Chapter 13: Algebra Play

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Exercise 13.1 — Number Pyramids

Q1 Without building the entire pyramid, find the number in the topmost row given the bottom row in each of these cases: (a) 4, 13, 8 (b) 7, 11, 3 (c) 10, 14, 25

SOLUTION

For a 3-row pyramid with bottom row [a, b, c], the top uses the pattern derived in the chapter: top = a + 2b + c.

(a) Bottom row: 4, 13, 8
Top = 4 + 2 × 13 + 8 = 4 + 26 + 8 = 38

(b) Bottom row: 7, 11, 3
Top = 7 + 2 × 11 + 3 = 7 + 22 + 3 = 32

(c) Bottom row: 10, 14, 25
Top = 10 + 2 × 14 + 25 = 10 + 28 + 25 = 63

Answer: (a) 38 (b) 32 (c) 63

Q2 Write an expression for the topmost row of a pyramid with 4 rows in terms of the values in the bottom row.

SOLUTION

Let the bottom row of a 4-row pyramid be a, b, c, d (from left to right).

Row 3 (second from bottom): a+b, b+c, c+d
Row 2: (a+b)+(b+c) = a+2b+c, (b+c)+(c+d) = b+2c+d
Row 1 (top): (a+2b+c) + (b+2c+d) = a + 3b + 3c + d

The coefficients 1, 3, 3, 1 match the third row of Pascal's triangle (since the top of an n-row pyramid uses C(n-1,0), C(n-1,1), ..., C(n-1,n-1) as coefficients of the bottom entries).

Answer: Top = a + 3b + 3c + d

Q3 Without building the entire pyramid, find the number in the topmost row given the bottom row in each of these cases: (a) 8, 19, 21, 13 (b) 7, 18, 19, 6 (c) 9, 7, 5, 11

SOLUTION

Using the formula from Q2: top = a + 3b + 3c + d

(a) Bottom row: 8, 19, 21, 13
Top = 8 + 3×19 + 3×21 + 13 = 8 + 57 + 63 + 13 = 141

(b) Bottom row: 7, 18, 19, 6
Top = 7 + 3×18 + 3×19 + 6 = 7 + 54 + 57 + 6 = 124

(c) Bottom row: 9, 7, 5, 11
Top = 9 + 3×7 + 3×5 + 11 = 9 + 21 + 15 + 11 = 56

Answer: (a) 141 (b) 124 (c) 56

Q4 If the first three Virahāṅka-Fibonacci numbers are written in the bottom row of a number pyramid with three rows, fill in the rest of the pyramid. What numbers appear in the grid? What is the number at the top? Are they all Virahāṅka-Fibonacci numbers?

SOLUTION

The Virahāṅka-Fibonacci sequence begins: 1, 2, 3, 5, 8, 13, 21, ...
The first three terms are 1, 2, 3.

Bottom row: 1, 2, 3
Row 2: 1+2 = 3, 2+3 = 5
Top: 3+5 = 8

All numbers in the pyramid: 1, 2, 3, 3, 5, 8.
Every one of these (1, 2, 3, 5, 8) appears in the Virahāṅka-Fibonacci sequence.
The number at the top is 8, which is the 5th term of the sequence.
Yes, every number that appears in the pyramid is a Virahāṅka-Fibonacci number.

Answer: Top = 8; all numbers in the pyramid (1, 2, 3, 5, 8) are Virahāṅka-Fibonacci numbers.

Q5 What can you say about the numbers in the pyramid and the number at the top in the following cases? (i) The first four Virahāṅka-Fibonacci numbers are written in the bottom row of a four row pyramid. (ii) The first 29 Virahāṅka-Fibonacci numbers are written in the bottom row of a 29 row pyramid.

SOLUTION

(i) First four Virahāṅka-Fibonacci numbers: 1, 2, 3, 5
Using top = a + 3b + 3c + d:
Top = 1 + 3×2 + 3×3 + 5 = 1 + 6 + 9 + 5 = 21
21 is the 7th Virahāṅka-Fibonacci number.

Building the full pyramid:
Row 3: 3, 5, 8
Row 2: 8, 13
Top: 21
Every number in the pyramid (1, 2, 3, 5, 3, 5, 8, 8, 13, 21) is a Virahāṅka-Fibonacci number.

(ii) The pattern from Q4 and Q5(i) is: for an n-row pyramid with the first n Virahāṅka-Fibonacci numbers at the bottom, the top is always the (2n−1)th term of the sequence.
For n = 29: top = the (2×29−1)th = 57th Virahāṅka-Fibonacci number.
All numbers in the pyramid are Virahāṅka-Fibonacci numbers.

Answer: (i) Top = 21 (the 7th Virahāṅka-Fibonacci number); all pyramid entries are Virahāṅka-Fibonacci. (ii) Top = 57th Virahāṅka-Fibonacci number; all pyramid entries are Virahāṅka-Fibonacci numbers.

