Chapter 1: Patterns in Mathematics

CONTENTS — 5 exercises · 17 questions

1.1	Patterns in Numbers	1 Q
1.2	Visualising Number Sequences	2 Q
1.3	Relations among Number Sequences	8 Q
1.4	Patterns in Shapes	1 Q
1.5	Relation to Number Sequences	5 Q

Exercise 1.1 — Patterns in Numbers

Q1 Rewrite each sequence of Table 1 in your notebook, along with the next three numbers in each sequence! After each sequence, write in your own words what is the rule for forming the numbers in the sequence.

Table 1 sequences:
1, 1, 1, 1, 1, 1, 1, ... (All 1's)
1, 2, 3, 4, 5, 6, 7, ... (Counting numbers)
1, 3, 5, 7, 9, 11, 13, ... (Odd numbers)
2, 4, 6, 8, 10, 12, 14, ... (Even numbers)
1, 3, 6, 10, 15, 21, 28, ... (Triangular numbers)
1, 4, 9, 16, 25, 36, 49, ... (Squares)
1, 8, 27, 64, 125, 216, ... (Cubes)
1, 2, 3, 5, 8, 13, 21, ... (Virahanka numbers)
1, 2, 4, 8, 16, 32, 64, ... (Powers of 2)
1, 3, 9, 27, 81, 243, 729, ... (Powers of 3)

SOLUTION

All 1's: 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, ... — Next three: 1, 1, 1. Rule: every term is 1.

Counting numbers: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, ... — Next three: 8, 9, 10. Rule: each term is 1 more than the previous.

Odd numbers: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, ... — Next three: 15, 17, 19. Rule: each term is 2 more than the previous.

Even numbers: 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, ... — Next three: 16, 18, 20. Rule: each term is 2 more than the previous.

Triangular numbers: 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, ... — Next three: 36, 45, 55. Rule: the n-th term is 1 + 2 + 3 + ... + n (add the next counting number each time).

Squares: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ... — Next three: 64, 81, 100. Rule: the n-th term is n x n.

Cubes: 1, 8, 27, 64, 125, 216, 343, 512, 729, ... — Next three: 343, 512, 729. Rule: the n-th term is n x n x n.

Virahanka numbers: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ... — Next three: 34, 55, 89. Rule: each term is the sum of the two preceding terms.

Powers of 2: 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, ... — Next three: 128, 256, 512. Rule: each term is double the previous.

Powers of 3: 1, 3, 9, 27, 81, 243, 729, 2187, 6561, 19683, ... — Next three: 2187, 6561, 19683. Rule: each term is three times the previous.

Answer: All 1's: 1,1,1 | Counting: 8,9,10 | Odd: 15,17,19 | Even: 16,18,20 | Triangular: 36,45,55 | Squares: 64,81,100 | Cubes: 343,512,729 | Virahanka: 34,55,89 | Powers of 2: 128,256,512 | Powers of 3: 2187,6561,19683

Exercise 1.2 — Visualising Number Sequences

Q1 Why are 1, 3, 6, 10, 15, … called triangular numbers? Why are 1, 4, 9, 16, 25, … called square numbers or squares? Why are 1, 8, 27, 64, 125, … called cubes?

SOLUTION

Triangular numbers (1, 3, 6, 10, 15, …): These numbers of dots can be arranged in the shape of an equilateral triangle. For example, 1 dot is a single point, 3 dots form a triangle of side 2, 6 dots form a triangle of side 3, and so on. That is why they are called triangular numbers.

Square numbers (1, 4, 9, 16, 25, …): These numbers of dots can be arranged in the shape of a square. For example, 1 dot is a 1×1 square, 4 dots form a 2×2 square, 9 dots form a 3×3 square, and so on. That is why they are called square numbers.

Cubes (1, 8, 27, 64, 125, …): These numbers of unit cubes can be stacked to form a larger cube. For example, 1 unit cube is a 1×1×1 cube, 8 form a 2×2×2 cube, 27 form a 3×3×3 cube, and so on. That is why they are called cubes.

Answer: Triangular numbers can be arranged as triangles of dots; square numbers as square grids of dots; cubes as cubic stacks of unit cubes.

Q2 What would you call the following sequence of numbers?
1, 7, 19, 37, ...
(The dots are arranged in hexagonal patterns as shown in the figures.)
What is the next number in the sequence?

SOLUTION

The sequence 1, 7, 19, 37, … is called hexagonal numbers, because the dots in each figure can be arranged in the shape of a regular hexagon.

To find the next number, observe the differences between consecutive terms:
7 - 1 = 6
19 - 7 = 12
37 - 19 = 18

The differences increase by 6 each time: 6, 12, 18, 24, …

Next difference = 24, so the next hexagonal number = 37 + 24 = 61.

Answer: They are called hexagonal numbers. The next number in the sequence is 61.

