Chapter 4: Exploring Algebraic Identities

CONTENTS 6 exercises · 25 questions

4.1	Visualising Identities — using $(a + b)^2 = a^2 + 2ab + b^2$	2 Q
4.2	Factorisation Using $(a + b)^2$ and $(a - b)^2$ Identities	2 Q
4.3	More Identities — $(a + b + c)^2$ and $(a - b)^2$	4 Q
4.4	Factorisation Using $(x + a)(x + b) = x^2 + (a + b)x + ab$ and Related Identities	3 Q
4.5	Simplification of Rational Expressions Using Algebraic Identities	1 Q
Exercises	Exploring Algebraic Identities	13 Q

Exercise 4.1 — Visualising Identities — using $(a + b)^2 = a^2 + 2ab + b^2$

Q1 Using the identity $(a + b)^2 = a^2 + 2ab + b^2$, expand the following: (i) $(7x + 4y)^2$ (ii) $(7x/5 + 3y/2)^2$ (iii) $(2.5p + 1.5q)^2$ (iv) $(3s/4 + 8t)^2$ (v) $(x + 1/(2y))^2$ (vi) $(1/x + 1/y)^2$

SOLUTION

Use the identity $(a + b)^2 = a^2 + 2ab + b^2$ for each part.

(i) $(7x + 4y)^2$
Here $a = 7x$, $b = 4y$.
$(7x + 4y)^2 = (7x)^2 + 2(7x)(4y) + (4y)^2$
$= 49x^2 + 56xy + 16y^2$

(ii) $(7x/5 + 3y/2)^2$
Here $a = 7x/5$, $b = 3y/2$.
$(7x/5 + 3y/2)^2 = (7x/5)^2 + 2(7x/5)(3y/2) + (3y/2)^2$
$= 49x^2/25 + 21xy/5 + 9y^2/4$

(iii) $(2.5p + 1.5q)^2$
Write $2.5 = 5/2$, $1.5 = 3/2$. So $a = 5p/2$, $b = 3q/2$.
$(5p/2 + 3q/2)^2 = (5p/2)^2 + 2(5p/2)(3q/2) + (3q/2)^2$
$= 25p^2/4 + 15pq/2 + 9q^2/4$

(iv) $(3s/4 + 8t)^2$
Here $a = 3s/4$, $b = 8t$.
$(3s/4 + 8t)^2 = (3s/4)^2 + 2(3s/4)(8t) + (8t)^2$
$= 9s^2/16 + 12st + 64t^2$

(v) $(x + 1/(2y))^2$
Here $a = x$, $b = 1/(2y)$.
$(x + 1/(2y))^2 = x^2 + 2(x)(1/(2y)) + (1/(2y))^2$
$= x^2 + x/y + 1/(4y^2)$

(vi) $(1/x + 1/y)^2$

Here a = 1/x, b = 1/y.
$(1/x + 1/y)^2 = (1/x)^2 + 2(1/x)(1/y) + (1/y)^2$
$= 1/x^2 + 2/(xy) + 1/y^2$
(i) $49x^2 + 56xy + 16y^2$ (ii) $49x^2/25 + 21xy/5 + 9y^2/4$ (iii) $25p^2/4 + 15pq/2 + 9q^2/4$ (iv) $9s^2/16 + 12st + 64t^2$ (v) $x^2 + x/y + 1/(4y^2)$ (vi) $1/x^2 + 2/(xy) + 1/y^2$

Q2 Using the same identity $(a + b)^2 = a^2 + 2ab + b^2$, find the values of: (i) $(64)^2$ (ii) $(105)^2$ (iii) $(205)^2$

SOLUTION

(i) $(64)^2$
Write 64 = 60 + 4, so a = 60, b = 4.
$(60 + 4)^2 = 60^2 + 2(60)(4) + 4^2$
$= 3600 + 480 + 16$
$= 4096$

(ii) $(105)^2$
Write 105 = 100 + 5, so a = 100, b = 5.
$(100 + 5)^2 = 100^2 + 2(100)(5) + 5^2$
$= 10000 + 1000 + 25$
$= 11025$

(iii) $(205)^2$
Write 205 = 200 + 5, so a = 200, b = 5.
$(200 + 5)^2 = 200^2 + 2(200)(5) + 5^2$
$= 40000 + 2000 + 25$
$= 42025$
(i) 4096 (ii) 11025 (iii) 42025

Exercise 4.2 — Factorisation Using (a + b)² and (a − b)² Identities

Q1 Factor completely: (i) $9x^2 + 24xy + 16y^2$ (ii) $4s^2 + 20st + 25t^2$ (iii) $49x^2 + 28xy + 4y^2$ (iv) $64p^2 + (32/3)pq + (4/9)q^2$ *(v) $3a^2 + 4ab + (4/3)b^2$ *(vi) $(9/5)s^2 + 6sv + 5v^2$

SOLUTION

Use $(a + b)^2 = a^2 + 2ab + b^2$ to recognise perfect-square trinomials.

(i) $9x^2 + 24xy + 16y^2$
$9x^2 = (3x)^2$, $16y^2 = (4y)^2$, $2(3x)(4y) = 24xy$. ✓
=> $9x^2 + 24xy + 16y^2 = (3x + 4y)^2$

(ii) $4s^2 + 20st + 25t^2$
$4s^2 = (2s)^2$, $25t^2 = (5t)^2$, $2(2s)(5t) = 20st$. ✓
=> $4s^2 + 20st + 25t^2 = (2s + 5t)^2$

(iii) $49x^2 + 28xy + 4y^2$
$49x^2 = (7x)^2$, $4y^2 = (2y)^2$, $2(7x)(2y) = 28xy$. ✓
=> $49x^2 + 28xy + 4y^2 = (7x + 2y)^2$

(iv) $64p^2 + (32/3)pq + (4/9)q^2$
$64p^2 = (8p)^2$, $4q^2/9 = (2q/3)^2$, $2(8p)(2q/3) = 32pq/3$. ✓
=> $64p^2 + (32/3)pq + (4/9)q^2 = (8p + 2q/3)^2$

