Chapter 10: Vector Algebra

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Exercise 10.1 — Exercise 10.1

Q1 Represent graphically a displacement of 40 km, 30° east of north.

SOLUTION

Draw a directed line segment (arrow) in the plane such that it makes an angle of 30° with the north direction (positive y-axis) toward the east (positive x-axis), with length proportional to 40 km.

The tail of the arrow is at the initial position; the head points in the direction 30° east of north.

Such a directed line segment represents the displacement vector of magnitude 40 km in the given direction.

Answer: A directed line segment of length 40 km inclined at 30° from north toward east represents the required displacement.

Q2 Classify the following measures as scalars and vectors.
(i) 10 kg (ii) 2 meters north-west (iii) 40° (iv) 40 watt (v) 10^(-19) coulomb (vi) 20 m/s²

SOLUTION

(i) 10 kg — mass has only magnitude, no direction. Scalar.

(ii) 2 meters north-west — has both magnitude (2 m) and direction (north-west). Vector.

(iii) 40° — an angle, has magnitude only. Scalar.

(iv) 40 watt — power has only magnitude. Scalar.

(v) 10^(-19) coulomb — electric charge has only magnitude. Scalar.

(vi) 20 m/s² — acceleration has both magnitude and direction. Vector.

Answer: Scalars: (i), (iii), (iv), (v). Vectors: (ii), (vi).

Q3 Classify the following as scalar and vector quantities.
(i) time period (ii) distance (iii) force (iv) velocity (v) work done

SOLUTION

(i) time period — has only magnitude. Scalar.
(ii) distance — has only magnitude (path length). Scalar.
(iii) force — has magnitude and direction. Vector.
(iv) velocity — has magnitude and direction. Vector.
(v) work done — defined as force dot displacement; result is a scalar. Scalar.

Answer: Scalars: (i) time period, (ii) distance, (v) work done. Vectors: (iii) force, (iv) velocity.

Q4 In Fig 10.6 (a square), identify the following vectors.
(i) Coinitial (ii) Equal (iii) Collinear but not equal

SOLUTION

In Fig 10.6 the four sides of the square carry the vectors a, b, c, d with the arrow directions shown: a along the top pointing right, b along the right side pointing down, c along the bottom pointing left, and d along the left side pointing down.
(i) Coinitial vectors have the same initial point. Vectors a and d both start from the top-left corner (a runs right along the top, d runs down along the left side), so a and d are coinitial.
(ii) Equal vectors have the same magnitude AND the same direction. b (right side) and d (left side) both point downward with the same length (sides of the square), so b and d are equal.
(iii) Collinear but not equal: a and c are horizontal, of equal magnitude, but point in opposite directions (a right, c left), so a and c are collinear but not equal.

Answer: (i) Coinitial: a and d. (ii) Equal: b and d. (iii) Collinear but not equal: a and c.

Q5 Answer the following as true or false.

(i) a and -a are collinear.

(ii) Two collinear vectors are always equal in magnitude.

(iii) Two vectors having same magnitude are collinear.

(iv) Two collinear vectors having the same magnitude are equal.

SOLUTION

(i) a and -a are parallel to the same line (they differ only in direction, not in the line they lie on). TRUE.

(ii) Collinear only means the vectors are parallel (lie along the same line). They can have different magnitudes. For example, 2i and 5i are collinear but not equal in magnitude. FALSE.

(iii) Having the same magnitude says nothing about direction. Two vectors of equal length can point in completely different directions and be non-collinear. For example, i and j both have magnitude 1 but are perpendicular, not collinear. FALSE.

(iv) Two collinear vectors having the same magnitude must have the same magnitude and either the same direction or opposite directions. If they are in opposite directions they are not equal (e.g., i and -i). So same magnitude alone does not guarantee equality. FALSE.

Answer: (i) True (ii) False (iii) False (iv) False

Exercise 10.2 — Exercise 10.2

Q1 Compute the magnitude of the following vectors:
$\vec{a} = \hat{i} + \hat{j} + \hat{k}$; $\vec{b} = 2\hat{i} - 7\hat{j} - 3\hat{k}$; $\vec{c} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$

SOLUTION

For $\vec{a} = \hat{i} + \hat{j} + \hat{k}$:	$\vec{a}$	$= \sqrt{(1^2 + 1^2 + 1^2)} = \sqrt{3}$.
For $\vec{b} = 2\hat{i} - 7\hat{j} - 3\hat{k}$:	$\vec{b}$	$= \sqrt{(4 + 49 + 9)} = \sqrt{62}$.
For $\vec{c} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}$:	$\vec{c}$	$= \sqrt{(1/3 + 1/3 + 1/3)} = \sqrt{1} = 1$.

Answer: $|\vec{a}| = \sqrt{3}$, $|\vec{b}| = \sqrt{62}$, $|\vec{c}| = 1$.

Q2 Write two different vectors having same magnitude.

SOLUTION

Any two vectors with equal magnitudes but different directions qualify. For example:
$\vec{a} = \hat{i} + \hat{j} + \hat{k}$ with $|\vec{a}| = \sqrt{3}$, and $\vec{b} = \hat{i} - \hat{j} - \hat{k}$ with $|\vec{b}| = \sqrt{3}$.
Both have magnitude $\sqrt{3}$ but different directions.

Answer: $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - \hat{j} - \hat{k}$ are two different vectors with the same magnitude $\sqrt{3}$ (any two correct examples suffice).

Q3 Write two different vectors having same direction.

SOLUTION

Two vectors have the same direction if one is a positive scalar multiple of the other.
For example: $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} + 2\hat{j} + 2\hat{k}$.
Both point in the direction (1, 1, 1) but have different magnitudes ($\sqrt{3}$ and $2\sqrt{3}$).

Answer: $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = 2(\hat{i} + \hat{j} + \hat{k}) = 2\hat{i} + 2\hat{j} + 2\hat{k}$ have the same direction (any valid positive-scalar-multiple pair is acceptable).

Q4 Find the values of x and y so that the vectors $2\hat{i} + 3\hat{j}$ and $x\hat{i} + y\hat{j}$ are equal.

SOLUTION

Two vectors are equal if and only if their corresponding components are equal.
Comparing $2\hat{i} + 3\hat{j}$ and $x\hat{i} + y\hat{j}$: $x = 2$ and $y = 3$.

Answer: $x = 2, y = 3$.

Q5 Find the scalar and vector components of the vector with initial point (2, 1) and terminal point (– 5, 7).

SOLUTION

The vector from initial point A(2, 1) to terminal point B(−5, 7) is:
$AB = (-5 - 2)\hat{i} + (7 - 1)\hat{j} = -7\hat{i} + 6\hat{j}$.
Scalar components: −7 (along x) and 6 (along y).
Vector components: $-7\hat{i}$ and $6\hat{j}$.

Answer: Scalar components: −7 and 6. Vector components: $-7\hat{i}$ and $6\hat{j}$.

Q6 Find the sum of the vectors $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$.

SOLUTION

$\vec{a} + \vec{b} + \vec{c} = (1 - 2 + 1)\hat{i} + (-2 + 4 - 6)\hat{j} + (1 + 5 - 7)\hat{k}$
$= 0\cdot\hat{i} + (-4)\hat{j} + (-1)\hat{k}$
$= -4\hat{j} - \hat{k}$.

Answer: $\vec{a} + \vec{b} + \vec{c} = -4\hat{j} - \hat{k}$.

Q7 Find the unit vector in the direction of the vector $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$.