Exercise 1.3 — Relations among Number Sequences

Q1 Can you find a similar pictorial explanation for why adding counting numbers up and down, i.e., 1, 1 + 2 + 1, 1 + 2 + 3 + 2 + 1, …, gives square numbers?

SOLUTION

Yes. Arrange dots in a square grid and divide them along the two diagonals.

For a 1×1 square: 1 dot → 1
For a 2×2 square: 4 dots → the diagonal strips going up then down give 1 + 2 + 1 = 4
For a 3×3 square: 9 dots → diagonal strips give 1 + 2 + 3 + 2 + 1 = 9
For a 4×4 square: 16 dots → diagonal strips give 1 + 2 + 3 + 4 + 3 + 2 + 1 = 16

In each n×n square grid, counting dots along the diagonal strips from the corner inward and back out gives the sum 1 + 2 + 3 + … + n + … + 3 + 2 + 1 = n^2.
Because such a picture can be drawn for a square of any size, this explains why adding counting numbers up and then down always gives a square number.

Answer: Yes — in an n×n square grid, the diagonal strips (going up to n, then back down to 1) count exactly n^2 dots, so the sum 1 + 2 + … + n + … + 2 + 1 = n^2.

Q2 By imagining a large version of your picture, or drawing it partially, as needed, can you see what will be the value of 1 + 2 + 3 + ... + 99 + 100 + 99 + ... + 3 + 2 + 1?

SOLUTION

From the pictorial pattern established above, the sum 1 + 2 + 3 + … + n + … + 3 + 2 + 1 = n^2.

Here the sum goes up to 100 and back down, so n = 100.

Therefore: 1 + 2 + 3 + ... + 99 + 100 + 99 + ... + 3 + 2 + 1 = 100^2 = 10,000.

Answer: 10,000

Q3 Which sequence do you get when you start to add the All 1's sequence up? What sequence do you get when you add the All 1's sequence up and down?

SOLUTION

Adding up the All 1's sequence:

1 = 1

1 + 1 = 2

1 + 1 + 1 = 3

1 + 1 + 1 + 1 = 4

This gives: 1, 2, 3, 4, 5, … — the Counting numbers sequence.

Adding the All 1's sequence up and then down:

1 = 1

1 + (1+1) + 1 = 4

1 + (1+1) + (1+1+1) + (1+1) + 1 = 9

1 + (1+1) + (1+1+1) + (1+1+1+1) + (1+1+1) + (1+1) + 1 = 16

This gives: 1, 4, 9, 16, 25, … — the Square numbers sequence.

Answer: Adding All 1's up gives Counting numbers (1, 2, 3, 4, …). Adding All 1's up and down gives Square numbers (1, 4, 9, 16, …).

Q4 Which sequence do you get when you start to add the counting numbers up? Can you give a smaller pictorial explanation?

SOLUTION

Adding up the counting numbers:

1 = 1

1 + 2 = 3

1 + 2 + 3 = 6

1 + 2 + 3 + 4 = 10

1 + 2 + 3 + 4 + 5 = 15

This gives: 1, 3, 6, 10, 15, … — the Triangular numbers sequence.

Pictorial explanation: Each triangular number counts the dots in a triangular arrangement. When we add 1+2+3+...+n, we are counting dots in a triangle with n rows, which is exactly the n-th triangular number.

Answer: Adding counting numbers up gives Triangular numbers: 1, 3, 6, 10, 15, …

Q5 What happens when you add up pairs of consecutive triangular numbers? That is, take 1 + 3, 3 + 6, 6 + 10, 10 + 15, … Which sequence do you get? Why? Can you explain it with a picture?

SOLUTION

Computing consecutive triangular pairs:

$1 + 3 = 4 = 2^2$

$3 + 6 = 9 = 3^2$

$6 + 10 = 16 = 4^2$

$10 + 15 = 25 = 5^2$

$15 + 21 = 36 = 6^2$

$21 + 28 = 49 = 7^2$

We get: 4, 9, 16, 25, 36, 49, … — the Square numbers sequence (starting from 2^2).

Pictorial explanation: The n-th triangular number is a triangle with n rows. The (n+1)-th triangular number is a triangle with (n+1) rows. When you fit these two triangles together (one pointing up, one pointing down), they form a square of side (n+1), containing $(n+1)^2$ dots. This shows $T(n) + T(n+1) = (n+1)^2$.

Answer: 1+3=4, 3+6=9, 6+10=16, 10+15=25, … — we get Square numbers (4, 9, 16, 25, …).

Q6 What happens when you start to add up powers of 2 starting with 1, i.e., take 1, 1 + 2, 1 + 2 + 4, 1 + 2 + 4 + 8, … ? Now add 1 to each of these numbers — what numbers do you get? Why does this happen?