*(v) $3a^2 + 4ab + (4/3)b^2$
Take out common factor 3: $3a^2 + 4ab + (4/3)b^2 = 3[a^2 + (4/3)ab + (4/9)b^2]$.
Inside the bracket: $a^2 = (a)^2$, $(4/9)b^2 = (2b/3)^2$, $2(a)(2b/3) = 4ab/3$. ✓
So $[a^2 + (4/3)ab + (4/9)b^2] = (a + 2b/3)^2$.
=> $3a^2 + 4ab + (4/3)b^2 = 3(a + 2b/3)^2$

*(vi) $(9/5)s^2 + 6sv + 5v^2$
Take out common factor 1/5: $(9/5)s^2 + 6sv + 5v^2 = (1/5)[9s^2 + 30sv + 25v^2]$.
Inside: $9s^2 = (3s)^2$, $25v^2 = (5v)^2$, $2(3s)(5v) = 30sv$. ✓
So $[9s^2 + 30sv + 25v^2] = (3s + 5v)^2$.
=> $(9/5)s^2 + 6sv + 5v^2 = (1/5)(3s + 5v)^2$
(i) $(3x + 4y)^2$ (ii) $(2s + 5t)^2$ (iii) $(7x + 2y)^2$ (iv) $(8p + 2q/3)^2$ (v) $3(a + 2b/3)^2$ (vi) $(1/5)(3s + 5v)^2$

Q2 Find the values of the following using the identity $(a - b)^2 = a^2 - 2ab + b^2$: (i) $(79)^2$ (ii) $(193)^2$ (iii) $(299)^2$

SOLUTION

Use $(a - b)^2 = a^2 - 2ab + b^2$ by writing each number as a round number minus a small number.

(i) $(79)^2$
Write $79 = 80 - 1$, so $a = 80$, $b = 1$.
$(80 - 1)^2 = 80^2 - 2(80)(1) + 1^2$
$= 6400 - 160 + 1$
$= 6241$

(ii) $(193)^2$

Write $193 = 200 - 7$, so $a = 200$, $b = 7$.

$(200 - 7)^2 = 200^2 - 2(200)(7) + 7^2$

$= 40000 - 2800 + 49$

$= 37249$

(iii) $(299)^2$

Write $299 = 300 - 1$, so $a = 300$, $b = 1$.

$(300 - 1)^2 = 300^2 - 2(300)(1) + 1^2$

$= 90000 - 600 + 1$

$= 89401$

(i) 6241 (ii) 37249 (iii) 89401

Exercise 4.3 — More Identities — $(a + b + c)^2$ and $(a - b)^2$

Q1 Find the following squares using one of the above identities. Determine which identity makes each calculation easier. (i) 117^2 (ii) 78^2 (iii) 198^2 (iv) 214^2 (v) 1104^2 (vi) 1120^2

SOLUTION

(i) 117^2

Use $(a - b)^2$ with $117 = 120 - 3$: $a = 120$, $b = 3$.

$117^2 = (120 - 3)^2 = 120^2 - 2(120)(3) + 3^2$

$= 14400 - 720 + 9$

$= 13689$

(ii) 78^2

Use $(a - b)^2$ with $78 = 80 - 2$: $a = 80$, $b = 2$.

$78^2 = (80 - 2)^2 = 80^2 - 2(80)(2) + 2^2$

$= 6400 - 320 + 4$

$= 6084$

(iii) 198^2

Use $(a - b)^2$ with $198 = 200 - 2$: $a = 200$, $b = 2$.

$198^2 = (200 - 2)^2 = 200^2 - 2(200)(2) + 2^2$

$= 40000 - 800 + 4$

$= 39204$

(iv) 214^2

Use $(a + b)^2$ with $214 = 200 + 14$: $a = 200$, $b = 14$.

$214^2 = (200 + 14)^2 = 200^2 + 2(200)(14) + 14^2$

$= 40000 + 5600 + 196$

$= 45796$

(v) 1104^2

Use $(a + b)^2$ with $1104 = 1100 + 4$: $a = 1100$, $b = 4$.

$1104^2 = (1100 + 4)^2 = 1100^2 + 2(1100)(4) + 4^2$

$= 1210000 + 8800 + 16$

$= 1218816$

(vi) 1120^2

Use $(a + b)^2$ with $1120 = 1100 + 20$: $a = 1100$, $b = 20$.

$1120^2 = (1100 + 20)^2 = 1100^2 + 2(1100)(20) + 20^2$

$= 1210000 + 44000 + 400$

$= 1254400$

(i) 13689 (ii) 6084 (iii) 39204 (iv) 45796 (v) 1218816 (vi) 1254400

Q2 Factor using suitable identities: (i) $16y^2 - 24y + 9$ (ii) $(9/4)s^2 + 6st + 4t^2$ (iii) $m^2/9 + mk/3 + k^2/4 + 3nk + 2mn + 9n^2$ (iv) $p^2/16 - 2 + 16/p^2$ (v) $9a^2 + 4b^2 + c^2 - 12ab + 6ac - 4bc$

SOLUTION

(i) $16y^2 - 24y + 9$
$16y^2 = (4y)^2$, $9 = 3^2$, $2(4y)(3) = 24y$. Recognise $(a - b)^2$ with $a = 4y$, $b = 3$.
$\Rightarrow 16y^2 - 24y + 9 = (4y - 3)^2$

(ii) $(9/4)s^2 + 6st + 4t^2$
$(9/4)s^2 = (3s/2)^2$, $4t^2 = (2t)^2$, $2(3s/2)(2t) = 6st$. ✓
Use $(a + b)^2$ with $a = 3s/2$, $b = 2t$.
$\Rightarrow (9/4)s^2 + 6st + 4t^2 = (3s/2 + 2t)^2$

(iii) $m^2/9 + mk/3 + k^2/4 + 3nk + 2mn + 9n^2$
Group as three squares and three cross terms to match $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$.
Try $a = m/3$, $b = k/2$, $c = 3n$:
$a^2 = m^2/9$, $b^2 = k^2/4$, $c^2 = 9n^2$, $2ab = mk/3$, $2bc = 3kn$, $2ca = 2mn$. ✓
$\Rightarrow m^2/9 + mk/3 + k^2/4 + 3nk + 2mn + 9n^2 = (m/3 + k/2 + 3n)^2$

