Chapter 1: A Square and A Cube

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Exercise 1.1 — Square Numbers and Square Roots

Q1 Which of the following numbers are not perfect squares?
(i) 2032 (ii) 2048 (iii) 1027 (iv) 1089

SOLUTION

A perfect square ends only in 0, 1, 4, 5, 6, or 9 in the units place. Using prime factorisation to confirm:

(i) $2032 = 2^4 \times 127$. The factor 127 appears only once (odd exponent), so the prime factors cannot be split into two equal groups. 2032 is NOT a perfect square.

(ii) $2048 = 2^{11}$. The exponent 11 is odd, so the factors cannot be split into two equal groups. 2048 is NOT a perfect square.

(iii) $1027 = 13 \times 79$. Both primes appear only once (odd exponents), so 1027 is NOT a perfect square.

(iv) $1089 = 3^2 \times 11^2$. All prime factors have even exponents, so $1089 = (3 \times 11)^2 = 33^2$. 1089 IS a perfect square.

Therefore, (i) 2032, (ii) 2048, and (iii) 1027 are not perfect squares.

Answer: (i) 2032, (ii) 2048, and (iii) 1027 are not perfect squares.

Q2 Which one among 64^2, 108^2, 292^2, 36^2 has last digit 4?

SOLUTION

The last digit of a square depends only on the last digit of the original number.

64^2 : last digit of 64 is 4. $4 \times 4 = 16$, so last digit of 64^2 is 6.
108^2: last digit of 108 is 8. $8 \times 8 = 64$, so last digit of 108^2 is 4.
292^2: last digit of 292 is 2. $2 \times 2 = 4$, so last digit of 292^2 is 4.
36^2 : last digit of 36 is 6. $6 \times 6 = 36$, so last digit of 36^2 is 6.

Verify: $108^2 = 11664$ (last digit 4 ✓), $292^2 = 85264$ (last digit 4 ✓).

Answer: 108^2 and 292^2 have last digit 4.

Q3 Given $125^2 = 15625$, what is the value of 126^2?
(i) 15625 + 126 (ii) 15625 + 26^2 (iii) 15625 + 253 (iv) 15625 + 251 (v) 15625 + 51^2

SOLUTION

Use the identity: $(n+1)^2 = n^2 + 2n + 1$, which can also be written as $(n+1)^2 = n^2 + n + (n+1)$.

Here n = 125, so: $126^2 = 125^2 + 125 + 126 = 15625 + 251$.

Verify: $126^2 = 15876$ and $15625 + 251 = 15876$ ✓.

Check other options: $15625 + 126 = 15751 \neq 15876$; $15625 + 26^2 = 15625 + 676 = 16301 \neq 15876$; $15625 + 253 = 15878 \neq 15876$; $15625 + 51^2 = 15625 + 2601 = 18226 \neq 15876$.

Option (iv) is correct.

Answer: (iv) $15625 + 251 = 15876 = 126^2$

Q4 Find the length of the side of a square whose area is 441 m^2.

SOLUTION

Side of a square = $\sqrt{}$(area).

Prime factorise 441: $441 = 3 \times 147 = 3 \times 3 \times 49 = 3^2 \times 7^2$.

Group into two equal sets: $441 = (3 \times 7) \times (3 \times 7) = 21 \times 21 = 21^2$.

Therefore $\sqrt{441} = 21$.

Answer: The side of the square is 21 m.

Q5 Find the smallest square number that is divisible by each of the following numbers: 4, 9, and 10.

SOLUTION

First find the LCM of 4, 9, and 10.

$4 = 2^2$, $9 = 3^2$, $10 = 2 \times 5$.
LCM = $2^2 \times 3^2 \times 5 = 180$.

Prime factorisation of 180: $2^2 \times 3^2 \times 5^1$.

For 180 to be a perfect square, every prime factor must have an even exponent. The factor 5 has exponent 1 (odd), so we must multiply by 5.

