Chapter 3: The World of Numbers

CONTENTS — 6 exercises · 43 questions

3.1	Natural Numbers and Early Counting Systems	4 Q
3.2	Integers and Brahmagupta's Laws	4 Q
3.3	Fractions, Rational Numbers, and Their Arithmetic	8 Q
3.4	Representation of Rational Numbers on the Number Line	6 Q
3.5	Decimal Expansions, Cyclic Numbers, and Irrational Numbers	5 Q
Exercises	The World of Numbers	16 Q

Exercise 3.1 — Natural Numbers and Early Counting Systems

Q1 A merchant in the port city of Lothal is exchanging bags of spices for copper ingots. He receives 15 ingots for every 2 bags of spices. If he brings 12 bags of spices to the market, how many copper ingots will he leave with?

SOLUTION

Rate: 15 ingots for every 2 bags => 15/2 ingots per bag.
Number of ingots for 12 bags = 12 × (15/2) = (12 × 15)/2 = 180/2 = 90.
The merchant leaves with 90 copper ingots.

Q2 Look at the sequence of numbers on one column of the Ishango bone: 11, 13, 17, 19. What do these numbers have in common? List the next three numbers that fit this pattern.

SOLUTION

Observe: 11, 13, 17, 19 are all prime numbers, and they are precisely the prime numbers between 10 and 20.
The pattern is: consecutive prime numbers.
The next prime after 19 is 23 (not divisible by 2, 3).
The next prime after 23 is 29 (not divisible by 2, 3, 5).
The next prime after 29 is 31 (not divisible by 2, 3, 5).
Next three numbers: 23, 29, 31.
All are prime numbers. Next three primes: 23, 29, 31.

Q3 We know that Natural Numbers are closed under addition (the sum of any two natural numbers is always a natural number). Are they closed under subtraction? Provide a couple of examples to justify your answer.

SOLUTION

Natural numbers are NOT closed under subtraction.
Closure under subtraction would require that for any two natural numbers a and b, the result a - b is also a natural number.
Counter-example 1: 3 - 5 = -2. Here 3 and 5 are natural numbers, but -2 is not.
Counter-example 2: 7 - 10 = -3. Here 7 and 10 are natural numbers, but -3 is not.
Even 5 - 5 = 0 is not a natural number (natural numbers start at 1).
Therefore, natural numbers are not closed under subtraction.
No. Natural numbers are not closed under subtraction. For example, 3 - 5 = -2 and 7 - 10 = -3, which are not natural numbers.

Q4 *Ancient Indians used the joints of their fingers to count, a practice still seen today. Each finger has 3 joints, and the thumb is used to count them. How many can you count on one hand? How does this relate to the ancient base-12 counting systems?

SOLUTION

On one hand, the thumb acts as a pointer and the four fingers (index, middle, ring, little) each have 3 joints.
Total countable joints = 4 fingers × 3 joints = 12.
Using the thumb as a pointer, one can count from 1 to 12 on a single hand.
This naturally produces a base-12 (duodecimal) system.
With the other hand tracking groups of 12 (one finger raised per completed group of 12), a person can count up to 5 × 12 = 60, or with both hands used this way up to 12 × 12 = 144.
Base 12 has practical advantages: 12 is divisible by 2, 3, 4, and 6, making fractions easier to represent than in base 10.
12 joints on one hand (4 fingers × 3 joints each, with thumb as pointer). This directly gives the base-12 (duodecimal) counting system used in ancient India and elsewhere.

Exercise 3.2 — Integers and Brahmagupta's Laws

Q1 The temperature in the high-altitude desert of Ladakh is recorded as 4 °C at noon. By midnight, it drops by 15 °C. What is the midnight temperature?

SOLUTION

Midnight temperature = Noon temperature − Drop = 4 − 15 = −11 °C.
Using integers: starting at +4, subtracting 15 gives 4 − 15 = −11.
−11 °C

Q2 A spice trader takes a loan (debt) of 850. The next day, he makes a profit (fortune) of 1,200. The following week, he incurs a loss of 450. Write this sequence as an equation using integers and calculate his final financial standing.

SOLUTION

Represent debt as negative, fortune as positive.
Loan = −850, Profit = +1200, Loss = −450.
Equation: (−850) + 1200 + (−450)
= 1200 − 850 − 450
= 1200 − 1300
= −100.
The trader is in debt by 100.
Final standing: −100 (a net debt of 100). Equation: (−850) + 1200 + (−450) = −100.

Q3 Calculate the following using Brahmagupta's laws: (i) (−12) × 5 (ii) (−8) × (−7) (iii) 0 − (−14) (iv) (−20) ÷ 4

SOLUTION

(i) (−12) × 5: A debt times a fortune is a debt. (−12) × 5 = −60.
(ii) (−8) × (−7): A debt times a debt is a fortune. (−8) × (−7) = +56.
(iii) 0 − (−14): Subtracting a negative is adding a positive. 0 − (−14) = 0 + 14 = 14.
(iv) (−20) ÷ 4: A debt shared equally among a fortune count is still a debt. (−20) ÷ 4 = −5.
(i) −60 (ii) 56 (iii) 14 (iv) −5

Q4 Explain, using a real-world example of debt, why subtracting a negative number is the same as adding a positive number (e.g., 10 − (−5) = 15).