(iv) $p^2/16 - 2 + 16/p^2$
$(p/4)^2 = p^2/16$, $(4/p)^2 = 16/p^2$, $2(p/4)(4/p) = 2$. Recognise $(a - b)^2$ with $a = p/4$, $b = 4/p$.
$\Rightarrow p^2/16 - 2 + 16/p^2 = (p/4 - 4/p)^2$

(v) $9a^2 + 4b^2 + c^2 - 12ab + 6ac - 4bc$
Match with $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$.
Try $a = 3a$, $b = -2b$, $c = c$:
$(3a)^2 = 9a^2$, $(-2b)^2 = 4b^2$, $c^2 = c^2$
$2(3a)(-2b) = -12ab$, $2(3a)(c) = 6ac$, $2(-2b)(c) = -4bc$. ✓
$\Rightarrow 9a^2 + 4b^2 + c^2 - 12ab + 6ac - 4bc = (3a - 2b + c)^2$
(i) $(4y - 3)^2$ (ii) $(3s/2 + 2t)^2$ (iii) $(m/3 + k/2 + 3n)^2$ (iv) $(p/4 - 4/p)^2$ (v) $(3a - 2b + c)^2$

Q3 Expand the following using the identity $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$: (i) $(p + 3q + 7r)^2$ (ii) $(3x - 2y + 4z)^2$

SOLUTION

(i) $(p + 3q + 7r)^2$
Here $a = p$, $b = 3q$, $c = 7r$.
$(p + 3q + 7r)^2 = p^2 + (3q)^2 + (7r)^2 + 2(p)(3q) + 2(3q)(7r) + 2(7r)(p)$
$= p^2 + 9q^2 + 49r^2 + 6pq + 42qr + 14pr$

(ii) $(3x - 2y + 4z)^2$
Here $a = 3x$, $b = -2y$, $c = 4z$.
$(3x - 2y + 4z)^2 = (3x)^2 + (-2y)^2 + (4z)^2 + 2(3x)(-2y) + 2(-2y)(4z) + 2(4z)(3x)$
$= 9x^2 + 4y^2 + 16z^2 - 12xy - 16yz + 24xz$
(i) $p^2 + 9q^2 + 49r^2 + 6pq + 42qr + 14pr$ (ii) $9x^2 + 4y^2 + 16z^2 - 12xy - 16yz + 24xz$

Q4 Is this an identity? $(a + b - c)^2 + (a - b + c)^2 + (a - b - c)^2 = 2a^2 + 2b^2 + 2c^2$

SOLUTION

Expand each square on the left-hand side (LHS).

$(a + b - c)^2 = a^2 + b^2 + c^2 + 2ab - 2ac - 2bc$
$(a - b + c)^2 = a^2 + b^2 + c^2 - 2ab + 2ac - 2bc$
$(a - b - c)^2 = a^2 + b^2 + c^2 - 2ab - 2ac + 2bc$

Sum the three expansions:

$$\begin{aligned}\text{LHS} &= (a^2 + b^2 + c^2 + 2ab - 2ac - 2bc) \\ &\quad + (a^2 + b^2 + c^2 - 2ab + 2ac - 2bc) \\ &\quad + (a^2 + b^2 + c^2 - 2ab - 2ac + 2bc) \\ &= 3a^2 + 3b^2 + 3c^2 + (2ab - 2ab - 2ab) + (-2ac + 2ac - 2ac) + (-2bc - 2bc + 2bc) \\ &= 3a^2 + 3b^2 + 3c^2 - 2ab - 2ac - 2bc\end{aligned}$$

$\text{RHS} = 2a^2 + 2b^2 + 2c^2$

$\text{LHS} - \text{RHS} = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc \neq 0$ in general.
Counterexample: $a = 1, b = 1, c = 1$ gives $\text{LHS} = 1 + 1 + 1 = 3$, $\text{RHS} = 6$. $\text{LHS} \neq \text{RHS}$.

Therefore, this is NOT an identity.
No, this is not an identity. $\text{LHS} = 3a^2 + 3b^2 + 3c^2 - 2ab - 2ac - 2bc \neq 2a^2 + 2b^2 + 2c^2$ in general.

Exercise 4.4 — Factorisation Using $(x + a)(x + b) = x^2 + (a + b)x + ab$ and Related Identities

Q1 Fill in the blanks to complete the following identities: (i) $s^2 - 11s + 24 = (__)(__)$ (ii) $(__)(x + 1) = 3x^2 - 4x - 7$ (iii) $10x^2 - 11x - 6 = (2x - __)(__ + 2)$ (iv) $6x^2 + 7x + 2 = (__)(__)$

SOLUTION

(i) $s^2 - 11s + 24 = (__)(__)$
Use $(x + a)(x + b) = x^2 + (a + b)x + ab$ with $x = s$.
Need $a + b = -11$ and $ab = 24$.
Try $a = -3$, $b = -8$: $a + b = -11$ ✓, $ab = 24$ ✓.
=> $s^2 - 11s + 24 = (s - 3)(s - 8)$

(ii) $(__)(x + 1) = 3x^2 - 4x - 7$
Leading coefficient is 3, so the missing factor has leading term $3x$.
Try $(3x - 7)(x + 1) = 3x^2 + 3x - 7x - 7 = 3x^2 - 4x - 7$. ✓
=> $(3x - 7)(x + 1) = 3x^2 - 4x - 7$

(iii) $10x^2 - 11x - 6 = (2x - __)(__ + 2)$
The blank in the second factor must start with $5x$ to get leading term $2x \times 5x = 10x^2$.
Try $(2x - a)(5x + 2)$: product $= 10x^2 + 4x - 5ax - 2a = 10x^2 + (4 - 5a)x - 2a$.
Match coefficients: $4 - 5a = -11$ => $a = 3$. Check constant: $-2(3) = -6$. ✓
=> $10x^2 - 11x - 6 = (2x - 3)(5x + 2)$

