Chapter 9: The Baudhayana-Pythagoras Theorem

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Exercise 9.1 — Hypotenuse of an Isosceles Right Triangle

Q1 Earlier, we saw a method to create a square with double the area of a given square paper. There is another method to do this in which two identical square papers are cut diagonally into two triangles each (pieces 1, 2, 3, 4). Can you arrange these pieces to create a square with double the area of either square?

SOLUTION

Each square is cut along one diagonal, giving two right triangles per square — four triangles in total.

Arrange the four triangles so that their hypotenuses (the diagonals of the original squares) form the sides of a new square.

Place the four triangles with their right-angle vertices pointing outward and hypotenuses facing inward. The four hypotenuses together form a square whose side equals the diagonal of the original square.

Area of new square = 4 × (area of one triangle) = 4 × (1/2 × area of original square) = 2 × area of original square.

Thus the new arrangement forms a square with double the area of either original square.

Answer: Arrange the four triangles with their hypotenuses forming the sides of a new tilted square; this square has double the area of either original square.

Q2 The length of the two equal sides of an isosceles right triangle is given. Find the length of the hypotenuse. Find bounds on the length of the hypotenuse such that they have at least one digit after the decimal point.
(i) 3 (ii) 4 (iii) 6 (iv) 8 (v) 9

SOLUTION

For an isosceles right triangle with equal sides of length a, the hypotenuse c satisfies $c^2 = a^2 + a^2 = 2a^2$.

(i) a = 3:
$c^2 = 2 \times 3^2 = 2 \times 9 = 18$
$c = \sqrt{18}$
Check: $4.2^2 = 17.64$ and $4.3^2 = 18.49$
So $4.2 < \sqrt{18} < 4.3$
The hypotenuse is $\sqrt{18}$ units, between 4.2 and 4.3.

(ii) a = 4:
$c^2 = 2 \times 4^2 = 2 \times 16 = 32$
$c = \sqrt{32}$
Check: $5.6^2 = 31.36$ and $5.7^2 = 32.49$
So $5.6 < \sqrt{32} < 5.7$
The hypotenuse is $\sqrt{32}$ units, between 5.6 and 5.7.

(iii) a = 6:
$c^2 = 2 \times 6^2 = 2 \times 36 = 72$
$c = \sqrt{72}$
Check: $8.4^2 = 70.56$ and $8.5^2 = 72.25$
So $8.4 < \sqrt{72} < 8.5$
The hypotenuse is $\sqrt{72}$ units, between 8.4 and 8.5.

(iv) a = 8:
$c^2 = 2 \times 8^2 = 2 \times 64 = 128$
$c = \sqrt{128}$
Check: $11.3^2 = 127.69$ and $11.4^2 = 129.96$
So $11.3 < \sqrt{128} < 11.4$
The hypotenuse is $\sqrt{128}$ units, between 11.3 and 11.4.

(v) a = 9:
$c^2 = 2 \times 9^2 = 2 \times 81 = 162$
$c = \sqrt{162}$

Check: $12.7^2 = 161.29$ and $12.8^2 = 163.84$

So $12.7 < \sqrt{162} < 12.8$

The hypotenuse is $\sqrt{162}$ units, between 12.7 and 12.8.

Answer: (i) $\sqrt{18}$, between 4.2 and 4.3 (ii) $\sqrt{32}$, between 5.6 and 5.7 (iii) $\sqrt{72}$, between 8.4 and 8.5 (iv) $\sqrt{128}$, between 11.3 and 11.4 (v) $\sqrt{162}$, between 12.7 and 12.8

Q3 The hypotenuse of an isosceles right triangle is 10. What are its other two sidelengths? [Hint: Find the area of the square composed of two such right triangles.]

SOLUTION

Let each equal side have length a and the hypotenuse be c = 10.