SOLUTION

Suppose you have 10 in your account (fortune of +10).
You have a debt of 5 recorded, meaning −5 is owed.
If that debt of 5 is cancelled (subtracted/removed), you are 5 richer.
Removing a debt of 5 from a fortune of 10: 10 − (−5) = 10 + 5 = 15.

In general: subtracting a negative number means taking away a debt, which increases your fortune — the same effect as adding the positive amount.
Verification: 10 − (−5) = 10 + 5 = 15.
Removing (subtracting) a debt of 5 from a fortune of 10 leaves you with 15.
Algebraically, 10 − (−5) = 10 + 5 = 15. Subtracting a negative is equivalent to adding the corresponding positive.

Exercise 3.3 — Fractions, Rational Numbers, and Their Arithmetic

Q1 Prove that the following rational numbers are equal: (i) 2/3 and 4/6 (ii) 5/4 and 10/8 (iii) −3/5 and −6/10 (iv) 9/3 and 3

SOLUTION

Two fractions a/b and c/d are equal if and only if a × d = b × c (cross-multiplication).

(i) 2/3 and 4/6: Cross-multiply: 2 × 6 = 12 and 3 × 4 = 12. Equal. (Alternatively, 4/6 = 2/3 by dividing numerator and denominator by 2.)

(ii) 5/4 and 10/8: Cross-multiply: 5 × 8 = 40 and 4 × 10 = 40. Equal. (Alternatively, 10/8 = 5/4 by dividing by 2.)

(iii) −3/5 and −6/10: Cross-multiply: (−3) × 10 = −30 and 5 × (−6) = −30. Equal. (Alternatively, −6/10 = −3/5 by dividing by 2.)

(iv) 9/3 and 3: Write 3 = 3/1. Cross-multiply: 9 × 1 = 9 and 3 × 3 = 9. Equal. (Alternatively, 9/3 = 3 by simplifying.)
All pairs are equal. Cross-multiplication confirms: (i) 12 = 12, (ii) 40 = 40, (iii) −30 = −30, (iv) 9 = 9.

Q2 Find the sum: (i) 2/5 + 3/10 (ii) 7/12 + 5/8 (iii) −4/7 + 3/14

SOLUTION

(i) 2/5 + 3/10: LCD = 10. 2/5 = 4/10. Sum = 4/10 + 3/10 = 7/10.

(ii) 7/12 + 5/8: LCD = 24. 7/12 = 14/24; 5/8 = 15/24. Sum = 14/24 + 15/24 = 29/24.

(iii) −4/7 + 3/14: LCD = 14. −4/7 = −8/14. Sum = −8/14 + 3/14 = −5/14.
(i) 7/10 (ii) 29/24 (iii) −5/14

Q3 Find the difference: (i) 5/6 − 1/4 (ii) 11/8 − 3/4 (iii) −7/9 − (−2/3)

SOLUTION

(i) 5/6 − 1/4: LCD = 12. 5/6 = 10/12; 1/4 = 3/12. Difference = 10/12 − 3/12 = 7/12.

(ii) 11/8 − 3/4: LCD = 8. 3/4 = 6/8. Difference = 11/8 − 6/8 = 5/8.

(iii) −7/9 − (−2/3) = −7/9 + 2/3. LCD = 9. 2/3 = 6/9. Result = −7/9 + 6/9 = −1/9.
(i) 7/12 (ii) 5/8 (iii) −1/9

Q4 Find the product: (i) 2/3 × 3/10 (ii) 7/11 × 5/8 (iii) −4/7 × 5/14

SOLUTION

(i) 2/3 × 3/10 = (2 × 3)/(3 × 10) = 6/30 = 1/5.

(ii) 7/11 × 5/8 = (7 × 5)/(11 × 8) = 35/88. (No common factors — 35 = 5 × 7, 88 = 8 × 11.)

(iii) (−4/7) × (5/14) = (−4 × 5)/(7 × 14) = −20/98 = −10/49.
(i) 1/5 (ii) 35/88 (iii) −10/49

Q5 Find the quotient: (i) 2/3 ÷ 3/10 (ii) 7/11 ÷ 5/8 (iii) −4/7 ÷ 5/14

SOLUTION

To divide by a fraction, multiply by its reciprocal.

(i) 2/3 ÷ 3/10 = 2/3 × 10/3 = 20/9.

(ii) 7/11 ÷ 5/8 = 7/11 × 8/5 = 56/55.

(iii) (−4/7) ÷ (5/14) = (−4/7) × (14/5) = (−4 × 14)/(7 × 5) = −56/35 = −8/5.
(i) 20/9 (ii) 56/55 (iii) −8/5

Q6 Show that: (1/2 + 3/4) × 8/3 = 1/2 × 8/3 + 3/4 × 8/3.

SOLUTION

LHS: (1/2 + 3/4) × 8/3.
First find 1/2 + 3/4: LCD = 4, so 1/2 = 2/4. Sum = 2/4 + 3/4 = 5/4.
LHS = 5/4 × 8/3 = 40/12 = 10/3.

RHS: 1/2 × 8/3 + 3/4 × 8/3.
1/2 × 8/3 = 8/6 = 4/3.
3/4 × 8/3 = 24/12 = 2.
RHS = 4/3 + 2 = 4/3 + 6/3 = 10/3.

LHS = RHS = 10/3. Hence the distributive law is verified.
Both sides equal 10/3. The distributive law a × (b + c) = a × b + a × c is verified for rational numbers.

Q7 Simplify the following using the distributive property: 7/9 × (6/7 − 3/4).