(iv) $6x^2 + 7x + 2 = (__)(__)$
Need two numbers with product $6 \times 2 = 12$ and sum 7: 3 and 4.
Split: $6x^2 + 3x + 4x + 2 = 3x(2x + 1) + 2(2x + 1) = (3x + 2)(2x + 1)$.
=> $6x^2 + 7x + 2 = (2x + 1)(3x + 2)$
(i) $(s - 3)(s - 8)$ (ii) $(3x - 7)(x + 1)$ (iii) $(2x - 3)(5x + 2)$ (iv) $(2x + 1)(3x + 2)$

Q2 Select and use the identity that will help you to find the following products without multiplying directly: (i) $(41)^2$ (ii) $(27)^2$ (iii) 23×17 (iv) $(135)^2$ (v) $(97)^2$ (vi) 18×29 (vii) 34×43 (viii) $(205)^2$

SOLUTION

(i) $(41)^2$
Write $41 = 40 + 1$. Use $(a + b)^2$ with $a = 40$, $b = 1$.
$(40 + 1)^2 = 1600 + 80 + 1 = 1681$

(ii) $(27)^2$
Write $27 = 30 - 3$. Use $(a - b)^2$ with $a = 30$, $b = 3$.
$(30 - 3)^2 = 900 - 180 + 9 = 729$

(iii) 23×17
Write $23 = 20 + 3$, $17 = 20 - 3$. Use $(a + b)(a - b) = a^2 - b^2$ with $a = 20$, $b = 3$.
$23 \times 17 = (20 + 3)(20 - 3) = 400 - 9 = 391$

(iv) $(135)^2$
Write $135 = 130 + 5$. Use $(a + b)^2$ with $a = 130$, $b = 5$.

$(130 + 5)^2 = 16900 + 1300 + 25 = 18225$

(v) $(97)^2$
Write $97 = 100 - 3$. Use $(a - b)^2$ with $a = 100$, $b = 3$.
$(100 - 3)^2 = 10000 - 600 + 9 = 9409$

(vi) 18×29
Write $29 = 30 - 1$. Use distributive: $18 \times (30 - 1) = 18 \times 30 - 18 \times 1$.
$= 540 - 18 = 522$

(vii) 34×43
Write $43 = 40 + 3$. Use distributive: $34 \times (40 + 3) = 34 \times 40 + 34 \times 3$.
$= 1360 + 102 = 1462$

(viii) $(205)^2$
Write $205 = 200 + 5$. Use $(a + b)^2$ with $a = 200$, $b = 5$.
$(200 + 5)^2 = 40000 + 2000 + 25 = 42025$
(i) 1681 (ii) 729 (iii) 391 (iv) 18225 (v) 9409 (vi) 522 (vii) 1462 (viii) 42025

Q3 Factor the following: (i) $9a^2 + b^2 + 4c^2 - 6ab + 12ac - 4bc$ (ii) $16s^2 + 25t^2 - 40st$ (iii) $r^2 - r - 42$ (iv) $49g^2 + 14gh + h^2$ (v) $64u^2 + 121v^2 + 4w^2 - 176uv - 32uw + 44vw$

SOLUTION

(i) $9a^2 + b^2 + 4c^2 - 6ab + 12ac - 4bc$
Match with $(p + q + r)^2 = p^2 + q^2 + r^2 + 2pq + 2qr + 2pr$.
Try $p = 3a$, $q = -b$, $r = 2c$:
$p^2 = 9a^2$, $q^2 = b^2$, $r^2 = 4c^2$
$2pq = 2(3a)(-b) = -6ab$, $2pr = 2(3a)(2c) = 12ac$, $2qr = 2(-b)(2c) = -4bc$. ✓
$\Rightarrow 9a^2 + b^2 + 4c^2 - 6ab + 12ac - 4bc = (3a - b + 2c)^2$

(ii) $16s^2 + 25t^2 - 40st$
$16s^2 = (4s)^2$, $25t^2 = (5t)^2$, $2(4s)(5t) = 40st$. Recognise $(a - b)^2$ with $a = 4s$, $b = 5t$.
$\Rightarrow 16s^2 + 25t^2 - 40st = (4s - 5t)^2$

(iii) $r^2 - r - 42$
Use $(x + a)(x + b) = x^2 + (a + b)x + ab$ with $x = r$.
Need $a + b = -1$ and $ab = -42$.
Try $a = 6$, $b = -7$: $a + b = -1$ ✓, $ab = -42$ ✓.
$\Rightarrow r^2 - r - 42 = (r + 6)(r - 7)$

(iv) $49g^2 + 14gh + h^2$
$49g^2 = (7g)^2$, $h^2 = h^2$, $2(7g)(h) = 14gh$. Recognise $(a + b)^2$ with $a = 7g$, $b = h$.
$\Rightarrow 49g^2 + 14gh + h^2 = (7g + h)^2$

(v) $64u^2 + 121v^2 + 4w^2 - 176uv - 32uw + 44vw$
Match with $(p + q + r)^2 = p^2 + q^2 + r^2 + 2pq + 2qr + 2pr$.
Try $p = 8u$, $q = -11v$, $r = -2w$:

$p^2 = 64u^2$, $q^2 = 121v^2$, $r^2 = 4w^2$

$2pq = 2(8u)(-11v) = -176uv$, $2pr = 2(8u)(-2w) = -32uw$, $2qr = 2(-11v)(-2w) = 44vw$. ✓

=> $64u^2 + 121v^2 + 4w^2 - 176uv - 32uw + 44vw = (8u - 11v - 2w)^2$

(i) $(3a - b + 2c)^2$ (ii) $(4s - 5t)^2$ (iii) $(r + 6)(r - 7)$ (iv) $(7g + h)^2$ (v) $(8u - 11v - 2w)^2$