SOLUTION

	$\vec{a}$	$= \sqrt{(1^2 + 1^2 + 2^2)} = \sqrt{(1 + 1 + 4)} = \sqrt{6}$.
Unit vector $\hat{a} = \vec{a}/$	$\vec{a}$	$= (\hat{i} + \hat{j} + 2\hat{k})/\sqrt{6} = (1/\sqrt{6})\hat{i} + (1/\sqrt{6})\hat{j} + (2/\sqrt{6})\hat{k}$.

Answer: $\hat{a} = (1/\sqrt{6})\hat{i} + (1/\sqrt{6})\hat{j} + (2/\sqrt{6})\hat{k}$.

Q8 Find the unit vector in the direction of vector PQ, where P and Q are the points (1, 2, 3) and (4, 5, 6), respectively.

SOLUTION

PQ = (4 − 1)î + (5 − 2)ĵ + (6 − 3)k̂ = 3î + 3ĵ + 3k̂.

	PQ	= √(9 + 9 + 9) = √27 = 3√3.
Unit vector = PQ/	PQ	= (3î + 3ĵ + 3k̂)/(3√3) = (1/√3)î + (1/√3)ĵ + (1/√3)k̂.

Answer: Unit vector in direction of PQ = (1/√3)î + (1/√3)ĵ + (1/√3)k̂.

Q9 For given vectors, a = 2î − ĵ + 2k̂ and b = −î + ĵ − k̂, find the unit vector in the direction of the vector a + b.

SOLUTION

a + b = (2 − 1)î + (−1 + 1)ĵ + (2 − 1)k̂ = î + 0·ĵ + k̂ = î + k̂.

	a + b	= √(1 + 0 + 1) = √2.
Unit vector = (a + b)/	a + b	= (î + k̂)/√2 = (1/√2)î + (1/√2)k̂.

Answer: Unit vector in direction of a + b = (1/√2)î + (1/√2)k̂.

Q10 Find a vector in the direction of vector 5î − ĵ + 2k̂ which has magnitude 8 units.

SOLUTION

Let a = 5î − ĵ + 2k̂. Then |a| = √(25 + 1 + 4) = √30.

Unit vector â = (5î − ĵ + 2k̂)/√30.

Vector of magnitude 8 in this direction = 8â = (8/√30)(5î − ĵ + 2k̂).

= (40/√30)î − (8/√30)ĵ + (16/√30)k̂.

Answer: (40/√30)î − (8/√30)ĵ + (16/√30)k̂.

Q11 Show that the vectors 2î − 3ĵ + 4k̂ and −4î + 6ĵ − 8k̂ are collinear.

SOLUTION

Two vectors are collinear if one is a scalar multiple of the other.

Let a = 2î − 3ĵ + 4k̂ and b = −4î + 6ĵ − 8k̂.

Notice that b = −2(2î − 3ĵ + 4k̂) = −2a.

Since b = −2a, the vectors are parallel (collinear).

Answer: b = −2a, so the vectors are collinear.

Q12 Find the direction cosines of the vector $\hat{i} + 2\hat{j} + 3\hat{k}$.

SOLUTION

Let $a = \hat{i} + 2\hat{j} + 3\hat{k}$. Then $|a| = \sqrt{(1 + 4 + 9)} = \sqrt{14}$.
Direction cosines: $l = 1/\sqrt{14}$, $m = 2/\sqrt{14}$, $n = 3/\sqrt{14}$.

Answer: Direction cosines are $l = 1/\sqrt{14}$, $m = 2/\sqrt{14}$, $n = 3/\sqrt{14}$.

Q13 Find the direction cosines of the vector joining the points A(1, 2, −3) and B(−1, −2, 1), directed from A to B.

SOLUTION

$AB = (-1 - 1)\hat{i} + (-2 - 2)\hat{j} + (1 - (-3))\hat{k} = -2\hat{i} - 4\hat{j} + 4\hat{k}$.
$|AB| = \sqrt{(4 + 16 + 16)} = \sqrt{36} = 6$.
Direction cosines: $l = -2/6 = -1/3$, $m = -4/6 = -2/3$, $n = 4/6 = 2/3$.

Answer: Direction cosines: −1/3, −2/3, 2/3.

Q14 Show that the vector $\hat{i} + \hat{j} + \hat{k}$ is equally inclined to the axes OX, OY and OZ.

SOLUTION

Let $a = \hat{i} + \hat{j} + \hat{k}$. Then $|a| = \sqrt{3}$.
The direction cosines of a are: $\cos\alpha = 1/\sqrt{3}$, $\cos\beta = 1/\sqrt{3}$, $\cos\gamma = 1/\sqrt{3}$.
Since $\cos\alpha = \cos\beta = \cos\gamma$, the angles α, β, γ which the vector makes with the x, y, and z axes are all equal.
Therefore, the vector $\hat{i} + \hat{j} + \hat{k}$ is equally inclined to all three coordinate axes.

Answer: The direction cosines of $\hat{i} + \hat{j} + \hat{k}$ are all equal to $1/\sqrt{3}$, so it makes the same angle ($\arccos(1/\sqrt{3})$) with OX, OY, and OZ.

Q15 Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively, in the ratio 2 : 1

(i) internally (ii) externally

SOLUTION

Let $p = \hat{i} + 2\hat{j} - \hat{k}$ and $q = -\hat{i} + \hat{j} + \hat{k}$.

(i) Internal division in ratio m:n = 2:1:

$OR = (m \cdot q + n \cdot p)/(m + n) = (2q + p)/3$

$= (2(-\hat{i} + \hat{j} + \hat{k}) + (\hat{i} + 2\hat{j} - \hat{k}))/3$

$= (-2\hat{i} + 2\hat{j} + 2\hat{k} + \hat{i} + 2\hat{j} - \hat{k})/3$

$= (-\hat{i} + 4\hat{j} + \hat{k})/3$

$= -(1/3)\hat{i} + (4/3)\hat{j} + (1/3)\hat{k}$.

(ii) External division in ratio m:n = 2:1:

$OR = (m \cdot q - n \cdot p)/(m - n) = (2q - p)/(2 - 1) = 2q - p$

$= 2(-\hat{i} + \hat{j} + \hat{k}) - (\hat{i} + 2\hat{j} - \hat{k})$

$= -2\hat{i} + 2\hat{j} + 2\hat{k} - \hat{i} - 2\hat{j} + \hat{k}$

$= -3\hat{i} + 3\hat{k}$.

Answer: (i) $R = -(1/3)\hat{i} + (4/3)\hat{j} + (1/3)\hat{k}$. (ii) $R = -3\hat{i} + 3\hat{k}$.

Q16 Find the position vector of the mid point of the vector joining the points P(2, 3, 4) and Q(4, 1, −2).

SOLUTION

Position vectors: $p = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $q = 4\hat{i} + \hat{j} - 2\hat{k}$.

Midpoint $= (p + q)/2 = ((2 + 4)\hat{i} + (3 + 1)\hat{j} + (4 - 2)\hat{k})/2$

$= (6\hat{i} + 4\hat{j} + 2\hat{k})/2 = 3\hat{i} + 2\hat{j} + \hat{k}$.

Answer: Position vector of mid point = $3\hat{i} + 2\hat{j} + \hat{k}$.

Q17 Show that the points A, B and C with position vectors $a = 3\hat{i} - 4\hat{j} - 4\hat{k}$, $b = 2\hat{i} - \hat{j} + \hat{k}$ and $c = \hat{i} - 3\hat{j} - 5\hat{k}$, respectively form the vertices of a right angled triangle.