SOLUTION

Computing cumulative sums of powers of 2:

$1 = 1$

$1 + 2 = 3$

$1 + 2 + 4 = 7$

$1 + 2 + 4 + 8 = 15$

$1 + 2 + 4 + 8 + 16 = 31$

We get: 1, 3, 7, 15, 31, …

Now add 1 to each:

$1 + 1 = 2$

$3 + 1 = 4$

7 + 1 = 8
15 + 1 = 16
31 + 1 = 32
We get: 2, 4, 8, 16, 32, … — the Powers of 2 sequence (starting from 2).

Why? Each cumulative sum 1 + 2 + 4 + … + 2^(n-1) = 2^n - 1 (one less than the next power of 2). So adding 1 always gives the next power of 2.
Refer to the picture on page 6: each new strip added has exactly one more dot than all previous strips combined, so the total is always one less than the next power of 2.

Answer: Cumulative sums give 1, 3, 7, 15, 31, … Adding 1 to each gives Powers of 2: 2, 4, 8, 16, 32, …

Q7 What happens when you multiply the triangular numbers by 6 and add 1? Which sequence do you get? Can you explain it with a picture?

SOLUTION

Triangular numbers: 1, 3, 6, 10, 15, 21, …

Multiplying each by 6 and adding 1:
(1 × 6) + 1 = 7
(3 × 6) + 1 = 19
(6 × 6) + 1 = 37
(10 × 6) + 1 = 61
(15 × 6) + 1 = 91
(21 × 6) + 1 = 127

We get: 7, 19, 37, 61, 91, … — the Hexagonal numbers sequence.

Explanation: Each hexagonal number can be formed by taking 6 copies of the corresponding triangular arrangement of dots (arranged around a centre), plus the 1 centre dot. This gives 6 × T(n) + 1 dots, which is the (n+1)-th hexagonal number.

Answer: Multiplying triangular numbers by 6 and adding 1 gives Hexagonal numbers: 7, 19, 37, 61, 91, …

Q8 What happens when you start to add up hexagonal numbers, i.e., take 1, 1 + 7, 1 + 7 + 19, 1 + 7 + 19 + 37, … ? Which sequence do you get? Can you explain it using a picture of a cube?

SOLUTION

Computing cumulative sums of hexagonal numbers (1, 7, 19, 37, 61, …):

1 = 1

1 + 7 = 8

1 + 7 + 19 = 27

1 + 7 + 19 + 37 = 64

1 + 7 + 19 + 37 + 61 = 125

We get: 1, 8, 27, 64, 125, … — the Cube numbers sequence.

Pictorial explanation using a cube: Each hexagonal number counts the dots visible on the outer 'shell' of a cube when you look at it from a corner. Adding up these shells from the innermost (1 dot) outward builds up a larger and larger cube. So the n-th cube number equals the sum of the first n hexagonal numbers.

Answer: Cumulative sums of hexagonal numbers give Cube numbers: 1, 8, 27, 64, 125, …

Exercise 1.4 — Patterns in Shapes

Q1 Can you recognise the pattern in each of the sequences in Table 3?

SOLUTION

Regular Polygons: Triangle, Quadrilateral, Pentagon, Hexagon, Heptagon, Octagon, Nonagon, Decagon — the number of sides increases by 1 each time: 3, 4, 5, 6, 7, 8, 9, 10, …

Complete Graphs (K2, K3, K4, K5, K6): each shape connects every pair of dots; the number of lines follows the triangular number sequence: 1, 3, 6, 10, 15, …

Stacked Squares: each shape is a square grid; the number of little squares follows the square number sequence: 1, 4, 9, 16, 25, …

Stacked Triangles: each shape stacks rows of triangles; the number of little triangles follows the square number sequence: 1, 4, 9, 16, 25, …

Koch Snowflake: each shape is obtained by replacing every line segment with a 'speed bump'. The number of segments follows 3 times Powers of 4: 3, 12, 48, 192, …

Answer: Regular Polygons: sides 3,4,5,6,… (counting numbers from 3). Complete Graphs: 1,3,6,10,15 (triangular). Stacked Squares and Stacked Triangles: 1,4,9,16,25 (squares). Koch Snowflake: 3,12,48,192 (3 × powers of 4).

Exercise 1.5 — Relation to Number Sequences

Q1 Count the number of sides in each shape in the sequence of Regular Polygons. Which number sequence do you get? What about the number of corners in each shape in the sequence of Regular Polygons? Do you get the same number sequence? Can you explain why this happens?

SOLUTION

Counting sides of each Regular Polygon:
Triangle: 3 sides
Quadrilateral: 4 sides
Pentagon: 5 sides
Hexagon: 6 sides
Heptagon: 7 sides
Octagon: 8 sides
Nonagon: 9 sides
Decagon: 10 sides
Sequence of sides: 3, 4, 5, 6, 7, 8, 9, 10, … — counting numbers starting at 3.