Q6 If the bottom row of an n row pyramid contains the first n Virahāṅka-Fibonacci numbers, what can we say about the numbers in the pyramid? What can we say about the number at the top?

SOLUTION

Let F_1, F_2, F_3, ... denote the Virahāṅka-Fibonacci sequence: 1, 2, 3, 5, 8, 13, ...

The pattern observed in Q4 and Q5 generalises as follows:
1. Every number that appears anywhere in the pyramid is a Virahāṅka-Fibonacci number.
2. The number at the top of the n-row pyramid is F_{2n−1}, the (2n−1)th term of the sequence.

Verification for small n:

n=2: bottom [1,2], top = 3 = F_3 (2×2−1 = 3) ✓

n=3: bottom [1,2,3], top = 8 = F_5 (2×3−1 = 5) ✓

n=4: bottom [1,2,3,5], top = 21 = F_7 (2×4−1 = 7) ✓

In general the top equals the (2n−1)th Virahāṅka-Fibonacci number.

Answer: All numbers in the pyramid are Virahāṅka-Fibonacci numbers. The number at the top is the (2n−1)th Virahāṅka-Fibonacci number.

Exercise 13.2 — The Largest Product

Q1 Fill the digits 1, 3, and 7 in [__][__] × [__] to make the largest product possible.

SOLUTION

From the algebraic argument in the chapter: given three digits p < q < r, the largest product is always qp × r — the largest digit is the multiplier, and the remaining two digits form the multiplicand in decreasing order.

Digits: 1 < 3 < 7, so p=1, q=3, r=7.
Largest product = 31 × 7.

Verify by listing all options:
13 × 7 = 91
31 × 7 = 217
17 × 3 = 51
71 × 3 = 213
13 × 7 = 91
37 × 1 = 37
73 × 1 = 73

Maximum is 31 × 7 = 217.

Answer: 31 × 7 = 217

Q2 Fill the digits 3, 5, and 9 in [__][__] × [__] to make the largest product possible.

SOLUTION

Digits: 3 < 5 < 9, so p=3, q=5, r=9.
Using the rule: largest digit as multiplier, remaining in decreasing order as multiplicand.
Largest product = 53 × 9.

Verify the key comparisons:
53 × 9 = 477
93 × 5 = 465
95 × 3 = 285

Maximum is 53 × 9 = 477.

Answer: 53 × 9 = 477

Exercise 13.3 — Decoding Divisibility Tricks

Q1 In the trick given above, what is the quotient when you divide by 9? Is there a relationship between the two numbers and the quotient?

SOLUTION

In the trick: start with a 2-digit number ab (digits a and b, b > a). Reverse to get ba. Find the difference ba − ab, then divide by 9.

Using algebra (as shown in the chapter):
ba − ab = (10b + a) − (10a + b) = 9b − 9a = 9(b − a)

Dividing by 9 gives the quotient = b − a.

Relationship: the quotient equals the difference of the two digits of the original number (larger digit minus smaller digit).

Example: start with 47 (a=4, b=7). Difference = 74 − 47 = 27. Quotient = 27 ÷ 9 = 3 = 7 − 4. ✓

Answer: Quotient = b − a (the difference between the two digits of the original number).

Q2 In the trick given above, instead of finding the difference of the two 2-digit numbers, find their sum. What will happen? Is the sum always divisible by 11? Can we justify this claim using algebra?

SOLUTION

Let the two-digit number be ab = 10a + b. Its reverse is ba = 10b + a.

Their sum: (10a + b) + (10b + a) = 11a + 11b = 11(a + b).

Since the sum equals 11(a + b), it is always divisible by 11, regardless of the digits chosen.

Verification with the examples given:
31 + 13 = 44 = 11 × 4 (a+b = 1+3 = 4) ✓
28 + 82 = 110 = 11 × 10 (a+b = 2+8 = 10) ✓
12 + 21 = 33 = 11 × 3 (a+b = 1+2 = 3) ✓

Answer: The sum of a 2-digit number and its reverse is always divisible by 11. Sum = 11(a + b).

Q3 Consider any 3-digit number, say abc (100a + 10b + c). Make two other 3-digit numbers from these digits by cycling these digits around, yielding bca and cab. Now add the three numbers. Using algebra, justify that the sum is always divisible by 37. Will it also always be divisible by 3?

SOLUTION

The three numbers are:
abc = 100a + 10b + c
bca = 100b + 10c + a
cab = 100c + 10a + b

Sum = (100a + 10b + c) + (100b + 10c + a) + (100c + 10a + b)
= (100a + a + 10a) + (10b + 100b + b) + (c + 10c + 100c)
= 111a + 111b + 111c
= 111(a + b + c)

Now 111 = 3 × 37.
So the sum = 3 × 37 × (a + b + c), which is divisible by both 37 and 3.