Exercise 4.5 — Simplification of Rational Expressions Using Algebraic Identities

Q1 Simplify the following rational expressions assuming that the expressions in the denominators are not equal to zero: (i) $(3p^2 - 3pq - 18q^2) / (p^2 + 3pq - 10q^2)$ (ii) $(n^3 - 3n^2m + 3nm^2 - m^3) / (5m^2 - 10mn + 5n^2)$ (iii) $(w^3 - v^3 + x^3 + 3wvx) / (w^2 + v^2 + x^2 - 2wv - 2vx + 2wx)$ (iv) $(4y^2 - 20yz + 25z^2) / (25z^2 - 4y^2)$ (v) $(x^2 + x - 6)(x^2 - 7x + 12) / ((x^2 - 6x + 8)(x^2 - 9))$ (vi) $(p^4 - 16) / (p^2 - 4p + 4)$

SOLUTION

(i) $(3p^2 - 3pq - 18q^2) / (p^2 + 3pq - 10q^2)$
Factor the numerator: $3p^2 - 3pq - 18q^2 = 3(p^2 - pq - 6q^2)$.
$p^2 - pq - 6q^2$: need factors of -6 summing to -1: -3 and $+2$.
$(p - 3q)(p + 2q) = p^2 + 2pq - 3pq - 6q^2 = p^2 - pq - 6q^2$. ✓
So numerator = $3(p - 3q)(p + 2q)$.

Factor the denominator: $p^2 + 3pq - 10q^2$.
Need factors of -10 summing to 3: $+5$ and -2. => $(p + 5q)(p - 2q)$. ✓

No common factors between numerator and denominator.
=> $3(p - 3q)(p + 2q) / ((p - 2q)(p + 5q))$

(ii) $(n^3 - 3n^2m + 3nm^2 - m^3) / (5m^2 - 10mn + 5n^2)$
Numerator: $n^3 - 3n^2m + 3nm^2 - m^3 = (n - m)^3$.
. [Identity: $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$, here $a = n$, $b = m$]
Denominator: $5m^2 - 10mn + 5n^2 = 5(m^2 - 2mn + n^2) = 5(m - n)^2$.

Note $(n - m) = -(m - n)$, so $(n - m)^3 = -(m - n)^3$.
Expression = $-(m - n)^3 / [5(m - n)^2] = -(m - n)/5 = (n - m)/5$.

(iii) $(w^3 - v^3 + x^3 + 3wvx) / (w^2 + v^2 + x^2 - 2wv - 2vx + 2wx)$
Use the identity: $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$.
Set $a = w$, $b = -v$, $c = x$:
$a^3 + b^3 + c^3 = w^3 - v^3 + x^3$, and $-3abc = -3 \cdot w \cdot (-v) \cdot x = 3wvx$. ✓
Numerator = $(w - v + x)(w^2 + v^2 + x^2 + wv + vx - wx)$.

Denominator: check $(w - v + x)^2 = w^2 + v^2 + x^2 - 2wv + 2wx - 2vx$. ✓

Result: $(w - v + x)(w^2 + v^2 + x^2 + wv + vx - wx) / (w - v + x)^2$
$= (w^2 + v^2 + x^2 + wv + vx - wx) / (w - v + x)$.

(iv) $(4y^2 - 20yz + 25z^2) / (25z^2 - 4y^2)$
Numerator: $(2y)^2 - 2(2y)(5z) + (5z)^2 = (2y - 5z)^2$.
Denominator: $(5z)^2 - (2y)^2 = (5z - 2y)(5z + 2y)$. [difference of squares]
Note $(2y - 5z)^2 = (5z - 2y)^2$.
Result: $(5z - 2y)^2 / [(5z - 2y)(5z + 2y)] = (5z - 2y) / (5z + 2y)$.

(v) $(x^2 + x - 6)(x^2 - 7x + 12) / ((x^2 - 6x + 8)(x^2 - 9))$
Factor each:
$x^2 + x - 6 = (x + 3)(x - 2)$

$x^2 - 7x + 12 = (x - 3)(x - 4)$
$x^2 - 6x + 8 = (x - 2)(x - 4)$
$x^2 - 9 = (x - 3)(x + 3)$

Numerator = $(x + 3)(x - 2)(x - 3)(x - 4)$.
Denominator = $(x - 2)(x - 4)(x - 3)(x + 3)$.
All four factors cancel. Result = 1.

(vi) $(p^4 - 16) / (p^2 - 4p + 4)$
Numerator: $p^4 - 16 = (p^2)^2 - 4^2 = (p^2 - 4)(p^2 + 4) = (p - 2)(p + 2)(p^2 + 4)$.
Denominator: $p^2 - 4p + 4 = (p - 2)^2$.

Result: $(p - 2)(p + 2)(p^2 + 4) / (p - 2)^2$
$= (p + 2)(p^2 + 4) / (p - 2)$.
(i) $3(p-3q)(p+2q)/((p-2q)(p+5q))$ (ii) $(n-m)/5$ (iii) $(w^2+v^2+x^2+wv+vx-wx)/(w-v+x)$ (iv) $(5z-2y)/(5z+2y)$ (v) 1 (vi) $(p+2)(p^2+4)/(p-2)$

Exercises — Exploring Algebraic Identities

Q1 Use suitable identities to find the following products: (i) $(-3x + 4)^2$ (ii) $(2s + 7)(2s - 7)$ (iii) $(p^2 + 1/2)(p^2 - 1/2)$ (iv) $(2n + 7)(2n - 7)$ (v) $(s - 2t)(s^2 + 2st + 4t^2)$ (vi) $(1/(2r) - 4r)^2$ (vii) $(-3m + 4k - l)^2$ (viii) $(x - y/3)^3$ (ix) $((7/2)k - (2/3)m)^3$

SOLUTION

(i) Using $(a + b)^2 = a^2 + 2ab + b^2$ with $a = -3x$, $b = 4$:
$(-3x + 4)^2 = (-3x)^2 + 2(-3x)(4) + 4^2 = 9x^2 - 24x + 16$.

(ii) Using $(a + b)(a - b) = a^2 - b^2$ with $a = 2s$, $b = 7$:
$(2s + 7)(2s - 7) = (2s)^2 - 7^2 = 4s^2 - 49$.

(iii) Using $(a + b)(a - b) = a^2 - b^2$ with $a = p^2$, $b = 1/2$:
$(p^2 + 1/2)(p^2 - 1/2) = (p^2)^2 - (1/2)^2 = p^4 - 1/4$.

