Chapter 11: Exploring Some Geometric Themes

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Exercise 11.1 — Sierpinski Gasket

Q1 Find the number of holes, and the triangles that remain at each step of the shape sequence that leads to the Sierpinski Triangle.

SOLUTION

Let T_n = number of triangles remaining at step n, and H_n = number of holes at step n.

At Step 0: We start with 1 equilateral triangle. No holes yet.
$T_0 = 1, H_0 = 0$

At each step, every remaining triangle is subdivided into 4 equal equilateral triangles by joining the midpoints, and the central one is removed. So each remaining triangle gives 3 new remaining triangles and 1 new hole.
=> T_{n+1} = 3 × T_n, and H_{n+1} = H_n + T_n

Step 1: $T_1 = 3 \times 1 = 3, H_1 = 0 + 1 = 1$
Step 2: $T_2 = 3 \times 3 = 9, H_2 = 1 + 3 = 4$
Step 3: $T_3 = 3 \times 9 = 27, H_3 = 4 + 9 = 13$

General formula for remaining triangles:
T_n = 3^n

General formula for holes:
H_n = 1 + 3 + 3² + ... + 3^(n-1) = (3^n − 1)/2 for n ≥ 1, and $H_0 = 0$

Verification: $H_3 = (3^3 - 1)/2 = (27 - 1)/2 = 13$ ✓

Answer: At step n: T_n = 3^n remaining triangles and H_n = (3^n − 1)/2 holes (with $H_0 = 0$). E.g., step 0: T=1, H=0; step 1: T=3, H=1; step 2: T=9, H=4; step 3: T=27, H=13.

Q2 Find the area of the region remaining at the nth step in each of the shape sequences that lead to the Sierpinski fractals. Take the area of the starting square/triangle to be 1 sq. unit.

SOLUTION

PART A — Sierpinski Carpet (starting shape: square of area 1 sq. unit):

At each step, the square is divided into 9 equal smaller squares and the central one is removed. So 8 out of 9 equal parts remain.

Area at Step 0 = 1
Area at Step 1 = 8/9 × 1 = 8/9
Area at Step 2 = 8/9 × 8/9 = $(8/9)^2$= 64/81
Area at Step n = (8/9)^n

As n increases, (8/9)^n → 0, so the area approaches 0.

PART B — Sierpinski Gasket (starting shape: equilateral triangle of area 1 sq. unit):

At each step, the triangle is divided into 4 equal equilateral triangles and the central one is removed. So 3 out of 4 equal parts remain.

Each small triangle has area = 1/4 of the larger one.

Area at Step 0 = 1
Area at Step 1 = 3 × (1/4) = 3/4
Area at Step 2 = 9 × (1/16) = 9/16 = $(3/4)^2$
Area at Step n = (3/4)^n

As n increases, (3/4)^n → 0, so the remaining area also approaches 0.

Answer: Sierpinski Carpet: area at step n = (8/9)^n. Sierpinski Gasket: area at step n = (3/4)^n. Both approach 0 as n → ∞.

Exercise 11.2 — Koch Snowflake

Q1 Find the number of sides in the nth step of the shape sequence that leads to the Koch Snowflake.

SOLUTION

At Step 0, the shape is an equilateral triangle with 3 sides.
$S_0 = 3$

At each step, every existing side is replaced by 4 smaller segments (the middle third is replaced by two sides of a smaller equilateral triangle, giving a 'bump'). So the number of sides multiplies by 4 at each step.
S_{n+1} = 4 × S_n

Step 0: $S_0 = 3$
Step 1: $S_1 = 4 \times 3 = 12$
Step 2: $S_2 = 4 \times 12 = 48$
Step 3: $S_3 = 4 \times 48 = 192$

General formula: S_n = 3 × 4^n

Answer: S_n = 3 × 4^n. At step 0: 3 sides; step 1: 12; step 2: 48; step 3: 192.

Q2 Find the perimeter of the shape at the nth step of the sequence. Take the starting equilateral triangle to have a sidelength of 1 unit.

SOLUTION

At Step 0, sidelength = 1 unit, number of sides = 3.
Perimeter at Step 0 = 3 × 1 = 3 units.

At each step, every side is divided into 3 equal parts and a bump is added over the middle part. So each side becomes 4 segments, each of length 1/3 of the previous side.

Length of each side at step n = (1/3)^n units.
Number of sides at step n = 3 × 4^n.