Counting corners of each Regular Polygon:
Triangle: 3 corners, Quadrilateral: 4, Pentagon: 5, Hexagon: 6, …
Sequence of corners: 3, 4, 5, 6, 7, 8, 9, 10, … — same sequence!

Why? In any closed (polygon) figure, each side ends at a corner and each corner is where two sides meet. So the number of sides always equals the number of corners (vertices).

Answer: Number of sides: 3, 4, 5, 6, 7, 8, 9, 10 (counting numbers from 3). Number of corners: same sequence. They are equal because in any polygon, sides and corners always match one-to-one.

Q2 Count the number of lines in each shape in the sequence of Complete Graphs. Which number sequence do you get? Can you explain why?

SOLUTION

Counting lines in each Complete Graph:
K2 (2 dots): 1 line
K3 (3 dots): 3 lines
K4 (4 dots): 6 lines

K5 (5 dots): 10 lines
K6 (6 dots): 15 lines

Sequence: 1, 3, 6, 10, 15, … — the Triangular numbers sequence.

Why? In K_n (n dots), every dot is connected to every other dot. The number of pairs of dots is n × (n-1) ÷ 2, which is the (n-1)-th triangular number. For example, K3: 3×2÷2 = 3, K4: 4×3÷2 = 6, K5: 5×4÷2 = 10, K6: 6×5÷2 = 15.

Answer: Number of lines: 1, 3, 6, 10, 15 — Triangular numbers. This is because K_n has n×(n-1)÷2 pairs of dots, each pair joined by one line.

Q3 How many little squares are there in each shape of the sequence of Stacked Squares? Which number sequence does this give? Can you explain why?

SOLUTION

Counting little squares in each Stacked Square shape:
Shape 1 (1×1 grid): 1 little square
Shape 2 (2×2 grid): 4 little squares
Shape 3 (3×3 grid): 9 little squares
Shape 4 (4×4 grid): 16 little squares
Shape 5 (5×5 grid): 25 little squares

Sequence: 1, 4, 9, 16, 25, … — the Square numbers sequence.

Why? The n-th shape is an n×n grid of little squares, which contains n × n = n^2 squares. This is exactly what square numbers represent.

Answer: 1, 4, 9, 16, 25 — Square numbers. The n-th shape is an n×n grid, containing n^2 little squares.

Q4 How many little triangles are there in each shape of the sequence of Stacked Triangles? Which number sequence does this give? Can you explain why? (Hint: In each shape in the sequence, how many triangles are there in each row?)

SOLUTION

Counting little triangles in each Stacked Triangles shape:
Shape 1 (1 row): 1 triangle
Shape 2 (2 rows): Row 1 has 1 triangle, Row 2 has 3 triangles (2 pointing up + 1 pointing down) → total 4
Shape 3 (3 rows): Rows have 1, 3, 5 triangles → total 9

Shape 4 (4 rows): Rows have 1, 3, 5, 7 triangles → total 16
Shape 5 (5 rows): Rows have 1, 3, 5, 7, 9 triangles → total 25

Sequence: 1, 4, 9, 16, 25, … — the Square numbers sequence.

Why? The n-th shape has n rows containing 1, 3, 5, …, (2n-1) triangles respectively. Their sum = 1 + 3 + 5 + … + (2n-1) = n^2 (sum of first n odd numbers). This is also the same as adding counting numbers up and down: 1, 1+2+1, 1+2+3+2+1, …

Answer: 1, 4, 9, 16, 25 — Square numbers. In the n-th shape, each row k has (2k-1) triangles, and 1+3+5+…+(2n-1) = n^2.

Q5 To get from one shape to the next shape in the Koch Snowflake sequence, one replaces each line segment '—' by a 'speed bump'. As one does this more and more times, the changes become tinier and tinier with very very small line segments. How many total line segments are there in each shape of the Koch Snowflake? What is the corresponding number sequence? (The answer is 3, 12, 48, ..., i.e., 3 times Powers of 4; this sequence is not shown in Table 1.)

SOLUTION

Starting shape: an equilateral triangle with 3 line segments.

Each step replaces every line segment '—' with a speed bump (4 smaller segments).
So each step multiplies the number of segments by 4.

Shape 1: 3 segments
Shape 2: 3 × 4 = 12 segments
Shape 3: 12 × 4 = 48 segments
Shape 4: 48 × 4 = 192 segments
Shape 5: 192 × 4 = 768 segments

Corresponding sequence: 3, 12, 48, 192, 768, …
This equals 3 × 1, 3 × 4, 3 × 16, 3 × 64, 3 × 256, … = 3 × (Powers of 4).

Answer: Total line segments: 3, 12, 48, 192, 768 — this is 3 times the Powers of 4 (3×1, 3×4, 3×16, 3×64, …).