Two copies of this isosceles right triangle together form a square of sidelength a. The square on the hypotenuse has area equal to twice the area of that square:

$c^2 = 2a^2$

Substituting c = 10:

$10^2 = 2a^2$

$100 = 2a^2$

$a^2 = 100/2 = 50$

$a = \sqrt{50} = 5\sqrt{2}$

Verify: $a^2 + a^2 = 50 + 50 = 100 = 10^2$. ✓

Answer: Each equal side has length $5\sqrt{2}$ (approximately 7.07 units).

Exercise 9.2 — Using Baudhayana's Theorem

Q1 If a right-angled triangle has shorter sides of lengths 5 cm and 12 cm, then what is the length of its hypotenuse? First draw the right-angled triangle with these sidelengths and measure the hypotenuse, then check your answer using Baudhayana's Theorem.

SOLUTION

Let the two shorter sides be a = 5 cm and b = 12 cm, and the hypotenuse be c.

By Baudhayana's Theorem: $a^2 + b^2 = c^2$

$5^2 + 12^2 = c^2$

$25 + 144 = c^2$

$169 = c^2$

$c = \sqrt{169} = 13$ cm

Answer: The hypotenuse is 13 cm.

Q2 If a right-angled triangle has a short side of length 8 cm and hypotenuse of length 17 cm, what is the length of the third side? Again, try drawing the triangle and measuring, and then check your answer using Baudhayana's Theorem.

SOLUTION

Let the known short side be a = 8 cm, the hypotenuse c = 17 cm, and the unknown side be b.

By Baudhayana's Theorem: $a^2 + b^2 = c^2$

$8^2 + b^2 = 17^2$

$64 + b^2 = 289$

$b^2 = 289 - 64 = 225$

$b = \sqrt{225} = 15$ cm

Answer: The third side is 15 cm.

Q3 Using the constructions you have now seen, how would you construct a square whose area is triple the area of a given square? Five times the area of a given square? (Baudhayana's Sulba-Sutra, Verse 1.10)

SOLUTION

Let the given square have sidelength s, so area = s^2.

Triple the area (area = $3s^2$):
We need a square of area $3s^2$.
Observe: $s^2 + (\sqrt{2} \cdot s)^2 = s^2 + 2s^2 = 3s^2$.
So draw a right triangle with legs s and the diagonal of the original square (length $s\sqrt{2}$).
The hypotenuse of this right triangle has length $s\sqrt{3}$, and the square on this hypotenuse has area $3s^2$.

Five times the area (area = $5s^2$):
We need a square of area $5s^2$.
Observe: $s^2 + (2s)^2 = s^2 + 4s^2 = 5s^2$.
So draw a right triangle with legs s and 2s.
The hypotenuse of this right triangle has length $s\sqrt{5}$, and the square on this hypotenuse has area $5s^2$.

Answer: Triple: draw a right triangle with legs s and $s\sqrt{2}$; the square on the hypotenuse has area $3s^2$. Five times: draw a right triangle with legs s and 2s; the square on the hypotenuse has area $5s^2$.

Q4 Let a, b and c denote the length of the sides of a right triangle, with c being the length of the hypotenuse. Find the missing sidelength in each of the following cases: (i) a = 5, b = 7 (ii) a = 8, b = 12 (iii) a = 9, c = 15 (iv) a = 7, b = 12 (v) a = 1.5, b = 3.5

SOLUTION

Using Baudhayana's Theorem: $a^2 + b^2 = c^2$.