(iv) Using $(a + b)(a - b) = a^2 - b^2$ with $a = 2n$, $b = 7$:
$(2n + 7)(2n - 7) = (2n)^2 - 7^2 = 4n^2 - 49$.

(v) Using $(a - b)(a^2 + ab + b^2) = a^3 - b^3$ with $a = s$, $b = 2t$:
$(s - 2t)(s^2 + 2st + 4t^2) = s^3 - (2t)^3 = s^3 - 8t^3$.

(vi) Using $(a - b)^2 = a^2 - 2ab + b^2$ with $a = 1/(2r)$, $b = 4r$:
$(1/(2r) - 4r)^2 = (1/(2r))^2 - 2\cdot(1/(2r))\cdot(4r) + (4r)^2$
$= 1/(4r^2) - 4 + 16r^2$.

(vii) Using $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$ with $a = -3m$, $b = 4k$, $c = -l$:
$= 9m^2 + 16k^2 + l^2 + 2(-3m)(4k) + 2(4k)(-l) + 2(-l)(-3m)$
$= 9m^2 + 16k^2 + l^2 - 24km - 8kl + 6lm$.

(viii) Using $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$ with $a = x$, $b = y/3$:
$(x - y/3)^3 = x^3 - 3x^2(y/3) + 3x(y/3)^2 - (y/3)^3$
$= x^3 - x^2y + xy^2/3 - y^3/27$.

(ix) Using $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$ with $a = (7/2)k$, $b = (2/3)m$:
$= (7/2\ k)^3 - 3(7/2\ k)^2(2/3\ m) + 3(7/2\ k)(2/3\ m)^2 - (2/3\ m)^3$
$= 343k^3/8 - 49k^2m/2 + 14km^2/3 - 8m^3/27$.

(i) $9x^2-24x+16$ (ii) $4s^2-49$ (iii) $p^4-1/4$ (iv) $4n^2-49$ (v) s^3-8t^3 (vi) $16r^2-4+1/(4r^2)$ (vii) $9m^2+16k^2+l^2-24km-8kl+6lm$ (viii) $x^3-x^2y+xy^2/3-y^3/27$ (ix) $343k^3/8-49k^2m/2+14km^2/3-8m^3/27$

Q2 Find the values using suitable identities: (i) 17×21 (ii) 104×96 (iii) 24×16 (iv) 147^3 (v) 199^3 (vi) 127^3 (vii) $(-107)^3$ (viii) $(-299)^3$

SOLUTION

(i) $17 \times 21 = (19 - 2)(19 + 2) = 19^2 - 2^2 = 361 - 4 = 357$.

(ii) $104 \times 96 = (100 + 4)(100 - 4) = 100^2 - 4^2 = 10000 - 16 = 9984$.

(iii) $24 \times 16 = (20 + 4)(20 - 4) = 20^2 - 4^2 = 400 - 16 = 384$.

(iv) $147^3 = (150 - 3)^3$. Using $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$:
$= 150^3 - 3(150^2)(3) + 3(150)(3^2) - 3^3$
$= 3375000 - 202500 + 4050 - 27 = 3176523$.

(v) $199^3 = (200 - 1)^3 = 200^3 - 3(200^2)(1) + 3(200)(1^2) - 1^3$
$= 8000000 - 120000 + 600 - 1 = 7880599$.

(vi) $127^3 = (130 - 3)^3 = 130^3 - 3(130^2)(3) + 3(130)(3^2) - 3^3$
$= 2197000 - 152100 + 3510 - 27 = 2048383$.

(vii) $(-107)^3 = -(107^3)$. $107^3 = (100 + 7)^3 = 100^3 + 3(100^2)(7) + 3(100)(7^2) + 7^3$
$= 1000000 + 210000 + 14700 + 343 = 1225043$. So $(-107)^3 = -1225043$.

(viii) $(-299)^3 = -(299^3)$. $299^3 = (300 - 1)^3 = 300^3 - 3(300^2)(1) + 3(300)(1^2) - 1^3$
$= 27000000 - 270000 + 900 - 1 = 26730899$. So $(-299)^3 = -26730899$.
(i) 357 (ii) 9984 (iii) 384 (iv) 3176523 (v) 7880599 (vi) 2048383 (vii) –1225043 (viii) –26730899

Q3 Factor the following algebraic expressions: (i) $4y^2 + 1 + 1/(16y^2)$ (ii) $9m^2 - 1/(25n^2)$ (iii) $27b^3 - 1/(64b^3)$ (iv) $x^2 + 5x/6 + 1/6$ (v) $27u^3 - 1/125 - 27u^2/5 + 9u/25$ (vi) $64y^3 + z^3/125$ (vii) $p^3 + 27q^3 + r^3 - 9pqr$ (viii) $9m^2 - 12m + 4$ (ix) $9x^3 - (8/3)y^3 + z^3/3 + 6xyz$ (x) $4x^2 + 9y^2 + 36z^2 + 12xz + 36yz + 24xy$ (xi) $27u^3 - 1/216 - 9u^2/2 + u/4$

SOLUTION

(i) $4y^2 + 1 + 1/(16y^2) = (2y)^2 + 2\cdot(2y)\cdot(1/(4y)) + (1/(4y))^2$
$= (2y + 1/(4y))^2$, using $a^2 + 2ab + b^2$.

(ii) $9m^2 - 1/(25n^2) = (3m)^2 - (1/(5n))^2 = (3m - 1/(5n))(3m + 1/(5n))$, using $a^2 - b^2$.

(iii) $27b^3 - 1/(64b^3) = (3b)^3 - (1/(4b))^3$. Using $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$:
$= (3b - 1/(4b))(9b^2 + 3/4 + 1/(16b^2))$.

(iv) $x^2 + 5x/6 + 1/6$: need $a + b = 5/6$ and $ab = 1/6$. Take $a = 1/2$, $b = 1/3$.
$x^2 + 5x/6 + 1/6 = (x + 1/2)(x + 1/3)$.

