the general solution of differential equation $(x^3 - 3xy^2)\,dx = (y^3 - 3x^2 y)\,dy$, where c is a parameter.

SOLUTION

Differentiate $x^2 - y^2 = c(x^2+y^2)^2$ implicitly:
$2x - 2y\,y' = c * 2(x^2+y^2)(2x + 2y\,y')$.
From the given relation: $c = (x^2-y^2)/(x^2+y^2)^2$.
Substitute c:
$2x - 2y\,y' = 2(x^2-y^2)/(x^2+y^2)^2 * (x^2+y^2) * 2(x + y\,y')$.
$2x - 2y\,y' = 4(x^2-y^2)(x + y\,y')/(x^2+y^2)$.
$(x^2+y^2)(x - y\,y') = 2(x^2-y^2)(x + y\,y')$.
$x^3 + xy^2 - x^2\,y\,y' - y^3\,y' = 2x^3 - 2xy^2 + 2x^2\,y\,y' - 2y^3\,y'$.
Rearranging: $y'(-x^2\,y - y^3 - 2x^2\,y + 2y^3) = 2x^3 - 2xy^2 - x^3 - xy^2$.
$y'(y^3 - 3x^2\,y) = x^3 - 3xy^2$.
$dy/dx = (x^3 - 3xy^2)/(y^3 - 3x^2\,y)$.
This is exactly $(x^3-3xy^2)dx = (y^3-3x^2y)dy$. Proved.

Answer: Proved: $x^2 - y^2 = c(x^2 + y^2)^2$ is the general solution of $(x^3 - 3xy^2)\,dx = (y^3 - 3x^2 y)\,dy$.

Q4 Find the general solution of the differential equation $dy/dx + \sqrt{(1 - y^2)/(1 - x^2)} = 0$.

SOLUTION

Rewrite: $dy/dx = -\sqrt{(1 - y^2)}/\sqrt{(1 - x^2)}$.
Separate variables: $dy/\sqrt{(1-y^2)} = -dx/\sqrt{(1-x^2)}$.
Integrate: $\sin^{-1}(y) = -\sin^{-1}(x) + C$.
$\sin^{-1}(x) + \sin^{-1}(y) = C$.

Answer: $\sin^{-1}(x) + \sin^{-1}(y) = C$

Q5 Show that the general solution of the differential equation $dy/dx + (y^2 + y + 1)/(x^2 + x + 1) = 0$ is given by $(x + y + 1) = A(1 - x - y - 2xy)$, where A is parameter.

SOLUTION

Separate variables: $dy/(y^{2+}y+1) + dx/(x^{2+}x+1) = 0$.
Write $y^{2+}y+1 = (y+1/2)^2 + 3/4$ and $x^{2+}x+1 = (x+1/2)^2 + 3/4$.
Integrate: $(2/\sqrt{3})\tan^{-1}((2y+1)/\sqrt{3}) + (2/\sqrt{3})\tan^{-1}((2x+1)/\sqrt{3}) = C_o$.
Let $A_o = \tan^{-1}((2x+1)/\sqrt{3})$ and $B_o = \tan^{-1}((2y+1)/\sqrt{3})$, so $A_o + B_o = C$.
$\tan(A_o + B_o) = (\tan A_o + \tan B_o)/(1 - \tan A_o \tan B_o)$.
$= [(2x+1+2y+1)/\sqrt{3}] / [1 - (2x+1)(2y+1)/3]$.
Numerator: $(2x+2y+2)/\sqrt{3} = 2(x+y+1)/\sqrt{3}$.
Denominator: $[3-(2x+1)(2y+1)]/3 = [3-4xy-2x-2y-1]/3 = 2(1-x-y-2xy)/3$.
$\tan(C) = [2(x+y+1)/\sqrt{3}] / [2(1-x-y-2xy)/3] = 3(x+y+1)/[\sqrt{3}(1-x-y-2xy)] = \sqrt{3}(x+y+1)/(1-x-y-2xy)$.
Since $\tan(C)$ is a constant (say $\tan(C) = \sqrt{3}*A$), this gives $(x+y+1) = A(1-x-y-2xy)$. Proved.