SOLUTION

$AB = b - a = (2-3)\hat{i} + (-1+4)\hat{j} + (1+4)\hat{k} = -\hat{i} + 3\hat{j} + 5\hat{k}$.

$BC = c - b = (1-2)\hat{i} + (-3+1)\hat{j} + (-5-1)\hat{k} = -\hat{i} - 2\hat{j} - 6\hat{k}$.

$CA = a - c = (3-1)\hat{i} + (-4+3)\hat{j} + (-4+5)\hat{k} = 2\hat{i} - \hat{j} + \hat{k}$.

	AB	² = 1 + 9 + 25 = 35.				
	BC	² = 1 + 4 + 36 = 41.				
	CA	² = 4 + 1 + 1 = 6.				
Check:	BC	² =	AB	² +	CA	² => 41 = 35 + 6 = 41. ✓

Alternatively, AB · CA = (−1)(2) + (3)(−1) + (5)(1) = −2 − 3 + 5 = 0.
Since AB · CA = 0, angle A = 90°. The triangle is right-angled at A.

Answer: Since |BC|² = |AB|² + |CA|² (i.e., 41 = 35 + 6) and AB · CA = 0, the triangle is right-angled at A.

Q18 In triangle ABC (Fig 10.18), which of the following is not true:
(A) AB + BC + CA = 0 (B) AB + BC − AC = 0 (C) AB + BC − CA = 0 (D) AB − CB + CA = 0

SOLUTION

By the triangle law of vector addition, going around the triangle A → B → C → A: AB + BC + CA = 0. So (A) is true.

(B) AB + BC − AC = 0. Since CA = −AC, we have AB + BC + CA = AB + BC − AC = 0. So (B) is true.

(C) AB + BC − CA = 0 means AB + BC = CA. But AB + BC = AC = −CA (not CA). So AB + BC − CA = AC − CA = −CA − CA = −2CA ≠ 0 in general. (C) is NOT true.

(D) AB − CB + CA = 0. Note CB = −BC, so AB − CB + CA = AB + BC + CA = 0. So (D) is true.

The statement that is not true is (C).

Answer: (C) AB + BC − CA = 0 is not true.

Q19 If a and b are two collinear vectors, then which of the following are incorrect:
(A) $\vec{b} = \lambda\vec{a}$ for some scalar λ
(B) $\vec{a} = \pm\vec{b}$
(C) the respective components of a and b are not proportional
(D) both the vectors a and b have same direction, but different magnitudes.

SOLUTION

Collinear vectors are parallel — one is a scalar multiple of the other: $\vec{b} = \lambda\vec{a}$ for some nonzero scalar λ.

(A) $\vec{b} = \lambda\vec{a}$ for some scalar λ. This is the definition of collinearity. CORRECT.

(B) $\vec{a} = \pm\vec{b}$ means $|\vec{a}|=|\vec{b}|$ and same/opposite direction. Collinear vectors need not have equal magnitudes, so this is not always true. INCORRECT.

(C) Components of collinear vectors must be proportional ($b_i = \lambda a_i$). Saying 'not proportional' contradicts collinearity. INCORRECT.

(D) Collinear vectors can have opposite directions ($\lambda < 0$) as well as same direction ($\lambda > 0$). Claiming they must have the same direction is too restrictive. INCORRECT.

Options (C) and (D) are incorrect.

Answer: (C) and (D) are incorrect.

Exercise 10.3 — Exercise 10.3

Q1 Find the angle between two vectors a and b with magnitudes √3 and 2, respectively having a · b = √6.

SOLUTION

The angle θ between two vectors satisfies cos θ = (a · b)/(|a||b|).
cos θ = √6/(√3 × 2) = √6/(2√3) = √(6/3)/2 = √2/2.
θ = arccos(√2/2) = π/4.

Answer: θ = π/4 (45°).

Q2 Find the angle between the vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$.

SOLUTION

Let $a = \hat{i} - 2\hat{j} + 3\hat{k}$ and $b = 3\hat{i} - 2\hat{j} + \hat{k}$.
a · b = (1)(3) + (−2)(−2) + (3)(1) = 3 + 4 + 3 = 10.

	a	= √(1 + 4 + 9) = √14.
	b	= √(9 + 4 + 1) = √14.

cos θ = 10/(√14 × √14) = 10/14 = 5/7.
θ = arccos(5/7).

Answer: θ = arccos(5/7).

Q3 Find the projection of the vector $\hat{i} - \hat{j}$ on the vector $\hat{i} + \hat{j}$.

SOLUTION

Projection of a on b is given by (a · b)/|b|.
$a = \hat{i} - \hat{j}$, $b = \hat{i} + \hat{j}$.
a · b = (1)(1) + (−1)(1) = 1 − 1 = 0.
|b| = √(1 + 1) = √2.
Projection = 0/√2 = 0.

Answer: Projection = 0.

Q4 Find the projection of the vector $\hat{i} + 3\hat{j} + 7\hat{k}$ on the vector $7\hat{i} - \hat{j} + 8\hat{k}$.

SOLUTION

Projection of a on b = $(a \cdot b)/|b|$.

$a = \hat{i} + 3\hat{j} + 7\hat{k}$, $b = 7\hat{i} - \hat{j} + 8\hat{k}$.

$a \cdot b = (1)(7) + (3)(-1) + (7)(8) = 7 - 3 + 56 = 60$.

$|b| = \sqrt{(49 + 1 + 64)} = \sqrt{114}$.

Projection = $60/\sqrt{114} = (60\sqrt{114})/114 = (10\sqrt{114})/19$.

Answer: Projection = $60/\sqrt{114} = 10\sqrt{114}/19$.

Q5 Show that each of the given three vectors is a unit vector:
$(1/7)(2\hat{i} + 3\hat{j} + 6\hat{k})$, $(1/7)(3\hat{i} - 6\hat{j} + 2\hat{k})$, $(1/7)(6\hat{i} + 2\hat{j} - 3\hat{k})$
Also, show that they are mutually perpendicular to each other.

SOLUTION

Let $u = (1/7)(2\hat{i} + 3\hat{j} + 6\hat{k})$, $v = (1/7)(3\hat{i} - 6\hat{j} + 2\hat{k})$, $w = (1/7)(6\hat{i} + 2\hat{j} - 3\hat{k})$.

Magnitudes:

	u	= $(1/7)\sqrt{(4 + 9 + 36)} = (1/7)\sqrt{49} = (1/7)(7) = 1$. ✓
	v	= $(1/7)\sqrt{(9 + 36 + 4)} = (1/7)\sqrt{49} = 1$. ✓
	w	= $(1/7)\sqrt{(36 + 4 + 9)} = (1/7)\sqrt{49} = 1$. ✓

Mutual perpendicularity (using unnormalized vectors $p = 2\hat{i}+3\hat{j}+6\hat{k}$, $q = 3\hat{i}-6\hat{j}+2\hat{k}$, $r = 6\hat{i}+2\hat{j}-3\hat{k}$):

$p \cdot q = (2)(3) + (3)(-6) + (6)(2) = 6 - 18 + 12 = 0$. => $u \perp v$ ✓

$p \cdot r = (2)(6) + (3)(2) + (6)(-3) = 12 + 6 - 18 = 0$. => $u \perp w$ ✓

$q \cdot r = (3)(6) + (-6)(2) + (2)(-3) = 18 - 12 - 6 = 0$. => $v \perp w$ ✓

All three are unit vectors and mutually perpendicular.