(i) a = 5, b = 7, find c:
$c^2 = 5^2 + 7^2 = 25 + 49 = 74$
$c = \sqrt{74}$

(ii) a = 8, b = 12, find c:
$c^2 = 8^2 + 12^2 = 64 + 144 = 208 = 16 \times 13$
$c = \sqrt{208} = 4\sqrt{13}$

(iii) a = 9, c = 15, find b:

$b^2 = c^2 - a^2 = 15^2 - 9^2 = 225 - 81 = 144$

$b = \sqrt{144} = 12$

(iv) a = 7, b = 12, find c:

$c^2 = 7^2 + 12^2 = 49 + 144 = 193$

$c = \sqrt{193}$

(v) a = 1.5, b = 3.5, find c:

$c^2 = 1.5^2 + 3.5^2 = 2.25 + 12.25 = 14.5$

$c = \sqrt{14.5} = \sqrt{(29/2)} = \sqrt{58} / 2$

Answer: (i) $c = \sqrt{74}$ (ii) $c = 4\sqrt{13}$ (iii) $b = 12$ (iv) $c = \sqrt{193}$ (v) $c = \sqrt{(29/2)}$

Exercise 9.3 — Right-Triangles Having Integer Sidelengths

Q1 Find 5 more Baudhayana triples using this idea. [The method: if the nth odd number (2n - 1) is a perfect square, then $(n - 1)^2 + (2n - 1) = n^2$ gives a Baudhayana triple.]

SOLUTION

The method: if the nth odd number 2n - 1 is itself a perfect square, then $(n-1)^2 + (2n-1) = n^2$, giving the Baudhayana triple $(\sqrt{(2n-1)}, n-1, n)$.

The book has already shown the cases 9 (n=5) and 25 (n=13). We find five more:

Case 1: $49 = 7^2$ is an odd square. $49 = 2 \times 25 - 1$, so n = 25.
$(25 - 1)^2 + 49 = 25^2$
$24^2 + 7^2 = 25^2$ => triple (7, 24, 25)
Verify: $576 + 49 = 625 = 25^2$. ✓

Case 2: $81 = 9^2$ is an odd square. $81 = 2 \times 41 - 1$, so n = 41.
$(41 - 1)^2 + 81 = 41^2$
$40^2 + 9^2 = 41^2$ => triple (9, 40, 41)
Verify: $1600 + 81 = 1681 = 41^2$. ✓

Case 3: $121 = 11^2$ is an odd square. $121 = 2 \times 61 - 1$, so n = 61.
$(61 - 1)^2 + 121 = 61^2$
$60^2 + 11^2 = 61^2$ => triple (11, 60, 61)
Verify: $3600 + 121 = 3721 = 61^2$. ✓

Case 4: $169 = 13^2$ is an odd square. $169 = 2 \times 85 - 1$, so n = 85.
$(85 - 1)^2 + 169 = 85^2$
$84^2 + 13^2 = 85^2$ => triple (13, 84, 85)
Verify: $7056 + 169 = 7225 = 85^2$. ✓

Case 5: $225 = 15^2$ is an odd square. $225 = 2 \times 113 - 1$, so n = 113.
$(113 - 1)^2 + 225 = 113^2$
$112^2 + 15^2 = 113^2$ => triple (15, 112, 113)
Verify: $12544 + 225 = 12769 = 113^2$. ✓

Answer: (7, 24, 25), (9, 40, 41), (11, 60, 61), (13, 84, 85), (15, 112, 113)

Q2 Does this method yield non-primitive Baudhayana triples? [Hint: Observe that among the triples generated, one of the smaller sidelengths is one less than the hypotenuse.]

SOLUTION

In every triple generated by this method, the two largest numbers are n-1 and n (consecutive integers).
For example: (3,4,5), (5,12,13), (7,24,25), (9,40,41), ...

In each case, the hypotenuse c = n and the larger leg b = n - 1 = c - 1.

Since b and c = b + 1 are consecutive integers, gcd(b, c) = 1.
Also, the smallest side is an odd number (the square root of the odd square used), and gcd(odd square root, n-1) needs checking.

For (3,4,5): gcd(3,4,5) = 1 — primitive.
For (5,12,13): gcd(5,12,13) = 1 — primitive.
For (7,24,25): gcd(7,24,25) = 1 — primitive.

In general, since c and c-1 are consecutive, they share no common factor. The smallest side is always odd, and one can verify it shares no common factor with the others.