Answer: Proved: $(x + y + 1) = A(1 - x - y - 2xy)$ is the general solution.

Q6 Find the equation of the curve passing through the point (0, pi/4) whose differential equation is sin x cos y dx + cos x sin y dy = 0.

SOLUTION

Divide by cos x cos y: tan x dx + tan y dy = 0.

Integrate: -log	cos x	- log	cos y	= C_1.
log	cos x cos y	= C_0.		

cos x cos y = A (constant).
Apply (0, pi/4): cos(0) cos(pi/4) = 1 * 1/sqrt(2) = 1/sqrt(2).
Equation of curve: cos x cos y = 1/sqrt(2).
Or: sqrt(2) cos x cos y = 1.

Answer: cos x cos y = 1/sqrt(2)

Q7 Find the particular solution of the differential equation (1 + e^(2x)) dy + (1 + y^2) e^x dx = 0, given that y = 1 when x = 0.

SOLUTION

Separate variables: dy/(1+y^2) = -e^x/(1+e^(2x)) dx.
For the right side, let t = e^x, dt = e^x dx: -e^x/(1+(e^x)^2) dx = -dt/(1+t^2).
Integrate: $\tan^{-1}(y) = -\tan^{-1}(e^x) + C$.
Apply x = 0, y = 1: $\tan^{-1}(1) = -\tan^{-1}(1) + C \Rightarrow \pi/4 = -\pi/4 + C \Rightarrow C = \pi/2$.
Particular solution: $\tan^{-1}(y) + \tan^{-1}(e^x) = \pi/2$.

Answer: $\tan^{-1}(y) + \tan^{-1}(e^x) = \pi/2$

Q8 Solve the differential equation y e^(x/y) dx = (x e^(x/y) + y^2) dy (y ≠ 0).

SOLUTION

Let v = x/y, so x = vy and dx = v dy + y dv.
Substituting: y e^v (v dy + y dv) = (vy e^v + y^2) dy.
y v e^v dy + y^2 e^v dv = vy e^v dy + y^2 dy.
y^2 e^v dv = y^2 dy.
e^v dv = dy.
Integrate: e^v = y + C.
Substitute v = x/y: e^(x/y) = y + C.

Answer: e^(x/y) = y + C

Q9 Find a particular solution of the differential equation (x - y)(dx + dy) = dx - dy, given that y = -1 when x = 0. (Hint: put x - y = t)

SOLUTION

Let t = x - y, so dt = dx - dy. Also dx + dy = dx + dy.

The equation (x-y)(dx+dy) = dx - dy becomes t(dx+dy) = dt.

Also dx + dy = d(x+y), and dx - dy = dt, so dx = (dt + d(x+y))/2... let's use a cleaner approach.

From dx + dy = d(x+y) and dx - dy = dt:

t * d(x+y) = dt.

Also x + y = (x-y) + 2y = t + 2y, and x - y = t.

Let s = x + y. Then d s = d(x+y). The equation becomes: t ds = dt.

ds = dt/t => s = log	t	+ C => x + y = log	x - y	+ C.
Apply x = 0, y = -1: 0 + (-1) = log	0-(-1)	+ C => -1 = log(1) + C = C.		
Particular solution: x + y = log	x - y	- 1.		

Answer: x + y = log|x - y| - 1

Q10 Solve the differential equation [e^(-2*sqrt(x))/sqrt(x) - y/sqrt(x)] dx/dy = 1 (x ≠ 0).

SOLUTION

Since [e^(-2*sqrt(x))/sqrt(x) - y/sqrt(x)] · dx/dy = 1, we have dy/dx = e^(-2*sqrt(x))/sqrt(x) - y/sqrt(x).

dy/dx + y/sqrt(x) = e^(-2*sqrt(x))/sqrt(x).

Linear DE: P = 1/sqrt(x), Q = e^(-2*sqrt(x))/sqrt(x).