Smallest perfect square = 180 × 5 = 900.

Verify: $900 = 2^2 \times 3^2 \times 5^2 = (2 \times 3 \times 5)^2 = 30^2$. Also 900 ÷ 4 = 225 ✓, 900 ÷ 9 = 100 ✓, 900 ÷ 10 = 90 ✓.

Answer: 900 (= 30^2) is the smallest perfect square divisible by 4, 9, and 10.

Q6 Find the smallest number by which 9408 must be multiplied so that the product is a perfect square. Find the square root of the product.

SOLUTION

Prime factorise 9408:
9408 ÷ 2 = 4704, ÷ 2 = 2352, ÷ 2 = 1176, ÷ 2 = 588, ÷ 2 = 294, ÷ 2 = 147, ÷ 3 = 49, ÷ 7 = 7, ÷ 7 = 1.
$9408 = 2^6 \times 3^1 \times 7^2$.

For a perfect square, all exponents must be even. The exponent of 3 is 1 (odd), so we need one more factor of 3.

Multiply by 3: 9408 × 3 = 28224.
$28224 = 2^6 \times 3^2 \times 7^2 = (2^3 \times 3 \times 7)^2 = (8 \times 3 \times 7)^2 = 168^2$.

Verify: $168^2 = 28224$ ✓.

Answer: Multiply by 3. The product is 28224 and its square root is 168.

Q7 How many numbers lie between the squares of the following numbers?
(i) 16 and 17 (ii) 99 and 100

SOLUTION

The number of integers strictly between n^2 and $(n+1)^2$ equals $(n+1)^2 - n^2 - 1 = 2n + 1 - 1 = 2n$.

(i) Between 16^2 and 17^2: n = 16, so count = 2 × 16 = 32.
Verify: $16^2 = 256$, $17^2 = 289$. Integers from 257 to 288 = 289 - 256 - 1 = 32 ✓.

(ii) Between 99^2 and 100^2: n = 99, so count = 2 × 99 = 198.
Verify: $99^2 = 9801$, $100^2 = 10000$. Integers from 9802 to 9999 = 10000 - 9801 - 1

= 198 ✓.

Answer: (i) 32 numbers lie between 16^2 and 17^2. (ii) 198 numbers lie between 99^2 and 100^2.

Q8 In the following pattern, fill in the missing numbers:

$1^2 + 2^2 + 2^2 = 3^2$

$2^2 + 3^2 + 6^2 = 7^2$

$3^2 + 4^2 + 12^2 = 13^2$

$4^2 + 5^2 + 20^2 = (___)^2$

$9^2 + 10^2 + (___)^2 = (___)^2$

SOLUTION

Observe the pattern in each row: $n^2 + (n+1)^2 + [n(n+1)]^2 = [n^2 + n + 1]^2$.

Check: n=1: $1^2 + 2^2 + (1\times2)^2 = 1 + 4 + 4 = 9 = 3^2$; and $1^2 + 1 + 1 = 3$ ✓.
n=2: $2^2 + 3^2 + (2\times3)^2 = 4 + 9 + 36 = 49 = 7^2$; and $2^2 + 2 + 1 = 7$ ✓.
n=3: $3^2 + 4^2 + (3\times4)^2 = 9 + 16 + 144 = 169 = 13^2$; and $3^2 + 3 + 1 = 13$ ✓.

Row 4 (n = 4):
Third term = 4 × 5 = 20 (already given). Right-hand side = $4^2 + 4 + 1 = 21$.
$4^2 + 5^2 + 20^2 = 16 + 25 + 400 = 441 = 21^2$ ✓.

Row 5 (n = 9):
Third term = 9 × 10 = 90. Right-hand side = $9^2 + 9 + 1 = 91$.
$9^2 + 10^2 + 90^2 = 81 + 100 + 8100 = 8281 = 91^2$ ✓.

Answer: $4^2 + 5^2 + 20^2 = (21)^2$; and $9^2 + 10^2 + (90)^2 = (91)^2$.