Therefore, this method always yields primitive Baudhayana triples — it never produces non-primitive ones.

Answer: No. This method always produces primitive Baudhayana triples; the two larger numbers in each triple are consecutive integers and therefore share no common factor.

Q3 Are there primitive triples that cannot be obtained through this method? If yes, give examples.

SOLUTION

Yes. This method only produces triples where the hypotenuse exceeds the larger leg by exactly 1.

Consider the primitive triple (8, 15, 17):
Verify: $8^2 + 15^2 = 64 + 225 = 289 = 17^2$. ✓
gcd(8, 15, 17) = 1, so it is primitive.
But 17 - 15 = 2, not 1, so it is NOT generated by this method.

Another example: (20, 21, 29):

$20^2 + 21^2 = 400 + 441 = 841 = 29^2$. ✓

gcd(20, 21, 29) = 1 — primitive.

29 - 21 = 8, not 1, so it cannot come from this method.

Thus there are infinitely many primitive Baudhayana triples not reachable by this odd-square method.

Answer: Yes. For example, (8, 15, 17) is primitive ($8^2 + 15^2 = 17^2$) but the hypotenuse exceeds the larger leg by 2, so it cannot be generated by this method. Similarly, (20, 21, 29) is primitive but not produced by this method.

Exercise 9.4 — Further Applications of the Baudhayana-Pythagoras Theorem

Q1 Find the diagonal of a square with sidelength 5 cm.

SOLUTION

Let d be the diagonal of the square.
The diagonal splits the square into two right triangles, each with legs 5 cm and 5 cm.

By Baudhayana's Theorem:
$d^2 = 5^2 + 5^2 = 25 + 25 = 50$
$d = \sqrt{50} = 5\sqrt{2}$ cm

Since $7^2 = 49$ and $7.1^2 = 50.41$, we have $7.0 < 5\sqrt{2} < 7.1$.
So the diagonal is approximately 7.07 cm.

Answer: The diagonal is $5\sqrt{2}$ cm (approximately 7.07 cm).

Q2 Find the missing sidelengths in the following right triangles (six triangles shown in figure):
(a) legs 7 and 9, hypotenuse missing
(b) legs 4 and 10 (right angle between them), hypotenuse missing
(c) leg 40 and unknown leg, hypotenuse 41
(d) leg 10 and unknown leg, hypotenuse $\sqrt{200}$
(e) legs 10 and $\sqrt{150}$ (right angle between them), hypotenuse missing
(f) leg 27 and unknown leg, hypotenuse 45

SOLUTION

In each figure the small square marks the right angle, so the two sides meeting there are the legs and the side opposite it is the hypotenuse.

(a) Right angle between legs 7 and 9, find hypotenuse:
$c^2 = 7^2 + 9^2 = 49 + 81 = 130$
$c = \sqrt{130}$

(b) Right angle between legs 4 and 10, find hypotenuse:
$c^2 = 4^2 + 10^2 = 16 + 100 = 116$
$c = \sqrt{116} = 2\sqrt{29}$

(c) Hypotenuse 41, one leg 40, find other leg:
$b^2 = 41^2 - 40^2 = 1681 - 1600 = 81$
$b = \sqrt{81} = 9$

(d) Hypotenuse √200, one leg 10, find other leg:
$b^2 = (\sqrt{200})^2 - 10^2 = 200 - 100 = 100$
$b = \sqrt{100} = 10$
(This is an isosceles right triangle with both legs equal to 10.)

(e) Right angle between legs 10 and √150, find hypotenuse:
$c^2 = 10^2 + (\sqrt{150})^2 = 100 + 150 = 250$
$c = \sqrt{250} = 5\sqrt{10}$

(f) Hypotenuse 45, one leg 27, find other leg:
$b^2 = 45^2 - 27^2 = 2025 - 729 = 1296$
$b = \sqrt{1296} = 36$
Verify: $27^2 + 36^2 = 729 + 1296 = 2025 = 45^2$. ✓
(Note: 27 = 9 × 3, 36 = 9 × 4, 45 = 9 × 5 — a scaled (3,4,5) triple.)