(v) $27u^3 - 27u^2/5 + 9u/25 - 1/125 = (3u)^3 - 3(3u)^2(1/5) + 3(3u)(1/5)^2 - (1/5)^3$
$= (3u - 1/5)^3$, using $(a - b)^3$.

(vi) $64y^3 + z^3/125 = (4y)^3 + (z/5)^3$. Using $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$:
$= (4y + z/5)(16y^2 - 4yz/5 + z^2/25)$.

(vii) $p^3 + 27q^3 + r^3 - 9pqr = p^3 + (3q)^3 + r^3 - 3(p)(3q)(r)$.
Using $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$ with $a = p$, $b = 3q$, $c = r$:
$= (p + 3q + r)(p^2 + 9q^2 + r^2 - 3pq - 3qr - pr)$.

(viii) $9m^2 - 12m + 4 = (3m)^2 - 2(3m)(2) + 2^2 = (3m - 2)^2$, using $a^2 - 2ab + b^2$.

(ix) Take out 1/3: $9x^3 - (8/3)y^3 + z^3/3 + 6xyz = (1/3)(27x^3 - 8y^3 + z^3 + 18xyz)$.
Now $27x^3 - 8y^3 + z^3 + 18xyz = (3x)^3 + (-2y)^3 + z^3 - 3(3x)(-2y)(z)$.
Using $a^3 + b^3 + c^3 - 3abc$ with $a = 3x$, $b = -2y$, $c = z$:

= (3x – 2y + z)(9x² + 4y² + z² + 6xy + 2yz – 3xz).
So the full factorisation is (1/3)(3x – 2y + z)(9x² + 4y² + z² + 6xy + 2yz – 3xz).

(x) The expression is 4x² + 9y² + 36z² + 12xz + 36yz + 24xy. Note: as printed, the intended perfect square is (2x + 3y + 6z)², whose expansion is (2x)² + (3y)² + (6z)² + 2(2x)(3y) + 2(3y)(6z) + 2(2x)(6z) = 4x² + 9y² + 36z² + 12xy + 36yz + 24xz. So the intended cross-terms are 24xz and 12xy (the textbook's 12xz + 24xy appears to be a print swap of these two coefficients).
Using (a + b + c)² with a = 2x, b = 3y, c = 6z:
= (2x + 3y + 6z)².

(xi) 27u³ – 9u²/2 + u/4 – 1/216 = (3u)³ – 3(3u)²(1/6) + 3(3u)(1/6)² – (1/6)³
= (3u – 1/6)³, using (a – b)³.
(i) (2y + 1/(4y))² (ii) (3m – 1/(5n))(3m + 1/(5n)) (iii) (3b – 1/(4b))(9b² + 3/4 + 1/(16b²)) (iv) (x + 1/2)(x + 1/3) (v) (3u – 1/5)³ (vi) (4y + z/5)(16y² – 4yz/5 + z²/25) (vii) (p + 3q + r)(p² + 9q² + r² – 3pq – 3qr – pr) (viii) (3m – 2)² (ix) (1/3)(3x – 2y + z)(9x² + 4y² + z² + 6xy + 2yz – 3xz) (x) (2x + 3y + 6z)² (xi) (3u – 1/6)³

Q4 Simplify the following (assume denominators are not equal to 0): (i) (4x² + 4x + 1)/(4x² – 1) (ii) 9(3a³ – 24b³)/(9a² – 36b²) (iii) (s³ + 125t³)/(s² – 2st – 35t²)

SOLUTION

(i) Numerator: 4x² + 4x + 1 = (2x + 1)². Denominator: 4x² – 1 = (2x – 1)(2x + 1).
So (4x² + 4x + 1)/(4x² – 1) = (2x + 1)² / [(2x – 1)(2x + 1)].
Cancel the common factor (2x + 1): = (2x + 1)/(2x – 1).

(ii) Numerator: 9(3a³ – 24b³) = 9·3(a³ – 8b³) = 27(a – 2b)(a² + 2ab + 4b²).
Denominator: 9a² – 36b² = 9(a² – 4b²) = 9(a – 2b)(a + 2b).
So the expression = 27(a – 2b)(a² + 2ab + 4b²) / [9(a – 2b)(a + 2b)].
Cancel 9 and (a – 2b): = 3(a² + 2ab + 4b²)/(a + 2b).

(iii) Numerator: s³ + 125t³ = (s + 5t)(s² – 5st + 25t²).
Denominator: s² – 2st – 35t² needs a + b = –2, ab = –35 → –7 and 5: = (s – 7t)(s + 5t).
So the expression = (s + 5t)(s² – 5st + 25t²) / [(s – 7t)(s + 5t)].
Cancel (s + 5t): = (s² – 5st + 25t²)/(s – 7t).
(i) (2x + 1)/(2x – 1) (ii) 3(a² + 2ab + 4b²)/(a + 2b) (iii) (s² – 5st + 25t²)/(s – 7t)

Q5 Find possible expressions for the length and breadth of each of the following rectangles whose areas are given by the following expressions in square units: (i) $25a^2 – 30ab + 9b^2$ (ii) $36s^2 – 49t^2$

SOLUTION

The area of a rectangle = length × breadth, so we factor the area into two factors.

(i) $25a^2 – 30ab + 9b^2 = (5a)^2 – 2(5a)(3b) + (3b)^2 = (5a – 3b)^2$.
So length = (5a – 3b) units and breadth = (5a – 3b) units (a square).

(ii) $36s^2 – 49t^2 = (6s)^2 – (7t)^2 = (6s – 7t)(6s + 7t)$.
So one possibility is length = (6s + 7t) units and breadth = (6s – 7t) units.
(i) length = breadth = (5a – 3b) units (ii) length = (6s + 7t) units, breadth = (6s – 7t) units

Q6 Find possible expressions for the length, breadth, and height of each of the following cuboids whose volumes are given by the following expressions in cubic units: (i) $6a^2 – 24b^2$ (ii) $3ps^2 – 15ps + 12p$

SOLUTION

The volume of a cuboid = length × breadth × height, so we factor the volume into three factors.