Q9 How many tiny squares are there in the following picture? Write the prime factorisation of the number of tiny squares.

SOLUTION

The picture shows a large square arrangement of motifs. Counting the grid:
The motifs are arranged in 9 columns × 9 rows = 81 large motifs.
Each large motif contains a 4 × 4 grid of tiny squares = 16 tiny squares.

Total tiny squares = 81 × 16 = 1296.

Prime factorise 1296:
1296 ÷ 2 = 648, ÷ 2 = 324, ÷ 2 = 162, ÷ 2 = 81.

81 ÷ 3 = 27, ÷ 3 = 9, ÷ 3 = 3, ÷ 3 = 1.

1296 = $2^4 \times 3^4$.

Note: 1296 = $(2^2 \times 3^2)^2 = 36^2 = 6^4$, connecting beautifully to the chapter theme.

Answer: 1296 tiny squares. Prime factorisation: 1296 = $2^4 \times 3^4$.

Exercise 1.2 — Cubic Numbers and Cube Roots

Q1 Find the cube roots of 27000 and 10648.

SOLUTION

Method: prime factorise, then split factors into three identical groups.

Cube root of 27000:
$27000 = 27 \times 1000 = 3^3 \times 10^3 = 3^3 \times (2 \times 5)^3 = 2^3 \times 3^3 \times 5^3$.
Group into three identical sets: $(2 \times 3 \times 5) \times (2 \times 3 \times 5) \times (2 \times 3 \times 5) = 30^3$.
Therefore $\sqrt[3]{27000} = 30$.
Verify: $30^3 = 30 \times 30 \times 30 = 27000$ ✓.

Cube root of 10648:
$10648 \div 2 = 5324, \div 2 = 2662, \div 2 = 1331$.
$1331 \div 11 = 121, \div 11 = 11, \div 11 = 1$.
$10648 = 2^3 \times 11^3$.
Group into three identical sets: $(2 \times 11) \times (2 \times 11) \times (2 \times 11) = 22^3$.
Therefore $\sqrt[3]{10648} = 22$.
Verify: $22^3 = 22 \times 22 \times 22 = 10648$ ✓.

Answer: $\sqrt[3]{27000} = 30$ and $\sqrt[3]{10648} = 22$.

Q2 What number will you multiply by 1323 to make it a cube number?

SOLUTION

Prime factorise 1323:
$1323 \div 3 = 441, \div 3 = 147, \div 3 = 49$.
$49 \div 7 = 7, \div 7 = 1$.
$1323 = 3^3 \times 7^2$.

For a perfect cube, every prime factor must appear in groups of 3. The prime 3 already has exponent 3 (complete). The prime 7 has exponent 2 (needs one more 7 to complete the triplet).

Multiply by 7: $1323 \times 7 = 9261 = 3^3 \times 7^3 = (3 \times 7)^3 = 21^3$.
Verify: $21^3 = 21 \times 21 \times 21 = 9261$ ✓.

Answer: Multiply by 7. The result is $9261 = 21^3$.

Q3 State true or false. Explain your reasoning.
(i) The cube of any odd number is even.
(ii) There is no perfect cube that ends with 8.
(iii) The cube of a 2-digit number may be a 3-digit number.
(iv) The cube of a 2-digit number may have seven or more digits.
(v) Cube numbers have an odd number of factors.

SOLUTION

(i) FALSE. The cube of an odd number is always odd. If n is odd then n × n is odd (odd × odd = odd), and n × n × n is again odd. Example: $3^3 = 27$ (odd), $5^3 = 125$ (odd).

(ii) FALSE. $2^3 = 8$ ends with 8. Also $12^3 = 1728$ ends with 8. So perfect cubes can end with 8.

(iii) FALSE. The smallest 2-digit number is 10, and $10^3 = 1000$ (4 digits, not 3). Since 10^3 already has 4 digits, no 2-digit number can have a 3-digit cube.