SOLUTION

Apply distributive law: 7/9 × (6/7 − 3/4) = 7/9 × 6/7 − 7/9 × 3/4.

7/9 × 6/7 = 42/63 = 6/9 = 2/3.
7/9 × 3/4 = 21/36 = 7/12.

Result = 2/3 − 7/12. LCD = 12: 2/3 = 8/12.
= 8/12 − 7/12 = 1/12.

Verification by bracketing first: 6/7 − 3/4; LCD = 28: 24/28 − 21/28 = 3/28.
7/9 × 3/28 = 21/252 = 1/12. Confirmed.

Q8 Find the rational number x such that: 5/6 (x + 3/5) = 5/6 · x + 1/2.

SOLUTION

Expand the LHS using the distributive property:
5/6 (x + 3/5) = 5/6 · x + 5/6 · 3/5.

Compute the constant term: 5/6 × 3/5 = (5 × 3)/(6 × 5) = 15/30 = 1/2.

So LHS = 5x/6 + 1/2, which equals RHS = 5x/6 + 1/2.

The equation is an identity — it holds for every rational number x.
This demonstrates that the distributive property (5/6)(x + 3/5) = (5/6)x + (5/6)(3/5) = (5/6)x + 1/2 is valid for all rational x.
The equation holds for all rational numbers x (it is an identity, verifying the distributive property: 5/6 × 3/5 = 1/2).

Exercise 3.4 — Representation of Rational Numbers on the Number Line

Q1 Represent the rational numbers 2/3, −5/4 and 1(1/2) on a single number line.

SOLUTION

Convert to decimals for placement: 2/3 ≈ 0.67, −5/4 = −1.25, 1(1/2) = 3/2 = 1.5.

Draw a number line with integers −2, −1, 0, 1, 2 marked.

2/3: lies between 0 and 1. Divide the segment [0, 1] into 3 equal parts; mark the 2nd division to the right of 0.

−5/4 = −1(1/4): lies between −2 and −1. Divide the segment [−2, −1] into 4 equal parts; mark the 3rd division to the right of −2 (i.e., 1 part to the left of −1).

1(1/2) = 3/2: lies between 1 and 2. Divide the segment [1, 2] into 2 equal parts; mark the midpoint.
On the number line: −5/4 = −1.25 lies between −2 and −1; 2/3 ≈ 0.67 lies between 0 and 1; 1(1/2) = 1.5 lies between 1 and 2.

Q2 Find three distinct rational numbers that lie strictly between −1/2 and 1/4.

SOLUTION

We need numbers r with −1/2 < r < 1/4.
Convert to a common denominator: −1/2 = −4/8, 1/4 = 2/8.
Integers between −4 and 2 (exclusive) at denominator 8: −3, −2, −1, 0, 1.
So three rational numbers: −3/8, −1/8, 1/8.

Verification: −1/2 = −4/8 < −3/8 < 1/4 = 2/8. True.
−4/8 < −1/8 < 2/8. True.
−4/8 < 1/8 < 2/8. True.
Three rational numbers strictly between −1/2 and 1/4: −3/8, −1/8, 1/8. (Other valid answers exist.)

Q3 Simplify the expression: (−1/4) + (5/12).

SOLUTION

LCD of 4 and 12 is 12.
−1/4 = −3/12.
(−3/12) + 5/12 = (−3 + 5)/12 = 2/12 = 1/6.

Q4 A tailor has 15(3/4) metres of fine silk. If making one kurta requires 2(1/4) metres of silk, exactly how many kurtas can he make?

SOLUTION

Convert mixed numbers to improper fractions.
Total silk: 15(3/4) = (15 × 4 + 3)/4 = 63/4 metres.
Per kurta: 2(1/4) = (2 × 4 + 1)/4 = 9/4 metres.
Number of kurtas = (63/4) ÷ (9/4) = 63/4 × 4/9 = 63/9 = 7.
7 kurtas

Q5 Find three rational numbers between 3.1415 and 3.1416.

SOLUTION

We need three rational numbers r with 3.1415 < r < 3.1416.
The gap is 3.1416 − 3.1415 = 0.0001.
Choose values in this interval by adding small increments:
r_1 = 3.14151 = 314151/100000
r_2 = 3.14153 = 314153/100000
r_3 = 3.14155 = 314155/100000
All three satisfy 3.1415 < r < 3.1416.
Three rational numbers between 3.1415 and 3.1416: 3.14151, 3.14153, 3.14155 (or equivalently 314151/100000, 314153/100000, 314155/100000).

Q6 *Can you think of other way(s) to find a rational number between any two rational numbers?

SOLUTION

Method 1 — Averaging (Mediant shortcut): Given two rational numbers a and b, their arithmetic mean (a + b)/2 always lies strictly between them. For example, between −1/2 and 1/4: (−1/2 + 1/4)/2 = (−2/4 + 1/4)/2 = (−1/4)/2 = −1/8.

Method 2 — Common denominator enumeration: Convert both fractions to a common denominator d; any fraction with denominator d whose numerator lies strictly between the two numerators is a rational number between them. If no numerator exists, use a larger denominator (e.g., 2d).

Method 3 — Decimal interpolation: Express both numbers as decimals, then pick any terminating decimal strictly between them (e.g., between 0.33 and 0.34, pick 0.335 = 67/200).