Perimeter at step n = (number of sides) × (sidelength)
= 3 × 4^n × (1/3)^n
= 3 × (4/3)^n

Verification:
$P_0 = 3 \times (4/3)^0 = 3$ ✓

$P_1 = 3 \times (4/3)^1 = 4$ ✓ (12 sides of length 1/3 each: 12 × 1/3 = 4)

$P_2 = 3 \times (4/3)^2 = 3 \times 16/9 = 16/3 \approx 5.33$

Since 4/3 > 1, the perimeter grows without bound as n → ∞.

The Koch Snowflake has infinite perimeter despite enclosing a finite area.

Answer: Perimeter at step n = 3 × (4/3)^n units. The perimeter grows without bound as n → ∞.

Exercise 11.3 — Nets of a Cube

Q1 Which of the following are the nets of a cube? First, try to answer by visualisation. Then, you may use cutouts and try. (Six hexomino shapes labelled (i) to (vi) are shown.)

SOLUTION

A net of a cube must have exactly 6 squares, and when folded, all 6 faces must close up to form a cube without any two squares landing on the same face.

Analysing each shape (fold it up mentally, tracking which cube face each square lands on):

(i) A bottom row of 4 squares, with 2 more squares forming an L above the left end (one directly above the second square of the row, and one further to its left):
When folded, two of these squares are forced onto the same face of the cube, so the shape does not close up. => NO, it is NOT a net of a cube.

(ii) A top row of 3 squares and a bottom row of 3 squares, offset so they share one column (a Z / staircase of two 3-strips):
When folded, the 6 squares close up correctly with all faces distinct. => YES, it is a net of a cube.

(iii) A staircase of three 2-square steps, each step shifted one column across and one row down:
When folded, all 6 faces close without overlap. => YES, it is a net of a cube.

(iv) A row of 4 squares with 1 square above and 1 square below the third square (a 1-4-1 arrangement):
When folded, the four in the row wrap around as four side faces and the two flaps become top and bottom. All 6 faces pair correctly. => YES, it is a net of a cube.

(v) A row of 4 squares with 2 squares stacked below the second square (a T with a long tail):
When folded, two squares land on the same face. => NO, it is NOT a net of a cube.

(vi) A row of 4 squares with 1 square above and 1 square below the second square (a 1-4-1 arrangement — the mirror image of (iv)):
When folded, the four in the row become four side faces and the two flaps become top and bottom, all distinct. => YES, it is a net of a cube.

Answer: (ii), (iii), (iv) and (vi) are nets of a cube. (i) and (v) are NOT nets of a cube.

Q2 Draw a net of a cuboid having sidelengths: (i) 5 cm, 3 cm, and 1 cm (ii) 6 cm, 3 cm, and 2 cm

SOLUTION

A cuboid with dimensions l × w × h has 6 rectangular faces: two l×w faces, two l×h faces, and two w×h faces.

(i) Cuboid 5 cm × 3 cm × 1 cm:
Faces: two 5×3 rectangles, two 5×1 rectangles, two 3×1 rectangles.

One standard net layout (cross/T shape):
• Draw a 5×1 rectangle in the centre-left (length-band top).
• Below it, draw a 5×3 rectangle (the main front face).
• Below that, another 5×1 rectangle.
• Below that, the final 5×3 rectangle (back face).
• Attach a 3×1 rectangle to the left of the middle 5×3.
• Attach another 3×1 rectangle to the right of the middle 5×3.
Total area = 2×(5×3) + 2×(5×1) + 2×(3×1) = 30 + 10 + 6 = 46 sq cm.

(ii) Cuboid 6 cm × 3 cm × 2 cm:
Faces: two 6×3 rectangles, two 6×2 rectangles, two 3×2 rectangles.

One standard net layout:
• Draw a 6×2 strip at the top.
• Below it, a 6×3 rectangle (front face).
• Below that, a 6×2 strip.
• Below that, the 6×3 rectangle (back face).
• Attach a 3×2 rectangle to the left of the front face.
• Attach another 3×2 rectangle to the right of the front face.
Total area = 2×(6×3) + 2×(6×2) + 2×(3×2) = 36 + 24 + 12 = 72 sq cm.

Answer: (i) Net for 5×3×1 cuboid: a cross-shaped layout with two 5×3 faces, two 5×1 strips, and two 3×1 strips; total net area = 46 sq cm. (ii) Net for 6×3×2 cuboid: similar cross layout; total net area = 72 sq cm.

Exercise 11.4 — Projections

Q1 Observe the front view, top view and side view of the different lines in Fig. 4.6. Is there any relation between their lengths?