Answer: Each vector has magnitude 1 (unit vector), and their pairwise dot products are all 0 (mutually perpendicular).

Q6 Find $|a|^2$ and $|b|^2$, if $(a + b) \cdot (a - b) = 8$ and $|a| = 8|b|$.

SOLUTION

$(a + b) \cdot (a - b) =$	a	² −	b	² = 8.				
Also given	a	= 8	b	, so	a	² = 64	b	².
Substituting: 64	b	² −	b	² = 8 => 63	b	² = 8 =>	b	² = 8/63.
	a	² = 64 × (8/63) = 512/63.						

Answer: $|a|^2 = 512/63$, $|b|^2 = 8/63$.

Q7 Evaluate the product $(3a - 5b) \cdot (2a + 7b)$.

SOLUTION

$(3a - 5b) \cdot (2a + 7b)$
$= 3a \cdot 2a + 3a \cdot 7b - 5b \cdot 2a - 5b \cdot 7b$
$= 6(a \cdot a) + 21(a \cdot b) - 10(b \cdot a) - 35(b \cdot b)$

= 6	a	² + 21(a · b) − 10(a · b) − 35	b	²
= 6	a	² + 11(a · b) − 35	b	².

Answer: $(3a - 5b) \cdot (2a + 7b) = 6|a|^2 + 11(a \cdot b) - 35|b|^2$.

Q8 Find the magnitude of two vectors a and b, having the same magnitude and such that the angle between them is 60° and their scalar product is 1/2.

SOLUTION

Given	a	=	b	, angle θ = 60°, and a · b = 1/2.				
a · b =	a		b	cos θ =	a	² cos 60° =	a	² × (1/2).

Given	a	=	b	, angle θ = 60°, and a · b = 1/2.				
So (1/2)	a	² = 1/2, giving	a	² = 1,	a	= 1.		
Therefore	a	=	b	= 1.				

Answer: |a| = |b| = 1.

Q9 Find |x|, if for a unit vector â, (x − â) · (x + â) = 12.

SOLUTION

(x − â) · (x + â) = x · x − â · â =	x	² −	â	².		
Since â is a unit vector,	â	= 1, so	â	² = 1.		
	x	² − 1 = 12 =>	x	² = 13 =>	x	= √13.

Answer: |x| = √13.

Q10 If a = 2î + 2ĵ − 3k̂, b = 3î − ĵ + 2k̂ are such that a + λb is perpendicular to b, then find the value of λ.

SOLUTION

For (a + λb) to be perpendicular to b, we need (a + λb) · b = 0.

a · b + λ|b|² = 0.

a · b = (2)(3) + (2)(−1) + (−3)(2) = 6 − 2 − 6 = −2.

|b|² = 9 + 1 + 4 = 14.

−2 + 14λ = 0 => λ = 2/14 = 1/7.

Answer: λ = 1/7.

Q11 Show that |a|b + |b|a is perpendicular to |a|b − |b|a, for any two nonzero vectors a and b.

SOLUTION

Compute the dot product (	a	b +	b	a) · (	a	b −	b	a):							
=	a	²(b · b) −	a		b	(b · a) +	b		a	(a · b) −	b	²(a · a)			
=	a	²	b	² −	a		b	(a · b) +	a		b	(a · b) −	b	²	a	²
=	a	²	b	² −	a	²	b	²								

= 0.

Since the dot product is 0, |a|b + |b|a is perpendicular to |a|b − |b|a.

Answer: (|a|b + |b|a) · (|a|b − |b|a) = 0, so they are perpendicular.

Q12 If a · a = 0 and a · b = 0, then what can be concluded about the vector b?

SOLUTION

a · a = |a|² = 0 implies |a| = 0, so a = 0 (the zero vector).

When a = 0, the condition a · b = 0 is automatically satisfied for any vector b.

Therefore, no conclusion can be drawn about b — it can be any vector.

Answer: Since a · a = 0 implies a = 0, the equation a · b = 0 holds for any b; b is an arbitrary vector.

Q13 If a, b, c are unit vectors such that a + b + c = 0, find the value of a · b + b · c + c · a.

SOLUTION

Since a + b + c = 0, take the dot product of both sides with itself:

	a + b + c	² = 0.				
	a	² +	b	² +	c	² + 2(a · b + b · c + c · a) = 0.
Since a, b, c are unit vectors:	a	=	b	=	c	= 1.

1 + 1 + 1 + 2(a · b + b · c + c · a) = 0.
2(a · b + b · c + c · a) = −3.
a · b + b · c + c · a = −3/2.

Answer: a · b + b · c + c · a = −3/2.

Q14 If either vector a = 0 or b = 0, then a · b = 0. But the converse need not be true. Justify your answer with an example.

SOLUTION

The forward statement: if a = 0 or b = 0, then a · b = |a||b|cos θ = 0 (since at least one magnitude is 0). TRUE.

Converse: 'if a · b = 0, then a = 0 or b = 0' is FALSE.

Counterexample: Let a = î and b = ĵ. Then a ≠ 0 and b ≠ 0, but a · b = î · ĵ = 0 (since they are perpendicular).

This shows that a · b = 0 does not imply either vector is zero — it only means the vectors are perpendicular (when both are nonzero).

Answer: Converse is false. Counterexample: a = î, b = ĵ; both nonzero yet î · ĵ = 0.

Q15 If the vertices A, B, C of a triangle ABC are (1, 2, 3), (−1, 0, 0), (0, 1, 2), respectively, then find ∠ABC. [∠ABC is the angle between the vectors BA and BC].

SOLUTION

A = (1,2,3), B = (−1,0,0), C = (0,1,2).
BA = A − B = (1−(−1))î + (2−0)ĵ + (3−0)k̂ = 2î + 2ĵ + 3k̂.
BC = C − B = (0−(−1))î + (1−0)ĵ + (2−0)k̂ = î + ĵ + 2k̂.
BA · BC = (2)(1) + (2)(1) + (3)(2) = 2 + 2 + 6 = 10.

	BA	= √(4 + 4 + 9) = √17.
	BC	= √(1 + 1 + 4) = √6.

cos(∠ABC) = 10/(√17 × √6) = 10/√102.
∠ABC = arccos(10/√102).

Answer: ∠ABC = arccos(10/√102).

Q16 Show that the points A(1, 2, 7), B(2, 6, 3) and C(3, 10, −1) are collinear.

SOLUTION

AB = (2−1)î + (6−2)ĵ + (3−7)k̂ = î + 4ĵ − 4k̂.
BC = (3−2)î + (10−6)ĵ + (−1−3)k̂ = î + 4ĵ − 4k̂.
Since AB = BC, both vectors are identical, meaning B is the midpoint of AC and all three points lie on the same line.
Also, AC = 2î + 8ĵ − 8k̂ = 2·AB, confirming A, B, C are collinear.

Answer: AB = BC = î + 4ĵ − 4k̂, so A, B, C are collinear.

Q17 Show that the vectors 2î − ĵ + k̂, î − 3ĵ − 5k̂ and 3î − 4ĵ − 4k̂ form the vertices of a right angled triangle.

SOLUTION

Let A have position vector a = 2î − ĵ + k̂, B have position vector b = î − 3ĵ − 5k̂, C have position vector c = 3î − 4ĵ − 4k̂.