Method 4 — Stern–Brocot / Mediant of fractions: The mediant (a+c)/(b+d) of fractions a/b and c/d (in lowest terms with a/b < c/d) always lies strictly between them.
Yes. The arithmetic mean (a+b)/2 always gives a rational number between a and b. Alternatively, convert to a common denominator and pick an intermediate numerator; or use decimal interpolation.

Exercise 3.5 — Decimal Expansions, Cyclic Numbers, and Irrational Numbers

Q1 Without performing long division, determine which of the following rational numbers will have terminating decimals and which will be repeating: 7/20, 4/15 and 13/250. Then check your answers by explicitly performing the long divisions and expressing these rational numbers as decimals.

SOLUTION

Rule: A fraction p/q in lowest terms has a terminating decimal if and only if the denominator q has no prime factors other than 2 and 5.

7/20: 20 = $2^2 \times 5$. Prime factors are only 2 and 5 => TERMINATING.
4/15: 15 = 3×5. Prime factor 3 is present => REPEATING.
13/250: 250 = 2×5^3. Prime factors are only 2 and 5 => TERMINATING.

Verification by long division:

7/20: Multiply numerator and denominator by 5 to get denominator 100: 7/20 = 35/100 = 0.35. TERMINATING. Confirmed.

4/15: Perform division — 40 ÷ 15 = 2 remainder 10; 100 ÷ 15 = 6 remainder 10; the remainder 10 repeats, so the digit 6 repeats. 4/15 = 0.2(6) = 0.2666... REPEATING. Confirmed.

13/250: Multiply numerator and denominator by 4 to get denominator 1000: 13/250 = 52/1000 = 0.052. TERMINATING. Confirmed.
7/20 = 0.35 (terminating); 4/15 = 0.2(6) (repeating); 13/250 = 0.052 (terminating).

Q2 Perform the long division for 1/13. Identify the repeating block of digits. Does it show cyclic properties if you evaluate 2/13? Now compute 3/13, 4/13, etc. What do you notice?

SOLUTION

Long division of 1 ÷ 13:
10 ÷ 13 = 0 r 10 | 100 ÷ 13 = 7 r 9 | 90 ÷ 13 = 6 r 12 | 120 ÷ 13 = 9 r 3 | 30 ÷ 13 = 2 r 4 | 40 ÷ 13 = 3 r 1 | remainder 1 repeats the cycle.
1/13 = 0.(076923) — repeating block is 076923 (6 digits).

Computing n/13 for n = 1 to 6:
1/13 = 0.(076923)
2/13 = 0.(153846)
3/13 = 0.(230769)
4/13 = 0.(307692)
5/13 = 0.(384615)
6/13 = 0.(461538)

Observation: The six repeating blocks split into two families of cyclic rotations. Family A (from 076923): 1/13 = 0.(076923), 3/13 = 0.(230769), 4/13 = 0.(307692). Family B (from 153846): 2/13 = 0.(153846), 5/13 = 0.(384615), 6/13 = 0.(461538). Within each family, each block is a cyclic rotation of the previous one.

Unlike 1/7 (which is a true cyclic number — all six multiples are rotations of a single block 142857), 1/13 produces two distinct 6-digit cycles. The repeating block length for 1/13 is 6, and there are 12/6 = 2 such distinct cycle families.
1/13 = 0.(076923). The 6-digit repeating block 076923 shows partial cyclic properties: the six decimal expansions of n/13 (n = 1 to 6) group into two families of cyclic rotations — {076923, 230769, 307692} and {153846, 384615, 461538}.

Q3 Classify the following numbers as rational or irrational: (i) √81 (ii) √12 (iii) 0.33333… (iv) 0.123451234512345… (v) 1.01001000100001… (vi) 23.560185612239874790120. Find the explicit fractions in case they are rational.

SOLUTION

(i) √81 = 9. Since 9 is a natural number, it is rational. Fraction: 9/1.

(ii) √12 = √(4 × 3) = 2√3. Since 3 is not a perfect square, √3 is irrational (proved by contradiction: if √3 = p/q in lowest terms, then $3q^2 = p^2$, so 3 | p, write p = 3k, then $3q^2 = 9k^2$, $q^2 = 3k^2$, so 3 | q — contradicts p, q coprime). Therefore √12 = 2√3 is IRRATIONAL.

(iii) 0.33333… = 0.(3). Let x = 0.(3). Then 10x = 3.(3). Subtract: 9x = 3, x = 1/3. RATIONAL. Fraction: 1/3.

(iv) 0.123451234512345… = 0.(12345). The block 12345 repeats. Let x = 0.(12345). Then 100000x = 12345.(12345). Subtract: 99999x = 12345, x = 12345/99999 = 4115/33333. RATIONAL. Fraction: 4115/33333.

(v) 1.01001000100001… The number of zeros between consecutive 1s increases by 1 each time (1 zero, then 2 zeros, then 3 zeros, …). There is no fixed repeating block, so this is a NON-TERMINATING, NON-REPEATING decimal => IRRATIONAL.

(vi) 23.560185612239874790120. This decimal terminates (ends in 0). Therefore it is RATIONAL. Fraction: 23560185612239874790120/1000000000000000000000 (simplify as needed; the exact rational form is this ratio).
(i) √81 = 9 — RATIONAL (= 9/1). (ii) √12 = 2√3 — IRRATIONAL. (iii) 0.(3) — RATIONAL (= 1/3). (iv) 0.(12345) — RATIONAL (= 4115/33333). (v) 1.01001000100001… — IRRATIONAL (no repeating block). (vi) 23.560185612239874790120 — RATIONAL (terminating decimal).