SOLUTION

From Fig. 4.6, three different line segments are shown at different orientations in space, and their projections onto the front, top, and side planes are drawn.

Key observation from the figure:
The length of the projection of a line is always less than or equal to the actual length of the line. Equality holds only when the line is parallel to the projection plane (i.e., the line lies flat with respect to that plane).

Formal reasoning: Consider a line segment AB of actual length l. Let AE be drawn perpendicular to the projection plane, so that AECD forms a rectangle, giving the projection DC = AE = p (the projected length). Since angle AEB = 90°, triangle AEB has AE = p and hypotenuse AB = l. By the Baudhayana-Pythagorean theorem:

$l^2 = p^2 + BE^2$

$\Rightarrow l^2 \geq p^2$ (since $BE^2 \geq 0$)

$\Rightarrow l \geq p$.

Equality l = p holds when BE = 0, i.e., when the line is parallel to the projection plane.

In Fig. 4.6:

- The horizontal line (parallel to the vertical plane) has its front-view projection equal to its full length. Its side-view projection is a point (zero length) because it points directly toward the side plane.
- Tilted lines have projections strictly shorter than their actual length.
- The sum of the squares of any two perpendicular projections of a line plus the square of the third projection equals the square of the actual length.

Answer: The projected length p is always ≤ the actual length l (i.e., $p \leq l$). They are equal only when the line is parallel to the projection plane.

Q2 Match each of the following objects with its projections (front, top, side views). Objects shown: mug, funnel, hammer, car, slide/ramp, chair, ceiling fan, tin can.

SOLUTION

We match each object to its projection row by identifying characteristic shapes in each view:

Car:
• Front view: car silhouette with wheels and body shape.
• Top view: rectangular car outline from above.
• Side view: car side profile.
=> Car matches projection Row 1.

Tin can:
• Front view: cylinder (rectangle with rounded top/bottom).
• Top view: circle with inner ring (lid and body).
• Side view: cylinder profile.
=> Tin can matches projection Row 2.

Slide/ramp:
• Front view: narrow rectangle (the slide post/frame seen from front).
• Top view: wide horizontal rectangle.
• Side view: A-frame / inclined ramp shape.
=> Slide/ramp matches projection Row 3.

Chair:
• Front view: chair-front outline (seat and back legs).
• Top view: square seat outline.
• Side view: chair-side profile (back, seat, front legs).
=> Chair matches projection Row 4.

Ceiling fan:
• Front view: horizontal disc/plate shape.
• Top view: fan blades radiating from centre (3-blade propeller shape).
• Side view: flat disc with motor housing.
=> Ceiling fan matches projection Row 5.

Funnel:
• Front view: V / inverted triangle shape.
• Top view: circle.
• Side view: V / inverted triangle shape.

=> Funnel matches projection Row 6.

Hammer:
• Front view: lollipop shape (thin handle, rectangular head at top).
• Top view: small head shape from above.
• Side view: full hammer profile (handle + head).
=> Hammer matches projection Row 7.

Mug:
• Front view: open-top mug silhouette.
• Top view: circle (the mouth of the mug).
• Side view: mug with handle visible.
=> Mug matches projection Row 8.

Answer: Car→Row 1, Tin can→Row 2, Slide/ramp→Row 3, Chair→Row 4, Ceiling fan→Row 5, Funnel→Row 6, Hammer→Row 7, Mug→Row 8.

Exercise 11.5 — Views of Cube Combinations and Shadows

Q1 Which solid corresponds to the given top view, front view, and side view? (Front view: C-shape / staircase; Top view: offset staircase from above; Side view: step/small block.)

SOLUTION

We examine the three given projection views and match them to one of the seven cube-stack solids shown (options i–vii).

Given views analysis:

• Front view: shows a C-shape (like a 2×2 block with the middle-right unit missing), indicating a step structure visible from the front.

• Top view: shows a staircase/L shape from above, meaning the solid has cubes arranged in a step pattern when seen from directly above.

• Side view: shows a small step block, consistent with a 2-step structure from the side.

Looking at all seven options:

• Option (iv) is a C-shaped arrangement of cubes (similar to the gold C-shape shown at the top right of the problem page). It produces:

— Front view: C / staircase shape ✓

— Top view: staircase/L arrangement ✓

— Side view: step/small block ✓

The other options produce different front, top, or side views that do not match all three simultaneously.

Answer: The solid in option (iv) corresponds to the given front, top, and side views.