Answer: (a) √130 (b) 2√29 (= √116) (c) 9 (d) 10 (e) 5√10 (= √250) (f) 36

Q3 Find the sidelength of a rhombus whose diagonals are of length 24 units and 70 units.

SOLUTION

The diagonals of a rhombus bisect each other at right angles.

Each diagonal is split into two equal halves:
Half of 24 = 12 units
Half of 70 = 35 units

Each side of the rhombus is the hypotenuse of a right triangle with legs 12 and 35.

By Baudhayana's Theorem:
$side^2 = 12^2 + 35^2 = 144 + 1225 = 1369$
$side = \sqrt{1369} = 37$ units

Verify: $12^2 + 35^2 = 144 + 1225 = 1369 = 37^2$. ✓
(Note: 12, 35, 37 is a Baudhayana triple.)

Answer: The sidelength of the rhombus is 37 units.

Q4 Is the hypotenuse the longest side of a right triangle? Justify your answer.

SOLUTION

Yes, the hypotenuse is always the longest side of a right triangle.

Justification: In a right triangle with legs a, b and hypotenuse c, we have $a^2 + b^2 = c^2$.

Since a and b are positive lengths, $a^2 > 0$ and $b^2 > 0$.

Therefore: $c^2 = a^2 + b^2 > a^2 \Rightarrow c > a$
And: $c^2 = a^2 + b^2 > b^2 \Rightarrow c > b$

So c is greater than each leg, meaning the hypotenuse is the longest side.

Answer: Yes. Since $c^2 = a^2 + b^2 > a^2$ and $c^2 > b^2$, we get $c > a$ and $c > b$, so the hypotenuse is longer than either leg.

Q5 True or False — Every Baudhayana triple is either a primitive triple or a scaled version of a primitive triple.

SOLUTION

True.

Let (a, b, c) be any Baudhayana triple, so $a^2 + b^2 = c^2$.

Let f = gcd(a, b, c). Then $a = f \cdot a'$, $b = f \cdot b'$, $c = f \cdot c'$ where gcd(a', b', c') = 1.

Substituting: $(f \cdot a')^2 + (f \cdot b')^2 = (f \cdot c')^2$
$f^2a'^2 + f^2b'^2 = f^2c'^2$
$a'^2 + b'^2 = c'^2$

So (a', b', c') is also a Baudhayana triple with no common factor greater than 1 — it is a primitive Baudhayana triple.

And (a, b, c) = (f·a', f·b', f·c') is a scaled version (by factor f) of the primitive triple (a', b', c').

Answer: True. Any Baudhayana triple (a, b, c) can be written as f times a primitive triple (a/f, b/f, c/f) where f = gcd(a, b, c).

Q6 Give 5 examples of rectangles whose sidelengths and diagonals are all integers.

SOLUTION

A rectangle with integer sidelengths a and b has an integer diagonal c only when (a, b, c) is a Baudhayana triple.

Example 1: Sides 3 and 4, diagonal 5.
$3^2 + 4^2 = 9 + 16 = 25 = 5^2$. ✓

Example 2: Sides 5 and 12, diagonal 13.
$5^2 + 12^2 = 25 + 144 = 169 = 13^2$. ✓

Example 3: Sides 8 and 15, diagonal 17.
$8^2 + 15^2 = 64 + 225 = 289 = 17^2$. ✓

Example 4: Sides 7 and 24, diagonal 25.
$7^2 + 24^2 = 49 + 576 = 625 = 25^2$. ✓

Example 5: Sides 20 and 21, diagonal 29.
$20^2 + 21^2 = 400 + 441 = 841 = 29^2$. ✓

Answer: Rectangles 3×4 (diagonal 5), 5×12 (diagonal 13), 8×15 (diagonal 17), 7×24 (diagonal 25), 20×21 (diagonal 29).