(iv) FALSE. The largest 2-digit number is 99, and $99^3 = 970299$ (6 digits). The cube of any 2-digit number has at most 6 digits, never 7 or more.

(v) FALSE. Perfect squares (not cubes) are the numbers with an odd number of factors. For example, $8 = 2^3$ has factors {1, 2, 4, 8} — that is 4 factors (even). $27 = 3^3$ has factors {1, 3, 9, 27} — also 4 factors (even). Cube numbers do not generally have an odd number of factors.

Answer: (i) False (ii) False (iii) False (iv) False (v) False

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Q4 You are told that 1331 is a perfect cube. Can you guess without factorisation what its cube root is? Similarly, guess the cube roots of 4913, 12167, and 32768.

SOLUTION

Use the digit-matching method. First, note the units digit of cubes:
$1^3=1$, $2^3=8$, $3^3=7(27)$, $4^3=4(64)$, $5^3=5(125)$, $6^3=6(216)$, $7^3=3(343)$, $8^3=2(512)$, $9^3=9(729)$, $0^3=0$.

$\sqrt[3]{1331}$:
- Units digit of 1331 is 1 => units digit of cube root is 1.
- Remaining part (ignore last 3 digits): 1. Since $1^3 = 1$, tens digit is 1.

=> Cube root = 11. Verify: $11^3 = 1331$ ✓.

$\sqrt[3]{4913}$:

- Units digit of 4913 is 3 => units digit of cube root is 7 (since $7^3 = 343$ ends in 3).
- Remaining part (ignore last 3 digits): 4. Since $1^3 = 1 \le 4 < 8 = 2^3$, tens digit is 1.

=> Cube root = 17. Verify: $17^3 = 4913$ ✓.

$\sqrt[3]{12167}$:

- Units digit of 12167 is 7 => units digit of cube root is 3 (since $3^3 = 27$ ends in 7).
- Remaining part (ignore last 3 digits): 12. Since $2^3 = 8 \le 12 < 27 = 3^3$, tens digit is 2.

=> Cube root = 23. Verify: $23^3 = 12167$ ✓.

$\sqrt[3]{32768}$:

- Units digit of 32768 is 8 => units digit of cube root is 2 (since $2^3 = 8$ ends in 8).
- Remaining part (ignore last 3 digits): 32. Since $3^3 = 27 \le 32 < 64 = 4^3$, tens digit is 3.

=> Cube root = 32. Verify: $32^3 = 32768$ ✓.

Answer: $\sqrt[3]{1331} = 11$, $\sqrt[3]{4913} = 17$, $\sqrt[3]{12167} = 23$, $\sqrt[3]{32768} = 32$.

Q5 Which of the following is the greatest? Explain your reasoning.
(i) $67^3 - 66^3$ (ii) $43^3 - 42^3$ (iii) $67^2 - 66^2$ (iv) $43^2 - 42^2$

SOLUTION

Use the identities for consecutive integers:
$a^3 - (a-1)^3 = 3a^2 - 3a + 1$ (difference of consecutive cubes)
$a^2 - (a-1)^2 = 2a - 1$ (difference of consecutive squares)

(i) $67^3 - 66^3 = 3(67^2) - 3(67) + 1 = 3 \times 4489 - 201 + 1 = 13467 - 200 = 13267$.
(ii) $43^3 - 42^3 = 3(43^2) - 3(43) + 1 = 3 \times 1849 - 129 + 1 = 5547 - 128 = 5419$.
(iii) $67^2 - 66^2 = 2(67) - 1 = 134 - 1 = 133$.
(iv) $43^2 - 42^2 = 2(43) - 1 = 86 - 1 = 85$.

Comparing: 13267 > 5419 > 133 > 85.

Reasoning: Differences of cubes grow as $3a^2$, while differences of squares grow only as $2a$. So for large values of a, cube differences greatly exceed square differences.

Answer: (i) $67^3 - 66^3 = 13267$ is the greatest.