AB = b − a = −î − 2ĵ − 6k̂.	AB	² = 1 + 4 + 36 = 41.
BC = c − b = 2î − ĵ + k̂.	BC	² = 4 + 1 + 1 = 6.
CA = a − c = −î + 3ĵ + 5k̂.	CA	² = 1 + 9 + 25 = 35.

Check right angle: BC · CA = (2)(−1) + (−1)(3) + (1)(5) = −2 − 3 + 5 = 0.
Since BC · CA = 0, the angle at C is 90°.
Also verify: |AB|² = |BC|² + |CA|² => 41 = 6 + 35 = 41. ✓
The given vectors form the vertices of a right-angled triangle with right angle at C.

Answer: BC · CA = 0, confirming a right angle at C; also |AB|² = |BC|² + |CA|² = 41.

Q18 If a is a nonzero vector of magnitude 'a' and λ a nonzero scalar, then λa is unit vector if

(A) λ = 1 (B) λ = −1 (C) a = |λ| (D) a = 1/|λ|

SOLUTION

λa is a unit vector if	λa	= 1.						
	λa	=	λ	×	a	=	λ	× a = 1.
Therefore a = 1/	λ	.						

The correct option is (D).

Answer: (D) a = 1/|λ|.

Exercise 10.4 — Exercise 10.4

Q1 Find $|\vec{a} \times \vec{b}|$, if $\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$.

SOLUTION

$\vec{a} \times \vec{b} =$	$\hat{i}$ $\hat{j}$ $\hat{k}$	
	1 -7 7	
	3 -2 2	

$\hat{i}$ component: $(-7)(2) - (7)(-2) = -14 + 14 = 0$.

$\hat{j}$ component: $-((1)(2) - (7)(3)) = -(2 - 21) = 19$.

$\hat{k}$ component: $(1)(-2) - (-7)(3) = -2 + 21 = 19$.

$\vec{a} \times \vec{b} = 0\hat{i} + 19\hat{j} + 19\hat{k}$.

$|\vec{a} \times \vec{b}| = \sqrt{(0 + 361 + 361)} = \sqrt{722} = 19\sqrt{2}$.

Answer: $|\vec{a} \times \vec{b}| = 19\sqrt{2}$.

Q2 Find a unit vector perpendicular to each of the vectors $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$, where $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$.

SOLUTION

$\vec{a} + \vec{b} = (3+1)\hat{i} + (2+2)\hat{j} + (2-2)\hat{k} = 4\hat{i} + 4\hat{j} + 0\hat{k}$.

$\vec{a} - \vec{b} = (3-1)\hat{i} + (2-2)\hat{j} + (2+2)\hat{k} = 2\hat{i} + 0\hat{j} + 4\hat{k}$.

$(\vec{a}+\vec{b}) \times (\vec{a}-\vec{b}) =$	$\hat{i}$ $\hat{j}$ $\hat{k}$	
	4 4 0	
	2 0 4	

$\hat{i}$: $(4)(4) - (0)(0) = 16$.

$\hat{j}$: $-((4)(4) - (0)(2)) = -(16 - 0) = -16$.

$\hat{k}$: $(4)(0) - (4)(2) = 0 - 8 = -8$.

$(\vec{a}+\vec{b}) \times (\vec{a}-\vec{b}) = 16\hat{i} - 16\hat{j} - 8\hat{k} = 8(2\hat{i} - 2\hat{j} - \hat{k})$.

$|(\vec{a}+\vec{b}) \times (\vec{a}-\vec{b})| = 8\sqrt{(4 + 4 + 1)} = 8 \times 3 = 24$.

Unit vector $= (16\hat{i} - 16\hat{j} - 8\hat{k})/24 = (2/3)\hat{i} - (2/3)\hat{j} - (1/3)\hat{k}$.

The other unit vector (opposite direction) is $-(2/3)\hat{i} + (2/3)\hat{j} + (1/3)\hat{k}$.

Answer: Unit vector $= \pm[(2/3)\hat{i} - (2/3)\hat{j} - (1/3)\hat{k}]$.

Q3 If a unit vector â makes angles π/3 with î, π/4 with ĵ and an acute angle θ with k̂, then find θ and hence, the components of â.

SOLUTION

Let â = cos(π/3)î + cos(π/4)ĵ + cos(θ)k̂.
For any unit vector, the sum of squares of direction cosines equals 1:
cos²(π/3) + cos²(π/4) + cos²(θ) = 1.
(1/2)² + (1/√2)² + cos²(θ) = 1.
1/4 + 1/2 + cos²(θ) = 1.
cos²(θ) = 1 − 3/4 = 1/4.
cos(θ) = 1/2 (taking positive value since θ is acute).
θ = π/3 (60°).

Components of â: cos(π/3) = 1/2, cos(π/4) = 1/√2, cos(π/3) = 1/2.
â = (1/2)î + (1/√2)ĵ + (1/2)k̂.

Answer: θ = π/3; â = (1/2)î + (1/√2)ĵ + (1/2)k̂.

Q4 Show that (a − b) × (a + b) = 2(a × b).

SOLUTION

(a − b) × (a + b)
= a × a + a × b − b × a − b × b
= 0 + a × b − (−(a × b)) − 0
= a × b + a × b
= 2(a × b).

Here we used: a × a = 0, b × b = 0, and b × a = −(a × b).

Answer: (a − b) × (a + b) = 2(a × b). Proved.

Q5 Find λ and μ if (2î + 6ĵ + 27k̂) × (î + λĵ + μk̂) = 0.

SOLUTION

For the cross product to be 0, the vectors must be parallel (scalar multiples of each other).
2î + 6ĵ + 27k̂ = t(î + λĵ + μk̂) for some scalar t.
From the î component: t = 2.
From the ĵ component: 6 = tλ = 2λ => λ = 3.
From the k̂ component: 27 = tμ = 2μ => μ = 27/2.

Verify: ratio 2/1 = 6/3 = 27/(27/2) = 2. ✓

Answer: λ = 3, μ = 27/2.

Q6 Given that a · b = 0 and a × b = 0. What can you conclude about the vectors a and b?

SOLUTION

a · b = 0 means either a = 0, or b = 0, or a ⊥ b.
a × b = 0 means either a = 0, or b = 0, or a is parallel to b.

A nonzero vector cannot be simultaneously perpendicular to and parallel to another nonzero vector.
Therefore, both conditions together imply that at least one of the vectors must be the zero vector.
Conclusion: Either a = 0 or b = 0 (or both).

Answer: Either a = 0 or b = 0 (or both).

Q7 Let the vectors a, b, c be given as $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. Then show that a × (b + c) = a × b + a × c.

SOLUTION

Let $a = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $b = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $c = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$.
$b + c = (b_1+c_1)\hat{i} + (b_2+c_2)\hat{j} + (b_3+c_3)\hat{k}$.

a × (b + c) =	$\hat{i}$ $\hat{j}$ $\hat{k}$	
	a_1 a_2 a_3	
	b_1+c_1 b_2+c_2 b_3+c_3	

$= \hat{i}[a_2(b_3+c_3) - a_3(b_2+c_2)] - \hat{j}[a_1(b_3+c_3) - a_3(b_1+c_1)] + \hat{k}[a_1(b_2+c_2) - a_2(b_1+c_1)]$

$= \hat{i}[(a_2b_3 - a_3b_2) + (a_2c_3 - a_3c_2)] - \hat{j}[(a_1b_3 - a_3b_1) + (a_1c_3 - a_3c_1)] + \hat{k}[(a_1b_2 - a_2b_1) + (a_1c_2 - a_2c_1)]$

= (a × b) + (a × c).