Q4 The number 0.(9) (which means 0.99999…) is a rational number. Using algebra (let x = 0.(9), multiply by 10, and subtract), explain why 0.(9) is exactly equal to 1.

SOLUTION

Let x = 0.999…
Multiply both sides by 10: 10x = 9.999…
Subtract the first equation from the second:
10x − x = 9.999… − 0.999…
9x = 9

x = 1.

Therefore 0.(9) = 1 exactly.

Alternative confirmation: We know 1/3 = 0.(3). Multiplying both sides by 3: 3 × 1/3 = 3 × 0.(3), giving 1 = 0.(9).
0.(9) = 1 exactly. Proof: let x = 0.999…; then 10x = 9.999…; subtracting gives 9x = 9, so x = 1.

Q5 *We have seen that the repeating block of 1/7 is a cyclic number. Try to find more numbers (n) whose reciprocals (1/n) produce decimals with repeating blocks that are cyclic.

SOLUTION

A cyclic number arises from 1/n when the length of the repeating block of 1/n equals n − 1. This means 10 is a primitive root modulo n (i.e., 10 generates the full multiplicative group modulo n).

1/7 = 0.(142857), block length 6 = 7 − 1. Cyclic number: 142857.
Verify: 142857 × 1 = 142857, × 2 = 285714, × 3 = 428571, × 4 = 571428, × 5 = 714285, × 6 = 857142 — all are cyclic rotations.

Next candidates (primes where 10 is a primitive root):
1/17: repeating block length = 16, block = 0588235294117647. Cyclic.
1/19: repeating block length = 18, block = 052631578947368421. Cyclic.

Non-examples: 1/13 has block length 6 < 12 = 13 − 1; it is NOT a cyclic number.
1/11 has block length 2 < 10; not cyclic.

So n = 7, 17, 19, 23, 29, … (primes for which 10 is a primitive root mod n) produce cyclic reciprocals.
Numbers n whose reciprocals produce cyclic repeating blocks include n = 7 (block 142857), n = 17 (block length 16), n = 19 (block length 18). These are primes for which 10 is a primitive root modulo n.

Exercises — The World of Numbers

Q1 Convert the following rational numbers into a terminating decimal or a non-terminating and repeating decimal, whichever the case may be, by the process of long division: (i) 3/50 (ii) 2/9

SOLUTION

(i) 3/50

Divide 3 by 50 by long division. Since 3 < 50, write 3 as 3.000 and continue.

50 goes into 30 zero times, into 300 six times (6 × 50 = 300), remainder 0.

So 3/50 = 0.06, which is a terminating decimal.

(Check on denominator: 50 = 2 × 5², i.e. only prime factors 2 and 5, so it must terminate — consistent.)

(ii) 2/9

Divide 2 by 9. 9 into 20 goes 2 times (2 × 9 = 18), remainder 2.

The remainder 2 keeps repeating, giving 2, 2, 2, … in every step.

So 2/9 = 0.2222… = 0.2, a non-terminating repeating decimal.

(Denominator 9 = 3², contains a prime other than 2 and 5, so it must be non-terminating repeating — consistent.)

(i) 3/50 = 0.06 (terminating); (ii) 2/9 = 0.2 = 0.2222… (non-terminating repeating)

Q2 Prove that √5 is an irrational number.

SOLUTION

We use proof by contradiction.

Assume, if possible, that √5 is rational. Then √5 = p/q, where p and q are integers, q ≠ 0, and p, q have no common factor other than 1 (fraction in lowest terms).

Squaring both sides: 5 = p²/q², so p² = 5q². … (1)

From (1), 5 divides p². Since 5 is prime, 5 must divide p.

So write p = 5k for some integer k.

Substitute into (1): (5k)² = 5q² => 25k² = 5q² => q² = 5k².

Hence 5 divides q², and since 5 is prime, 5 divides q.

But now 5 divides both p and q, so 5 is a common factor of p and q.

This contradicts our assumption that p and q have no common factor other than 1.

Therefore our assumption is wrong, and √5 cannot be written as p/q.

Hence √5 is irrational.

√5 is irrational (proved by contradiction: assuming √5 = p/q in lowest terms forces 5 to divide both p and q — a contradiction).

Q3 Convert the following decimal numbers into the form p/q: (i) 12.6 (ii) 0.0120 (iii) 3.052 (iv) 1.235 (v) 0.23 (vi) 2.05 (vii) 2.125 (viii) 3.125 (ix) 2.1625

SOLUTION

Convert each decimal to the form p/q in lowest terms. A bar over digits (written here as a repeated block in brackets) means those digits repeat forever.

(i) 12.6 (terminating) = 126/10 = 63/5.
(ii) 0.0120 (terminating) = 120/10000 = 3/250.

(iii) 3.0(52) = 3.0525252... (the block '52' repeats). Let x = 3.0525252...
One digit ('0') lies between the point and the repeating block, so line up the repeats: 10x = 30.525252... and 1000x = 3052.525252...
Subtract: 1000x - 10x = 3052.5252... - 30.5252... = 3022, so 990x = 3022 and x = 3022/990 = 1511/495.

(iv) 1.2(35) = 1.2353535... ('35' repeats). 10x = 12.353535..., 1000x = 1235.353535...; 1000x - 10x = 1223, so 990x = 1223 and x = 1223/990.

(v) 0.(23) = 0.232323... ('23' repeats). 100x = 23.2323...; 100x - x = 23, so 99x = 23 and x = 23/99.