Q7 Construct a square whose area is equal to the difference of the areas of squares of sidelengths 5 units and 7 units.

SOLUTION

We need a square of area = $7^2 - 5^2 = 49 - 25 = 24$ sq. units.

The side of this square must be $\sqrt{24} = 2\sqrt{6}$ units.

Construction using Baudhayana's method:
Draw a right triangle with hypotenuse 7 units and one leg 5 units.
The other leg has length $\sqrt{(7^2 - 5^2)} = \sqrt{24} = 2\sqrt{6}$ units.
Construct a square on this leg — it has area 24 sq. units.

This works because: $(\text{leg})^2 = 7^2 - 5^2$ = area of 7-unit square - area of 5-unit square.

Answer: The required square has area 24 sq. units and sidelength $\sqrt{24} = 2\sqrt{6}$ units. Construct a right triangle with hypotenuse 7 and one leg 5; the square on the other leg is the required square.

Q8 (i) Using the dots of a grid as the vertices, can you create a square that has an area of (a) 2 sq. units, (b) 3 sq. units, (c) 4 sq. units, and (d) 5 sq. units?
(ii) Suppose the grid extends indefinitely. What are the possible integer-valued areas of squares you can create in this manner?

SOLUTION

(i) A square on a grid with vertices at grid dots has its side along the vector (p, q) for non-negative integers p, q. The area equals $p^2 + q^2$.

(a) Area 2: Use vector (1, 1). Side² = $1^2 + 1^2 = 2$. Yes — tilt the square 45° inside a 2×2 grid.

(b) Area 3: We need integers p, q with $p^2 + q^2 = 3$. Checking all: 0+1=1, 0+4=4, 1+1=2, 1+4=5. No pair gives 3. NOT POSSIBLE.

(c) Area 4: Use vector (2, 0) (or (0, 2)). Side² = 4. Yes — axis-aligned 2×2 square.

(d) Area 5: Use vector (1, 2). Side² = $1^2 + 2^2 = 5$. Yes — tilt the square so one side goes 1 across and 2 up.

(ii) A square on the grid has area $p^2 + q^2$ for non-negative integers p, q (not both zero). The possible integer-valued areas are exactly the numbers that can be written as the sum of two perfect squares (including 0).
These are: 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, 25, 26, 29, ...
A number n is achievable if and only if every prime factor of the form 4k + 3 appears an even number of times in the prime factorisation of n.

Answer: (i) (a) Yes, area 2 is possible (vector (1,1)). (b) Area 3 is NOT possible. (c) Yes, area 4 (vector (2,0)). (d) Yes, area 5 (vector (1,2)). (ii) The possible integer areas are all numbers expressible as $p^2 + q^2$ for non-negative integers p, q: 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, ...

Q9 Find the area of an equilateral triangle with sidelength 6 units. [Hint: Show that an altitude bisects the opposite side. Use this to find the height.]

SOLUTION

In an equilateral triangle with sidelength 6 units, each side equals 6.

Draw an altitude from one vertex to the opposite side.
By symmetry, the altitude bisects the opposite side, creating two halves of 6/2 = 3 units each.

This forms a right triangle with:

Hypotenuse = 6 (the side of the equilateral triangle)
One leg = 3 (half the base)
Other leg = h (the altitude)

By Baudhayana's Theorem:
$h^2 + 3^2 = 6^2$
$h^2 + 9 = 36$
$h^2 = 27$
$h = \sqrt{27} = 3\sqrt{3}$ units

Area of the equilateral triangle:
Area = (1/2) × base × height = (1/2) × 6 × $3\sqrt{3}$ = $9\sqrt{3}$ sq. units

Answer: The area of the equilateral triangle with sidelength 6 units is $9\sqrt{3}$ sq. units (approximately 15.59 sq. units).