Q2 Find the number of cubes in this stack of identical cubes. (An isometric image of a stepped pyramid arrangement is shown.)

SOLUTION

The image shows a triangular wall of cubes, one cube deep, viewed isometrically. Reading it row by row from the top, each row down has one more cube than the row above:

Top row: 1 cube

2nd row: 2 cubes

3rd row: 3 cubes

Bottom row: 4 cubes

Total = 1 + 2 + 3 + 4 = 10 cubes.

This is the 4th triangular number, $T_4 = 4 \times 5 / 2 = 10$ ✓

Answer: 10 cubes in total (rows of 1 + 2 + 3 + 4 from top to bottom).

Exercise 11.6 — Drawing on Isometric Grids

Q1 In addition to the 5 ways shown in Fig. 4.8, are there any additional ways of gluing four cubes together along faces? Can you visualise and draw these as well?

SOLUTION

The 5 shapes in Fig. 4.8 are the 5 flat tetrominoes (all four cubes coplanar):
(i) I-shape: 4 cubes in a straight row
(ii) O-shape: 2×2 square
(iii) L-shape: 3 in a row with 1 on the end of one side
(iv) S/Z-shape: 2 on each level, offset by 1
(v) T-shape: 3 in a row with 1 on the middle

In 3 dimensions, the fourth cube does not have to be in the same plane as the other three. When we allow 3D arrangements, there are 3 additional non-flat tetracubes:

(vi) Right-handed screw: Take the L-shape (3 in a row with 1 on the side), and instead of the side cube being in the same layer, place it one level up or rotated out of the flat plane. This gives a twisted shape.

(vii) Left-handed screw: The mirror image of (vi). Unlike the flat tetrominoes, this cannot be flipped to match (vi), so it counts as a distinct shape.

(viii) Right-angle tower: Take 3 cubes in a row in one direction, and place the 4th cube on top of one end — oriented perpendicular to the plane of the other 3 in 3D.

Total distinct tetracubes in 3D = 5 flat + 3 non-flat = 8.

These 3 extra shapes can be drawn on isometric grid paper by representing the 'out-of-plane' cube along the height axis while the other 3 cubes lie along the length or depth axis.

Answer: Yes, there are 3 additional non-flat arrangements: a right-handed screw, a left-handed screw (its mirror image), and a right-angle tower. Total distinct ways = 8 tetracubes.

Q2 Observe this impossible triangle (Penrose triangle made of cubes). (i) Would it be possible to build a model out of actual cubes? What are the front, top, and side profiles of this impossible triangle? (iii) Why does the illusion work?

SOLUTION

(i) Can it be built from actual cubes?
NO. The figure is a Penrose/impossible triangle — an optical illusion. It cannot be physically constructed from actual cubes.

The reason: if you trace the top edge of the triangle starting from any corner and follow it around all three corners, you would end up higher than where you started, which is physically impossible in 3D space. The figure violates the consistency of height in three dimensions.

Front, top, and side profiles of the impossible triangle:
• Front view: A square frame / rectangle (the two visible arms of the triangle form an L or rectangular border).
• Top view: Similarly a square frame / L-shaped outline.
• Side view: A similar rectangular frame.
Each individual view looks like a flat rectangular frame, which is a physically possible shape. No single projection reveals the impossibility — you need to mentally combine all three to detect the contradiction.

(ii) Recreating on the isometric grid:
Draw the three arms of the triangle using the three axis directions of the isometric grid (height |, depth /, length \). Each arm is drawn as a rectangular bar along one axis, and they appear to connect at corners — but the connection at the third corner is geometrically inconsistent in 3D.

(iii) Why does the illusion work?
The illusion works because every local portion of the figure — each individual corner and each individual arm — is a physically realisable arrangement of cubes. Our brain interprets each local part as a valid 3D structure and attempts to assemble them globally.

However, the global assembly is impossible: if you follow the 'up' direction along the height axis around the triangle, you find you have gone up by one step at each of the three corners, which would require a net gain in height of 3 units after one complete loop — but you end up back at the start. This 3D contradiction is hidden from any single viewpoint.

The isometric drawing style (equal projected lengths in all three directions) makes the deception especially convincing because it feels like an accurate 3D representation.

Answer: (i) No, it cannot be built from actual cubes — it is a Penrose impossible triangle. Each individual 2D view looks like a rectangular frame. (iii) The illusion works because every local portion is a valid cube arrangement, but the global 3D assembly is self-contradictory (the height increases by 1 at each of the 3 corners, giving a net height gain of 3 in a closed loop).