(vi) 2.0(5) = 2.05555... ('5' repeats). 10x = 20.5555..., 100x = 205.5555...; 100x - 10x = 185, so 90x = 185 and x = 185/90 = 37/18.

(vii) 2.12(5) = 2.125555... ('5' repeats). 100x = 212.5555..., 1000x = 2125.5555...; 1000x - 100x = 1913, so 900x = 1913 and x = 1913/900.

(viii) 3.12(5) = 3.125555... ('5' repeats). 100x = 312.5555..., 1000x = 3125.5555...; 1000x - 100x = 2813, so 900x = 2813 and x = 2813/900.

(ix) 2.(1625) = 2.16251625... ('1625' repeats). 10000x = 21625.1625...; 10000x - x = 21623, so 9999x = 21623 and x = 21623/9999.
(i) 63/5; (ii) 3/250; (iii) 1511/495; (iv) 1223/990; (v) 23/99; (vi) 37/18; (vii) 1913/900; (viii) 2813/900; (ix) 21623/9999.

Q4 Locate the following rational numbers on the number line: (i) 0.532 (ii) 1.15

SOLUTION

(i) 0.532 (terminating) lies between 0 and 1.
Divide 0 to 1 into 10 equal parts; 0.532 lies between 0.5 and 0.6. Divide 0.5 to 0.6 into 10 parts; it lies between 0.53 and 0.54. Divide that into 10 parts; the 2nd mark from 0.53 is 0.532. Mark it.

(ii) 1.1(5) = 1.15555... (the '5' repeats), which equals 52/45 (since 10x = 11.5555..., 100x = 115.5555..., so 90x = 104, x = 104/90 = 52/45).
So it is a little MORE than 1.155 and lies between 1.15 and 1.16 — not at 1.15 exactly.
Locate it by successive magnification: 1.1555... is between 1 and 2, then between 1.1 and 1.2, then between 1.15 and 1.16; within 1.15-1.16 it sits just past the midpoint (at 1.1555...). Mark

that point.

(i) 0.532 is located by successive tenths between 0.53 and 0.54 (2nd sub-mark). (ii) 1.1(5) = 1.15555... = 52/45 lies between 1.15 and 1.16, just past 1.155 — not at 1.15.

Q5 Find 6 rational numbers between 3 and 4.

SOLUTION

Write 3 and 4 with a common denominator large enough to leave at least 6 whole numbers between the numerators. Using denominator 7:

3 = 21/7 and 4 = 28/7.

The integers strictly between 21 and 28 are 22, 23, 24, 25, 26, 27 — exactly six of them. Placing each over 7 gives 6 rational numbers between 3 and 4:

22/7, 23/7, 24/7, 25/7, 26/7, 27/7.

As decimals these are about 3.14, 3.29, 3.43, 3.57, 3.71, 3.86 — all between 3 and 4.

22/7, 23/7, 24/7, 25/7, 26/7, 27/7 (six rationals between 3 = 21/7 and 4 = 28/7).

Q6 Find 5 rational numbers between 2/5 and 3/5.

SOLUTION

Write both with a denominator large enough to leave at least 5 gaps. Multiply numerator and denominator by 6:

2/5 = 12/30 and 3/5 = 18/30.

The numbers between 12/30 and 18/30 with denominator 30 are 13/30, 14/30, 15/30, 16/30, 17/30 — exactly 5 numbers.

In lowest terms: 13/30, 7/15, 1/2, 8/15, 17/30.

Each lies strictly between 2/5 (=0.4) and 3/5 (=0.6).

Five such numbers are 13/30, 7/15, 1/2, 8/15, 17/30.

Q7 Find 5 rational numbers between 1/6 and 2/5.

SOLUTION

Take the LCM of 6 and 5, which is 30, and write both fractions with denominator 30:

1/6 = 5/30 and 2/5 = 12/30.

The numbers between 5/30 and 12/30 with denominator 30 are 6/30, 7/30, 8/30, 9/30, 10/30, 11/30 (six numbers). Pick any 5.

For example 6/30, 7/30, 8/30, 9/30, 10/30.

In lowest terms: 1/5, 7/30, 4/15, 3/10, 1/3.

Each lies strictly between 1/6 (≈0.167) and 2/5 (=0.4).

Five such numbers are 1/5, 7/30, 4/15, 3/10, 1/3 (i.e. 6/30 … 10/30).

Q8 If $x/3 + x/5 = 16/15$, find the rational number x.

SOLUTION

Take the LCM of the denominators 3 and 5 on the left, which is 15:
$x/3 + x/5 = 5x/15 + 3x/15 = 8x/15$.
So $8x/15 = 16/15$.
Multiply both sides by 15: $8x = 16$.
Divide by 8: $x = 2$.
Check: $2/3 + 2/5 = 10/15 + 6/15 = 16/15$. ✓
$x = 2$

Q9 Let a and b be two non-zero rational numbers such that $a + 1/b = 0$. Without assigning any numerical values, determine whether ab is positive or negative. Justify your answer.

SOLUTION

Given $a + 1/b = 0$, so $a = -1/b$.
Multiply both sides by b: $ab = -1/b \times b = -1$.
So $ab = -1$.
Since -1 is a negative number, ab is negative.
(In words: $a = -1/b$ means a and 1/b are opposite in sign, hence a and b are opposite in sign, so their product ab is negative — in fact exactly -1.)
ab is negative; in fact $ab = -1$.