Hence a × (b + c) = a × b + a × c.

Answer: a × (b + c) = a × b + a × c. Proved by component expansion.

Q8 If either a = 0 or b = 0, then a × b = 0. Is the converse true? Justify your answer with an example.

SOLUTION

Forward: If a = 0 or b = 0, then |a × b| = |a||b|sin θ = 0 (since at least one magnitude is 0). TRUE.

Converse: 'If a × b = 0, then a = 0 or b = 0' is FALSE.

Counterexample: Let a = î and b = 2î. Both are nonzero.
a × b = î × 2î = 2(î × î) = 2 × 0 = 0.

So a × b = 0 even though neither a nor b is zero — they are parallel (collinear), which also makes the cross product zero.

Answer: The converse is false. Counterexample: a = î, b = 2î; both nonzero but î × 2î = 0.

Q9 Find the area of the triangle with vertices A(1, 1, 2), B(2, 3, 5) and C(1, 5, 5).

SOLUTION

AB = (2−1)î + (3−1)ĵ + (5−2)k̂ = î + 2ĵ + 3k̂.
AC = (1−1)î + (5−1)ĵ + (5−2)k̂ = 0î + 4ĵ + 3k̂.

AB × AC =	î ĵ k̂	
	1 2 3	
	0 4 3	

î: (2)(3) − (3)(4) = 6 − 12 = −6.
ĵ: −((1)(3) − (3)(0)) = −3.
k̂: (1)(4) − (2)(0) = 4.

AB × AC = −6î − 3ĵ + 4k̂.
|AB × AC| = √(36 + 9 + 16) = √61.

Area of triangle = (1/2)|AB × AC| = √61/2.

Answer: Area = √61/2 sq. units.

Q10 Find the area of the parallelogram whose adjacent sides are determined by the vectors a = î − ĵ + 3k̂ and b = 2î − 7ĵ + k̂.

SOLUTION

Area of parallelogram = |a × b|.

a × b =	î ĵ k̂	
	1 -1 3	
	2 -7 1	

î: (−1)(1) − (3)(−7) = −1 + 21 = 20.
ĵ: −((1)(1) − (3)(2)) = −(1 − 6) = 5.
k̂: (1)(−7) − (−1)(2) = −7 + 2 = −5.

a × b = 20î + 5ĵ − 5k̂.
|a × b| = √(400 + 25 + 25) = √450 = 15√2.

Answer: Area = 15√2 sq. units.

Q11 Let the vectors a and b be such that |a| = 3 and |b| = √2/3, then a × b is a unit vector, if the angle between a and b is
(A) π/6 (B) π/4 (C) π/3 (D) π/2

SOLUTION

	a × b	=	a		b	sin θ = 3 × (√2/3) × sin θ = √2 sin θ.
For a × b to be a unit vector:	a × b	= 1.				

√2 sin θ = 1 => sin θ = 1/√2 => θ = π/4.
The correct option is (B).

Answer: (B) π/4.

Q12 Area of a rectangle having vertices A, B, C and D with position vectors $-\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}$, $\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}$, $\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$ and $-\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$, respectively is
(A) 1/2 (B) 1 (C) 2 (D) 4

SOLUTION

$AB = (\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}) - (-\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}) = 2\hat{i}.$	AB	= 2.		
$BC = (\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}) - (\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}) = -\hat{j}.$	BC	= 1.		
Area of rectangle =	AB	×	BC	= 2 × 1 = 2.

The correct option is (C).

Answer: (C) 2.

Misc — Miscellaneous Exercise

Q1 Write down a unit vector in XY-plane, making an angle of 30° with the positive direction of x-axis.

SOLUTION

A unit vector in the XY-plane making angle θ with the positive x-axis is cos θ $\hat{i}$ + sin θ $\hat{j}$.

For θ = 30°: cos 30° = √3/2 and sin 30° = 1/2.

Required unit vector = (√3/2)$\hat{i}$ + (1/2)$\hat{j}$.

Answer: (√3/2)$\hat{i}$ + (1/2)$\hat{j}$.

Q2 Find the scalar components and magnitude of the vector joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.

SOLUTION

The vector PQ from P to Q is:

$PQ = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$.

Scalar components: $(x_2 - x_1)$, $(y_2 - y_1)$, $(z_2 - z_1)$.

Magnitude: $|PQ| = \sqrt{((x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2)}$.

Answer: Scalar components: (x_2-x_1), (y_2-y_1), (z_2-z_1). Magnitude = $\sqrt{((x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2)}$.

Q3 A girl walks 4 km towards west, then she walks 3 km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.

SOLUTION

Set up: east = +$\hat{i}$, north = +$\hat{j}$.

Step 1: 4 km west = −4$\hat{i}$.

Step 2: 3 km in direction '30° east of north' means the direction is 30° from north toward east.

x-component (east): 3 sin 30° = 3 × (1/2) = 3/2.

y-component (north): 3 cos 30° = 3 × (√3/2) = 3√3/2.

Step 2 vector = (3/2)$\hat{i}$ + (3√3/2)$\hat{j}$.

Total displacement = (−4 + 3/2)$\hat{i}$ + (3√3/2)$\hat{j}$ = (−5/2)$\hat{i}$ + (3√3/2)$\hat{j}$.

Magnitude: |d| = √((−5/2)² + (3√3/2)²) = √(25/4 + 27/4) = √(52/4) = √13 km.

Answer: Displacement = −(5/2)î + (3√3/2)ĵ with magnitude √13 km.

Q4 If ā = b̄ + c̄, then is it true that |ā| = |b̄| + |c̄|? Justify your answer.

SOLUTION

The statement is NOT always true.

By the triangle inequality: |b + c| ≤ |b| + |c|, with equality only when b and c point in the same direction.

In general, if b and c are not parallel (same direction), then |b + c| < |b| + |c|.

Counterexample: Let b = î and c = ĵ. Then a = b + c = î + ĵ.
|a| = √2, |b| + |c| = 1 + 1 = 2. Since √2 ≠ 2, the statement is false.

Therefore, |ā| = |b̄| + |c̄| is not true in general.

Answer: No, it is not true in general. |b + c| ≤ |b| + |c| (triangle inequality), with equality only when b and c are in the same direction.

Q5 Find the value of x for which x(î + ĵ + k̂) is a unit vector.

SOLUTION

	x(î + ĵ + k̂)	=	x	×	(î + ĵ + k̂)	=	x	× √3.
For this to be a unit vector:	x	√3 = 1 =>	x	= 1/√3.				

Therefore x = ±1/√3 = ±√3/3.

Answer: x = ±1/√3.

Q6 Find a vector of magnitude 5 units, and parallel to the resultant of the vectors ā = 2î + ĵ + k̂ and b̄ = î − 2ĵ − k̂.

SOLUTION

Resultant r = ā + b̄ = (2+1)î + (1−2)ĵ + (1−1)k̂ = 3î − ĵ + 0k̂ = 3î − ĵ.
|r| = √(9 + 1) = √10.
Unit vector r̂ = (3î − ĵ)/√10.
Required vector = 5r̂ = (15/√10)î − (5/√10)ĵ = (3√10/2)î − (√10/2)ĵ.

Simplification: $15/\sqrt{10} = 15\sqrt{10}/10 = 3\sqrt{10}/2$, and $5/\sqrt{10} = \sqrt{10}/2$.