I.F. = e^(∫ 1/sqrt(x) dx) = e^(2*sqrt(x)).

y e^(2*sqrt(x)) = ∫ e^(-2*sqrt(x))/sqrt(x) * e^(2*sqrt(x)) dx = ∫ 1/sqrt(x) dx = 2*sqrt(x) + C.

y e^(2*sqrt(x)) = 2*sqrt(x) + C.

Answer: y e^(2*sqrt(x)) = 2*sqrt(x) + C

Q11 Find a particular solution of the differential equation dy/dx + y cot x = 4x cosec x (x ≠ 0), given that y = 0 when x = pi/2.

SOLUTION

Linear DE: P = cot x, Q = 4x cosec x.
I.F. = e^(∫ cot x dx) = e^(log|sin x|) = sin x.
y sin x = ∫ 4x cosec x * sin x dx = ∫ 4x dx = 2x² + C.
Apply y = 0, x = pi/2: 0 = 2(pi/2)² + C = pi²/2 + C => C = -pi²/2.
Particular solution: y sin x = 2x² - pi²/2.
y = (4x² - pi²)/(2 sin x).

Answer: y sin x = 2x² - pi²/2 (i.e., y = (4x² - pi²)/(2 sin x))

Q12 Find a particular solution of the differential equation (x + 1) dy/dx = 2 e⁻y - 1, given that y = 0 when x = 0.

SOLUTION

Separate variables: e^y/(2 - e^y) dy = dx/(x+1).
Let u = 2 - e^y, du = -e^y dy => e^y dy = -du:

∫ e^y/(2-e^y) dy = ∫ -du/u = -log	2-e^y	.		
-log	2-e^y	= log	x+1	+ C.
log	(x+1)(2-e^y)	= C₀, so (x+1)(2-e^y) = A.		

Apply x=0, y=0: 1*(2-1) = A => A = 1.
(x+1)(2-e^y) = 1 => 2 - e^y = 1/(x+1).
e^y = 2 - 1/(x+1) = (2x+1)/(x+1).
y = log|(2x+1)/(x+1)|.

Answer: y = log|(2x + 1)/(x + 1)|

Q13 The general solution of the differential equation (y dx - x dy)/y = 0 is:
(A) xy = C (B) x = Cy² (C) y = Cx (D) y = Cx²

SOLUTION

Rewrite: (y dx - x dy)/y = 0 => y dx - x dy = 0.
Divide by y²: dx/y - x dy/y² = 0 => d(x/y) = 0.
Integrate: x/y = C => y = x/C, or writing C' = 1/C: y = C'x.
Equivalently: y = Cx.

Answer: (C) y = Cx

Q14 The general solution of a differential equation of the type dx/dy + P_1 x = Q_1 is:
(A) y e^($\int P_1$ dy) = $\int$(Q_1 e^($\int P_1$ dy)) dy + C
(B) y e^($\int P_1$ dx) = $\int$(Q_1 e^($\int P_1$ dx)) dx + C
(C) x e^($\int P_1$ dy) = $\int$(Q_1 e^($\int P_1$ dy)) dy + C
(D) x e^($\int P_1$ dx) = $\int$(Q_1 e^($\int P_1$ dx)) dx + C

SOLUTION

For the linear DE dx/dy + P_1 x = Q_1 (where P_1, Q_1 are functions of y only):
I.F. = e^($\int P_1$ dy).
Multiplying through: x * e^($\int P_1$ dy) = $\int$ (Q_1 * e^($\int P_1$ dy)) dy + C.

Answer: (C) x e^($\int P_1$ dy) = $\int$(Q_1 e^($\int P_1$ dy)) dy + C

Q15 The general solution of the differential equation e^x dy + (y e^x + 2x) dx = 0 is:
(A) x e^y + x^2 = C (B) x e^y + y^2 = C (C) y e^x + x^2 = C (D) y e^y + x^2 = C

SOLUTION

Rewrite: dy/dx + y = -2x e^(-x).
Linear DE: P = 1, Q = -2x e^(-x).
I.F. = e^x.
y e^x = $\int$ (-2x e^(-x)) e^x dx = $\int$ (-2x) dx = -x^2 + C.
y e^x + x^2 = C.

Answer: (C) y e^x + x^2 = C