Q10 A rational number has a terminating decimal expansion whose last non-zero digit occurs in the 4th decimal place. Show that such a number can be written in the form $p/10^4$, where p is an integer not divisible by 10. Is it necessary that the denominator of this rational number, when written in the lowest form, is divisible by 2^4 or 5^4? Give reasons.

SOLUTION

Part 1: Since the decimal terminates and its last non-zero digit is in the 4th decimal place, the number has exactly 4 digits after the decimal point (with the 4th being non-zero).
Multiplying the number by $10^4 = 10000$ shifts the decimal point 4 places right and gives a whole number p. So the number = $p/10^4$.
Because the 4th decimal digit is non-zero, p does NOT end in 0, i.e. p is not divisible by 10.
Hence the number can be written as $p/10^4$ with p an integer not divisible by 10.

Part 2: Now $10^4 = 2^4 \times 5^4$. Write the number in lowest form by cancelling the common factors of p and 10^4.
Since p is not divisible by 10, p is not divisible by both 2 and 5 at the same time. So at most one of the factors 2^4 or 5^4 can be reduced.
• If p is coprime to 10, no cancellation: denominator = $2^4 \times 5^4$ (divisible by both 2^4 and 5^4).
• If p is divisible only by 2 (say by 2^a, $a \leq 4$), the 2's cancel down but the full 5^4 remains: denominator is divisible by 5^4. Example: $2/10000 = 1/5000 = 1/(2^3 \times 5^4)$.

• If p is divisible only by 5 (say by 5^b, b ≤ 4), the 5's cancel down but the full 2^4 remains: denominator is divisible by 2^4. Example: 5/10000 = 1/2000 = $1/(2^4 \times 5^3)$.

In every case the denominator in lowest form is divisible by 2^4 OR by 5^4 (at least one of them).

So yes — it is necessary that the lowest-form denominator is divisible by 2^4 or 5^4, but it need not be divisible by both.

The number = $p/10^4 = p/(2^4 \times 5^4)$ with p not divisible by 10. Yes, the lowest-form denominator must be divisible by 2^4 OR 5^4 (at least one), though not necessarily by both — because p, not being divisible by 10, can cancel at most one of 2^4 or 5^4.

Q11 Without performing division, determine whether the decimal expansion of 18/125 is terminating or non-terminating. If it terminates, state the number of decimal places.

SOLUTION

First reduce 18/125 to lowest form. $18 = 2 \times 3^2$ and $125 = 5^3$, which share no common factor, so 18/125 is already in lowest form.

Rule: a rational number in lowest form terminates if and only if its denominator's prime factorisation is of the form 2^m × 5^n (only 2s and 5s).

Here denominator = $125 = 5^3$ (only the prime 5). So the decimal expansion is terminating.

Number of decimal places = the larger of the powers of 2 and 5 in the denominator = max(0, 3) = 3.

(Verification: 18/125 = 18×8 / (125×8) = 144/1000 = 0.144 — 3 decimal places.)

Terminating, with 3 decimal places (18/125 = 0.144).

Q12 A rational number in its lowest form has denominator $2^3 \times 5$. How many decimal places will its decimal expansion have? Explain your answer.

SOLUTION

The denominator is $2^3 \times 5$, which contains only the primes 2 and 5, so the decimal expansion terminates.

The number of decimal places equals the larger of the two exponents in the denominator's 2^m × 5^n form.

Here m = 3 (power of 2) and n = 1 (power of 5), so number of places = max(3, 1) = 3.

Reason: to convert to a power of 10 we must make the powers of 2 and 5 equal to the larger exponent. $2^3 \times 5 = 2^3 \times 5^1$; multiply numerator and denominator by 5^2 to get denominator $2^3 \times 5^3 = (2\times5)^3 = 10^3 = 1000$, giving exactly 3 decimal places.

3 decimal places (denominator $2^3\times5$ becomes $10^3 = 1000$, so max(3,1) = 3 places).

Q13 Let a = 7/12 and b = 5/6. Express both a and b in the form k_1/m and k_2/m where k_1, k_2 and m are integers and $k_2 - k_1 > 6$. Using the same denominator m, write exactly five distinct rational numbers lying between a and b keeping an integer numerator. Explain why the condition $k_2 - k_1 > n + 1$ is necessary to find n such rational numbers between the two rational numbers a and b using this method.

SOLUTION

First get a common denominator for a = 7/12 and b = 5/6: 5/6 = 10/12, so a = 7/12, b = 10/12.
Here $k_2 - k_1 = 10 - 7 = 3$, which is NOT greater than 6.
To make $k_2 - k_1 > 6$, scale both fractions up. Multiply numerator and denominator by 3, so m = 36:
a = 7/12 = 21/36 ($k_1 = 21$), b = 5/6 = 30/36 ($k_2 = 30$).
Now $k_2 - k_1 = 30 - 21 = 9 > 6$, as required.
The integers strictly between 21 and 30 are 22, 23, 24, 25, 26, 27, 28, 29 (eight of them). Choose any five to get five rational numbers between a and b:
22/36, 23/36, 24/36, 25/36, 26/36 (i.e. in lowest form 11/18, 23/36, 2/3, 25/36, 13/18).