Example: a=1, b=2, c=3 → 123 + 231 + 312 = 666 = 111 × 6 = 3 × 37 × 6 ✓

Answer: Sum = 111(a + b + c) = 3 × 37 × (a + b + c). Always divisible by both 37 and 3.

Q4 Consider any 3-digit number, say abc. Make it a 6-digit number by repeating the digits, that is abcabc. Divide this number by 7, then by 11, and finally by 13. What do you get? Try this with other numbers. Figure out why it works.

SOLUTION

The 6-digit number abcabc = abc × 1000 + abc = abc × 1001.

Key fact: 7 × 11 × 13 = 1001.

So abcabc = abc × (7 × 11 × 13).

Dividing by 7: gives abc × 11 × 13
Dividing that by 11: gives abc × 13
Dividing that by 13: gives abc

The final result is always the original 3-digit number abc.

Example: abc = 135 → abcabc = 135135
135135 ÷ 7 = 19305

19305 ÷ 11 = 1755
1755 ÷ 13 = 135 ✓

Answer: Dividing abcabc by 7, then 11, then 13 always gives back the original number abc, because abcabc = abc × 1001 and 7 × 11 × 13 = 1001.

Q5 There are 3 shrines, each with a magical pond in the front. If anyone dips flowers into these magical ponds, the number of flowers doubles. A person has some flowers. He dips them all in the first pond and then places some flowers in shrine 1. Next, he dips the remaining flowers in the second pond and places some flowers in shrine 2. Finally, he dips the remaining flowers in the third pond and then places them all in shrine 3. If he placed an equal number of flowers in each shrine, how many flowers did he start with? How many flowers did he place in each shrine?

SOLUTION

Let x = initial flowers and k = flowers placed at each shrine.

After pond 1 (doubles): 2x. Place k at shrine 1. Remaining: 2x − k.
After pond 2 (doubles): 2(2x − k) = 4x − 2k. Place k at shrine 2. Remaining: 4x − 3k.
After pond 3 (doubles): 2(4x − 3k) = 8x − 6k. Place ALL at shrine 3, so k = 8x − 6k.

Solving: k + 6k = 8x => 7k = 8x => x = 7k/8.
For the smallest whole-number solution: k = 8, x = 7.

Verification with x=7, k=8:
Start: 7 flowers.
After pond 1: 14. Place 8 at shrine 1. Remaining: 6.
After pond 2: 12. Place 8 at shrine 2. Remaining: 4.
After pond 3: 8. Place all 8 at shrine 3. ✓ (8 = 8 = 8)

Answer: He started with 7 flowers and placed 8 flowers at each shrine.

Q6 A farm has some horses and hens. The total number of heads of these animals is 55 and the total number of legs is 150. How many horses and how many hens are on the farm?

SOLUTION

Let h = number of horses and n = number of hens.

Equation 1 (heads): h + n = 55
Equation 2 (legs): 4h + 2n = 150

From Equation 1: n = 55 − h
Substitute into Equation 2: 4h + 2(55 − h) = 150
4h + 110 − 2h = 150
2h = 40
h = 20
n = 55 − 20 = 35

Alternative (hint method): If all 55 animals were hens: 55 × 2 = 110 legs.
Actual legs = 150. Difference = 150 − 110 = 40.
Each horse has 2 extra legs compared to a hen, so horses = 40 ÷ 2 = 20.
Hens = 55 − 20 = 35.

Check: 20 + 35 = 55 heads ✓; 20×4 + 35×2 = 80 + 70 = 150 legs ✓

Answer: 20 horses and 35 hens.

Q7 A mother is 5 times her daughter's age. In 6 years' time, the mother will be 3 times her daughter's age. How old is the daughter now?

SOLUTION

Let the daughter's current age = d years.
Then the mother's current age = 5d years.

In 6 years: daughter = d + 6, mother = 5d + 6.
Condition: 5d + 6 = 3(d + 6)
5d + 6 = 3d + 18
2d = 12
d = 6

Check: Daughter = 6, Mother = 30 (= 5 × 6) ✓
In 6 years: Daughter = 12, Mother = 36 = 3 × 12 ✓

Answer: The daughter is 6 years old now.

Q8 Two friends, Gauri and Naina, are cowherds. One day, they pass each other on the road with their cows. Gauri says to Naina, 'You have twice as many cows as I do'. Naina says, 'That's true, but if I gave you three of my cows, we would each have the same number of cows'. How many cows do Gauri and Naina have?

SOLUTION

Let Gauri's cows = g. Then Naina's cows = 2g (from Gauri's statement).