Answer: Required vector = $(3\sqrt{10}/2)\hat{i} - (\sqrt{10}/2)\hat{j}$.

Q7 If $\bar{a} = \hat{i} + \hat{j} + \hat{k}$, $\bar{b} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\bar{c} = \hat{i} - 2\hat{j} + \hat{k}$, find a unit vector parallel to the vector $2\bar{a} - \bar{b} + 3\bar{c}$.

SOLUTION

$2\bar{a} = 2\hat{i} + 2\hat{j} + 2\hat{k}$.

$\bar{b} = 2\hat{i} - \hat{j} + 3\hat{k}$.

$3\bar{c} = 3\hat{i} - 6\hat{j} + 3\hat{k}$.

$2\bar{a} - \bar{b} + 3\bar{c} = (2-2+3)\hat{i} + (2+1-6)\hat{j} + (2-3+3)\hat{k} = 3\hat{i} - 3\hat{j} + 2\hat{k}$.

$|2\bar{a} - \bar{b} + 3\bar{c}| = \sqrt{(9 + 9 + 4)} = \sqrt{22}$.

Unit vector = $(3\hat{i} - 3\hat{j} + 2\hat{k})/\sqrt{22} = (3/\sqrt{22})\hat{i} - (3/\sqrt{22})\hat{j} + (2/\sqrt{22})\hat{k}$.

Answer: Unit vector = $(3/\sqrt{22})\hat{i} - (3/\sqrt{22})\hat{j} + (2/\sqrt{22})\hat{k}$.

Q8 Show that the points A(1, −2, −8), B(5, 0, −2) and C(11, 3, 7) are collinear, and find the ratio in which B divides AC.

SOLUTION

$AB = (5-1)\hat{i} + (0+2)\hat{j} + (-2+8)\hat{k} = 4\hat{i} + 2\hat{j} + 6\hat{k} = 2(2\hat{i} + \hat{j} + 3\hat{k})$.

$BC = (11-5)\hat{i} + (3-0)\hat{j} + (7+2)\hat{k} = 6\hat{i} + 3\hat{j} + 9\hat{k} = 3(2\hat{i} + \hat{j} + 3\hat{k})$.

Since $AB = 2(2\hat{i} + \hat{j} + 3\hat{k})$ and $BC = 3(2\hat{i} + \hat{j} + 3\hat{k})$, both are scalar multiples of $(2\hat{i} + \hat{j} + 3\hat{k})$.

Therefore A, B, C are collinear.

B divides AC in ratio AB : BC = 2 : 3.

Answer: A, B, C are collinear; B divides AC in the ratio 2 : 3.

Q9 Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $(2\bar{a} + \bar{b})$ and $(\bar{a} - 3\bar{b})$ externally in the ratio 1 : 2. Also, show that P is the mid point of the line segment RQ.

SOLUTION

Let $p = 2\bar{a} + \bar{b}$ and $q = \bar{a} - 3\bar{b}$.

External division in ratio 1:2 means R divides PQ externally in ratio 1:2.

Formula for external division: $OR = (1 \cdot q - 2 \cdot p)/(1 - 2) = (q - 2p)/(-1) = 2p - q$.

OR = 2(2ā + b̄) − (ā − 3b̄) = 4ā + 2b̄ − ā + 3b̄ = 3ā + 5b̄.

Verification that P is the midpoint of RQ:
Midpoint of RQ = (OR + OQ)/2 = ((3ā + 5b̄) + (ā − 3b̄))/2 = (4ā + 2b̄)/2 = 2ā + b̄ = p = OP. ✓

Therefore P is the midpoint of RQ.

Answer: Position vector of R = 3ā + 5b̄. P is indeed the midpoint of RQ since (OR + OQ)/2 = OP.

Q10 The two adjacent sides of a parallelogram are 2î − 4ĵ + 5k̂ and î − 2ĵ − 3k̂. Find the unit vector parallel to its diagonal. Also, find its area.

SOLUTION

Let a = 2î − 4ĵ + 5k̂ and b = î − 2ĵ − 3k̂.

The two diagonals of the parallelogram are (a + b) and (a − b).

a + b = (2+1)î + (−4−2)ĵ + (5−3)k̂ = 3î − 6ĵ + 2k̂.
|a + b| = √(9 + 36 + 4) = √49 = 7.
Unit vector along diagonal (a + b): (3î − 6ĵ + 2k̂)/7 = (3/7)î − (6/7)ĵ + (2/7)k̂.

Area of parallelogram =	a × b	.
a × b =	î ĵ k̂	
	2 -4 5	
	1 -2 -3	

î: (−4)(−3) − (5)(−2) = 12 + 10 = 22.
ĵ: −((2)(−3) − (5)(1)) = −(−6 − 5) = 11.
k̂: (2)(−2) − (−4)(1) = −4 + 4 = 0.
a × b = 22î + 11ĵ + 0k̂ = 11(2î + ĵ).
|a × b| = 11√(4 + 1) = 11√5.

Answer: Unit vector along diagonal = (3/7)î − (6/7)ĵ + (2/7)k̂. Area = 11√5 sq. units.

Q11 Show that the direction cosines of a vector equally inclined to the axes OX, OY and OZ are ±(1/√3, 1/√3, 1/√3).

SOLUTION

If a vector is equally inclined to OX, OY, OZ, its direction cosines satisfy l = m = n.
Using the identity $l^2 + m^2 + n^2 = 1$:
$3l^2 = 1 \Rightarrow l^2 = 1/3 \Rightarrow l = \pm 1/\sqrt{3}$.
So l = m = n = ±1/√3.

The direction cosines are (1/√3, 1/√3, 1/√3) or (−1/√3, −1/√3, −1/√3),
which can be written as ±(1/√3, 1/√3, 1/√3).

Answer: Direction cosines are ±(1/√3, 1/√3, 1/√3). Proved.

Q12 Let $\bar{a} = \hat{i} + 4\hat{j} + 2\hat{k}$, $\bar{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\bar{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. Find a vector $\bar{d}$ which is perpendicular to both $\bar{b}$ and $\bar{c}$, and $\bar{a} \cdot \bar{d} = 15$.

SOLUTION

A vector perpendicular to both $\bar{b}$ and $\bar{c}$ is parallel to $\bar{b} \times \bar{c}$.

$\bar{b} \times \bar{c} =$	$\hat{i}$ $\hat{j}$ $\hat{k}$	
	3 -2 7	
	2 -1 4	

$\hat{i}$: (−2)(4) − (7)(−1) = −8 + 7 = −1.
$\hat{j}$: −((3)(4) − (7)(2)) = −(12 − 14) = 2.
$\hat{k}$: (3)(−1) − (−2)(2) = −3 + 4 = 1.

$\bar{b} \times \bar{c} = -\hat{i} + 2\hat{j} + \hat{k}$.

Let $\bar{d} = t(\bar{b} \times \bar{c}) = t(-\hat{i} + 2\hat{j} + \hat{k})$.
Condition: $\bar{a} \cdot \bar{d} = 15$.
$\bar{a} \cdot \bar{d} = t[(1)(-1) + (4)(2) + (2)(1)] = t[-1 + 8 + 2] = 9t = 15$.
$t = 15/9 = 5/3$.

$\bar{d} = (5/3)(-\hat{i} + 2\hat{j} + \hat{k}) = -(5/3)\hat{i} + (10/3)\hat{j} + (5/3)\hat{k}$.