Why the gap condition is necessary: with $a = k_1/m$ and $b = k_2/m$, the rational numbers between them having the SAME denominator m and integer numerators are exactly $(k_1+1)/m$, $(k_1+2)/m$, …, $(k_2-1)/m$.
The count of these integer numerators strictly between k_1 and k_2 is $(k_2 - k_1 - 1)$.
To pick n distinct such numbers we need $(k_2 - k_1 - 1) \geq n$, i.e. $k_2 - k_1 \geq n + 1$.
The strict form $k_2 - k_1 > n + 1$ stated in the question is a safe sufficient condition: it guarantees strictly more than n integer numerators lie between k_1 and k_2, so n of them can always be chosen with room to spare. (For n = 5 we needed $k_2 - k_1 > 6$, met by 9 above.)
a = 21/36, b = 30/36 (m = 36, $k_2 - k_1 = 9 > 6$). Five numbers: 22/36, 23/36, 24/36, 25/36, 26/36 = 11/18, 23/36, 2/3, 25/36, 13/18. The gap $k_2 - k_1$ must exceed n+1 because only the k_2-k_1-1 integers strictly between k_1 and k_2 give valid same-denominator numerators, and we need at least n of them.

Q14 Three rational numbers x, y, z satisfy x + y + z = 0 and xy + yz + zx = 0. Show that all the rational numbers x, y, z must be simultaneously zero.

SOLUTION

Recall the identity: $(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx)$.
Substitute the given conditions x + y + z = 0 and xy + yz + zx = 0:
$(0)^2 = x^2 + y^2 + z^2 + 2(0)$.
So $0 = x^2 + y^2 + z^2$.
Since x, y, z are rational numbers, x^2, y^2, z^2 are each ≥ 0 (a square is never negative).
A sum of non-negative numbers equals 0 only when each term is 0.
Therefore $x^2 = 0$, $y^2 = 0$, $z^2 = 0$, which gives x = 0, y = 0, z = 0.
Hence x, y, z are simultaneously zero.
Since $(x+y+z)^2 = x^2+y^2+z^2 + 2(xy+yz+zx)$, the two conditions give $x^2+y^2+z^2 = 0$; as each square

is ≥ 0, all must be 0, so x = y = z = 0.

Q15 Show that the rational number (a + b)/2 lies between the rational numbers a and b.

SOLUTION

Without loss of generality assume a < b (if a = b there is nothing between them; if a > b just swap the labels).
We must show a < (a + b)/2 < b.

Show (a + b)/2 > a:
(a + b)/2 – a = (a + b – 2a)/2 = (b – a)/2.
Since b > a, we have b – a > 0, so (b – a)/2 > 0, hence (a + b)/2 – a > 0, i.e. (a + b)/2 > a.

Show (a + b)/2 < b:
b – (a + b)/2 = (2b – a – b)/2 = (b – a)/2.
Again b – a > 0, so (b – a)/2 > 0, hence b – (a + b)/2 > 0, i.e. (a + b)/2 < b.

Combining, a < (a + b)/2 < b, so the mean (a + b)/2 lies between a and b.
Also, since a and b are rational, (a + b)/2 is rational (rationals are closed under addition and division by 2). This also proves rational numbers are dense.
(a + b)/2 – a = (b – a)/2 > 0 and b – (a + b)/2 = (b – a)/2 > 0 (for a < b), so a < (a+b)/2 < b; hence the mean lies between a and b.

Q16 Find the lengths of the hypotenuses of all the right triangles in Fig. 3.14, which is referred to as the square root spiral.

SOLUTION

The square root spiral is built from right triangles that each have both perpendicular sides (legs) of length 1 unit for the first triangle, and thereafter the hypotenuse of the previous triangle together with a new unit leg.
By the Pythagoras theorem, hypotenuse = $\sqrt{(leg_1^2 + leg_2^2)}$.

Triangle 1: legs 1 and 1 → hypotenuse = $\sqrt{(1^2 + 1^2)} = \sqrt{2}$.
Triangle 2: legs $\sqrt{2}$ and 1 → hypotenuse = $\sqrt{((\sqrt{2})^2 + 1^2)} = \sqrt{(2 + 1)} = \sqrt{3}$.
Triangle 3: legs $\sqrt{3}$ and 1 → hypotenuse = $\sqrt{(3 + 1)} = \sqrt{4} = 2$.
Triangle 4: legs $\sqrt{4}$ and 1 → hypotenuse = $\sqrt{(4 + 1)} = \sqrt{5}$.
Triangle 5: legs $\sqrt{5}$ and 1 → hypotenuse = $\sqrt{(5 + 1)} = \sqrt{6}$.
Triangle 6: legs $\sqrt{6}$ and 1 → hypotenuse = $\sqrt{(6 + 1)} = \sqrt{7}$.
Triangle 7: legs $\sqrt{7}$ and 1 → hypotenuse = $\sqrt{(7 + 1)} = \sqrt{8} = 2\sqrt{2}$.
In general the n-th triangle has hypotenuse $\sqrt{(n + 1)}$. The lengths therefore form the sequence $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$, … (each successive hypotenuse squared is one more than the previous).
(Approx values: $\sqrt{2} \approx 1.414$, $\sqrt{3} \approx 1.732$, $\sqrt{4} = 2$, $\sqrt{5} \approx 2.236$, $\sqrt{6} \approx 2.449$, $\sqrt{7} \approx 2.646$, $\sqrt{8} \approx 2.828$.)
The successive hypotenuses are $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$ (=2), $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$, … ; in general the n-th

triangle's hypotenuse is √(n+1). (Exact count depends on the number of triangles drawn in Fig. 3.14.)