After Naina gives 3 cows to Gauri:
Gauri has g + 3, Naina has 2g − 3.
These are equal: g + 3 = 2g − 3
6 = g

So Gauri has g = 6 cows, Naina has 2g = 12 cows.

Check: Naina (12) = 2 × Gauri (6) ✓
After transfer: Gauri = 9, Naina = 9 ✓

Answer: Gauri has 6 cows and Naina has 12 cows.

Q9 I run a small dosa cart and my expenses are as follows: Rent for the dosa cart is 5000 per day. The cost of making one dosa (including all the ingredients and fuel) is 10. (i) If I can sell 100 dosas a day, what should be the selling price of my dosa to make a profit of 2000? (ii) If my customers are willing to pay only 50 for a dosa, how many dosas should I aim to sell in a day to make a profit of 2000?

SOLUTION

(i) Let the selling price per dosa = p rupees.
Revenue = 100p
Total cost = Rent + ingredient cost = 5000 + 100 × 10 = 6000
Profit = Revenue − Cost = 100p − 6000 = 2000
100p = 8000
p = 80

(ii) Let the number of dosas to sell = q.
Selling price = 50 per dosa.
Revenue = 50q
Total cost = 5000 + 10q

Profit = 50q − (5000 + 10q) = 40q − 5000 = 2000
40q = 7000
q = 175

Check (ii): Revenue = 50×175 = 8750; Cost = 5000 + 175×10 = 6750; Profit = 2000 ✓

Answer: (i) Selling price should be 80 per dosa. (ii) Must sell 175 dosas per day.

Q10 Evaluate the following sequence of fractions: 1/3, (1+3)/(5+7), (1+3+5)/(7+9+11). What do you observe? Can you explain why this happens?

SOLUTION

Compute each fraction:
Term 1: 1/3
Term 2: (1+3)/(5+7) = 4/12 = 1/3
Term 3: (1+3+5)/(7+9+11) = 9/27 = 1/3

Observation: every fraction equals 1/3.

Explanation using algebra:
The numerator of the nth term is the sum of the first n odd numbers = n^2.

The denominator of the nth term is a sum of n consecutive odd numbers.
The odd numbers used in the denominators go: 3; 5,7; 7,9,11; ...
The nth denominator starts at the odd number (2n+1) and has n terms:
Denominator of term n = sum of n consecutive odds starting at (2n+1)
= n × (middle term) ... or directly:
= (2n+1) + (2n+3) + ... = n×(2n+1) + 2×(0+1+...+(n-1))
= n(2n+1) + n(n-1) = $2n^2 + n + n^2 - n = 3n^2$.

So each fraction = $n^2/(3n^2)$ = 1/3.

Answer: Every fraction in the sequence equals 1/3. The nth term equals $n^2/(3n^2)$ = 1/3.

Q11 Karim and the Genie: Karim agrees to a deal where his coins double each time he goes around a banyan tree, but he must pay 8 coins to the genie after each round. After 3 rounds, his coins doubled on the third round but he was left with exactly 8 coins (the amount he owed). (i) How many coins did Karim initially have? (ii) For what cost per round should Karim agree to the deal, if he wants to increase the number of coins he has? (iii) Through its magical powers, the genie knows the number of coins that Karim has. How should the genie set the cost per round so that it gets all of Karim's coins?

SOLUTION

Let x = Karim's initial coins, c = cost per round (here c = 8).

After round 1 (double then pay c): $2x - c$
After round 2: $2(2x - c) - c = 4x - 3c$
After doubling in round 3 (before paying): $2(4x - 3c) = 8x - 6c$

He is left with exactly 8 (= c) coins after doubling on round 3, which is all he owes:
$8x - 6c = c \Rightarrow 8x = 7c$

(i) With c = 8: $8x = 7 \times 8 = 56 \Rightarrow x = 7$.
Karim initially had 7 coins.

Verification: Start 7 → after round 1: 14−8=6 → after round 2: 12−8=4 → after doubling in round 3: 8 (= exactly c) ✓

(ii) After 3 complete rounds (paying c each time), Karim has $8x - 7c$ coins.
For this to exceed his starting amount x:
$8x - 7c > x \Rightarrow 7x > 7c \Rightarrow x > c$
So Karim should agree to the deal only if the cost per round is strictly less than his initial number of coins.
With x = 7, the cost per round must be at most 6 coins.

(iii) For the genie to end up with all of Karim's coins, Karim must finish with 0:
$8x - 7c = 0 \Rightarrow c = 8x/7$
The genie should set the cost per round to 8x/7 (where x is Karim's initial coins).
In this problem x = 7, so c = 8. ✓

Answer: (i) Karim started with 7 coins. (ii) Cost per round must be less than 7 (i.e., at most 6) for Karim to gain. (iii) The genie should set cost = 8x/7 (where x = Karim's initial coins) to get everything.