Answer: $\bar{d} = -(5/3)\hat{i} + (10/3)\hat{j} + (5/3)\hat{k}$.

Q13 The scalar product of the vector $\hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of vectors $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to one. Find the value of λ.

SOLUTION

Sum of the two vectors: $(2+\lambda)\hat{i} + (4+2)\hat{j} + (-5+3)\hat{k} = (2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}$.

Let $v = (2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}$.

$|v| = \sqrt{((2+\lambda)^2 + 36 + 4)} = \sqrt{((2+\lambda)^2 + 40)}$.

Unit vector along v: $v/|v|$.

Scalar product with $(\hat{i} + \hat{j} + \hat{k})$:

$(\hat{i} + \hat{j} + \hat{k}) \cdot (v/|v|) = [(2+\lambda) + 6 + (-2)] / \sqrt{((2+\lambda)^2 + 40)} = (\lambda + 6)/\sqrt{((2+\lambda)^2 + 40)} = 1$.

Therefore: $\lambda + 6 = \sqrt{((2+\lambda)^2 + 40)}$.

Square both sides: $(\lambda+6)^2 = (2+\lambda)^2 + 40$.

$\lambda^2 + 12\lambda + 36 = \lambda^2 + 4\lambda + 4 + 40$.

$8\lambda = 8$.

$\lambda = 1$.

Answer: $\lambda = 1$.

Q14 If $\bar{a}$, $\bar{b}$, $\bar{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\bar{a} + \bar{b} + \bar{c}$ is equally inclined to $\bar{a}$, $\bar{b}$ and $\bar{c}$.

SOLUTION

Given: $\bar{a} \cdot \bar{b} = \bar{b} \cdot \bar{c} = \bar{c} \cdot \bar{a} = 0$ and $|\bar{a}| = |\bar{b}| = |\bar{c}| = k$ (say).

	$\bar{a} + \bar{b} + \bar{c}$	$^2 =$	$\bar{a}$	$^2 +$	$\bar{b}$	$^2 +$	$\bar{c}$	$^2 + 2(\bar{a}\cdot\bar{b} + \bar{b}\cdot\bar{c} + \bar{c}\cdot\bar{a}) = 3k^2 + 0 = 3k^2$.
So	$\bar{a} + \bar{b} + \bar{c}$	$= k\sqrt{3}$.						

Angle α with $\bar{a}$:

$\cos\alpha = (\bar{a} + \bar{b} + \bar{c}) \cdot \bar{a} / ($	$\bar{a} + \bar{b} + \bar{c}$	$\times$	$\bar{a}$	$)$
$= ($	$\bar{a}$	$^2 + \bar{a}\cdot\bar{b} + \bar{a}\cdot\bar{c}) / (k\sqrt{3} \times k)$		

$= k^2 / (k^2\sqrt{3}) = 1/\sqrt{3}$.

By symmetry, the angle with $\bar{b}$ and $\bar{c}$ also has cos = $1/\sqrt{3}$.

Since all three angles are equal (= $\arccos(1/\sqrt{3})$), the vector $\bar{a} + \bar{b} + \bar{c}$ is equally

inclined to ā, b̄ and c̄.

Answer: Angle of (ā + b̄ + c̄) with each of ā, b̄, c̄ equals arccos(1/√3). Proved.

Q15 Prove that (ā + b̄) · (ā + b̄) = |ā|² + |b̄|², if and only if ā, b̄ are perpendicular, given ā ≠ 0, b̄ ≠ 0.

SOLUTION

Expand the left side:

(ā + b̄) · (ā + b̄) = ā · ā + 2ā · b̄ + b̄ · b̄ = |ā|² + 2(ā · b̄) + |b̄|².

So: (ā + b̄) · (ā + b̄) =	ā	² +	b̄	²				
<=>	ā	² + 2(ā · b̄) +	b̄	² =	ā	² +	b̄	²

<=> 2(ā · b̄) = 0

<=> ā · b̄ = 0

<=> ā ⊥ b̄ (since ā ≠ 0 and b̄ ≠ 0).

Therefore the statement holds if and only if ā and b̄ are perpendicular.

Answer: (ā + b̄) · (ā + b̄) = |ā|² + |b̄|² iff ā · b̄ = 0 iff ā ⊥ b̄. Proved.

Q16 If θ is the angle between two vectors ā and b̄, then ā · b̄ ≥ 0 only when (A) 0 < θ < π/2 (B) 0 ≤ θ ≤ π/2 (C) 0 < θ < π (D) 0 ≤ θ ≤ π

SOLUTION

ā · b̄ =	ā		b̄	cos θ.
For ā · b̄ ≥ 0, we need cos θ ≥ 0 (since	ā	,	b̄	> 0 for nonzero vectors).

cos θ ≥ 0 when 0 ≤ θ ≤ π/2.

The correct option is (B).

Answer: (B) 0 ≤ θ ≤ π/2.

Q17 Let ā and b̄ be two unit vectors and θ is the angle between them. Then ā + b̄ is a unit vector if
(A) θ = π/4 (B) θ = π/3 (C) θ = π/2 (D) θ = 2π/3

SOLUTION

	ā + b̄	² =	ā	² + 2(ā · b̄) +	b̄	² = 1 + 2cos θ + 1 = 2 + 2cos θ.
For	ā + b̄	= 1 (unit vector):	ā + b̄	² = 1.		

2 + 2cos θ = 1 => 2cos θ = −1 => cos θ = −1/2 => θ = 2π/3.
The correct option is (D).

Answer: (D) θ = 2π/3.

Q18 The value of î · (ĵ × k̂) + ĵ · (î × k̂) + k̂ · (î × ĵ) is
(A) 0 (B) −1 (C) 1 (D) 3

SOLUTION

ĵ × k̂ = î, so î · (ĵ × k̂) = î · î = 1.
î × k̂ = −ĵ (since k̂ × î = ĵ, so î × k̂ = −k̂ × î = −ĵ), so ĵ · (î × k̂) = ĵ · (−ĵ) = −1.
î × ĵ = k̂, so k̂ · (î × ĵ) = k̂ · k̂ = 1.
Total = 1 + (−1) + 1 = 1.
The correct option is (C).

Answer: (C) 1.

Q19 If θ is the angle between any two vectors ā and b̄, then |ā · b̄| = |ā × b̄| when θ is equal to
(A) 0 (B) π/4 (C) π/2 (D) π

SOLUTION

	$\vec{a} \cdot \vec{b}$ =	$\vec{a}$	$\vec{b}$	$\cos\theta$ and $\vec{a} \times \vec{b}$ =	$\vec{a}$	$\vec{b}$	$\sin\theta$ (with $0 \le \theta \le \pi$, so $\sin\theta \ge 0$).
Setting them equal:	$\vec{a}$	$\vec{b}$	$\cos\theta$ =	$\vec{a}$	$\vec{b}$	$\sin\theta$.	
Dividing both sides by	$\vec{a}$	$\vec{b}$	(assuming both nonzero):	$\cos\theta$ =	$\sin\theta$.		
For $0 \le \theta \le \pi$: $\sin\theta \ge 0$, and	$\cos\theta$ =	$\sin\theta$ gives $\tan\theta = 1$ (when $\cos\theta > 0$) => $\theta = \pi/4$.					
(At $\theta = 0$ or π, $\sin\theta = 0$ and	$\cos\theta$ =	1, so they are not equal unless both vectors are zero.)					

The correct option is (B).

Answer: (B) $\theta = \pi/4$.