(i) $6a^2 – 24b^2 = 6(a^2 – 4b^2) = 6(a – 2b)(a + 2b)$.
So the three dimensions can be 6, (a – 2b) and (a + 2b) units.

(ii) $3ps^2 – 15ps + 12p = 3p(s^2 – 5s + 4)$. For $s^2 – 5s + 4$, a + b = –5 and ab = 4 → –1 and –4.
$= 3p(s – 1)(s – 4)$.
So the three dimensions can be 3p, (s – 1) and (s – 4) units.
(i) 6, (a – 2b), (a + 2b) units (ii) 3p, (s – 1), (s – 4) units

Q7 The village playground is shaped as a square of side 40 metres. A path of width s metres is created around the playground for people to walk. Find an expression for the area of the path in terms of s.

SOLUTION

The playground is a square of side 40 m, so its area = $40^2 = 1600$ sq. m.
The path of width s runs all around the outside of the square, so the outer figure is a larger square whose side = 40 + s + s = (40 + 2s) m.
Area of the outer square = $(40 + 2s)^2$.
Using $(a + b)^2$: $(40 + 2s)^2 = 40^2 + 2(40)(2s) + (2s)^2 = 1600 + 160s + 4s^2$.
Area of the path = (area of outer square) – (area of playground)
$= (1600 + 160s + 4s^2) – 1600 = 4s^2 + 160s$ sq. m.
Area of the path = $(4s^2 + 160s)$ square metres

Q8 If a number plus its reciprocal equals 10/3, find the number.

SOLUTION

Let the number be x. Then $x + 1/x = 10/3$.
Multiply both sides by 3x: $3x^2 + 3 = 10x$, so $3x^2 – 10x + 3 = 0$.
Split the middle term: $–10x = –9x – x$, and $(–9)(–1) = 9 = 3 \times 3$ (product of $a \cdot c = 3 \cdot 3$).
$3x^2 – 9x – x + 3 = 3x(x – 3) – 1(x – 3) = (3x – 1)(x – 3) = 0$.
So $3x – 1 = 0$ or $x – 3 = 0$, giving $x = 1/3$ or $x = 3$.
Both are valid (each is the reciprocal of the other).
The number is 3 (or its reciprocal 1/3)

Q9 A rectangular pool has area $2x^2 + 7x + 3$ square hastas. If its width is $2x + 1$ hastas, find its length. (Hasta was a unit used to measure length.)

SOLUTION

Area = length × width, so length = area ÷ width = $(2x^2 + 7x + 3)/(2x + 1)$.
Factor the area: for $2x^2 + 7x + 3$, split $7x$ using product $2 \times 3 = 6 \rightarrow 6x + x$.
$2x^2 + 7x + 3 = 2x^2 + 6x + x + 3 = 2x(x + 3) + 1(x + 3) = (2x + 1)(x + 3)$.
So length = $(2x + 1)(x + 3)/(2x + 1) = x + 3$.
Cancelling the common factor $(2x + 1)$.
Length = $(x + 3)$ hastas

Q10 If both $x – 2$ and $x – 1/2$ are factors of $px^2 + 5x + r$, show that $p = r$.

SOLUTION

Let $f(x) = px^2 + 5x + r$.
Since $x – 2$ is a factor, $f(2) = 0$: $p(2)^2 + 5(2) + r = 0 \Rightarrow 4p + 10 + r = 0$. …(1)
Since $x – 1/2$ is a factor, $f(1/2) = 0$: $p(1/2)^2 + 5(1/2) + r = 0 \Rightarrow p/4 + 5/2 + r = 0$.
Multiply this by 4: $p + 10 + 4r = 0$. …(2)
Subtract (2) from (1): $(4p + 10 + r) – (p + 10 + 4r) = 0$
$\Rightarrow 3p – 3r = 0 \Rightarrow 3p = 3r \Rightarrow p = r$.
Hence $p = r$, as required.
$p = r$ (both equal –2 here, since $4p + 10 + r = 0$ with $p = r$ gives $5p = –10$)

Q11 If $a + b + c = 5$ and $ab + bc + ca = 10$, then prove that $a^3 + b^3 + c^3 – 3abc = –25$.

SOLUTION

Use the identity $a^3 + b^3 + c^3 – 3abc = (a + b + c)(a^2 + b^2 + c^2 – ab – bc – ca)$.
First find $a^2 + b^2 + c^2$. From $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$:
$5^2 = a^2 + b^2 + c^2 + 2(10) \Rightarrow 25 = a^2 + b^2 + c^2 + 20 \Rightarrow a^2 + b^2 + c^2 = 5$.
Now $a^2 + b^2 + c^2 – ab – bc – ca = 5 – 10 = –5$.
So $a^3 + b^3 + c^3 – 3abc = (a + b + c)(a^2 + b^2 + c^2 – ab – bc – ca) = 5 \times (–5) = –25$.

Hence proved.

$a^3 + b^3 + c^3 - 3abc = 5 \times (-5) = -25$

Q12 By factoring the expression, check that $n^3 - n$ is always divisible by 6 for all natural numbers n. Give reasons.

SOLUTION

Factor: $n^3 - n = n(n^2 - 1) = n(n - 1)(n + 1) = (n - 1) \cdot n \cdot (n + 1)$.

So $n^3 - n$ is the product of three consecutive natural numbers $(n - 1)$, n, $(n + 1)$.

Reason 1 (divisible by 2): Among any two consecutive integers, at least one is even, so among three consecutive integers at least one is even. Hence the product is divisible by 2.

Reason 2 (divisible by 3): Among any three consecutive integers, exactly one is a multiple of 3. Hence the product is divisible by 3.

Since the product is divisible by both 2 and 3, and 2 and 3 are coprime, it is divisible by $2 \times 3 = 6$.

Therefore $n^3 - n$ is always divisible by 6 for every natural number n.

(Check: $n = 2 \rightarrow 6$, $n = 3 \rightarrow 24$, $n = 4 \rightarrow 60$, $n = 5 \rightarrow 120$ — all multiples of 6.)

$n^3 - n = (n - 1)n(n + 1)$ is a product of 3 consecutive integers, so it is divisible by both 2 and 3, hence by 6.

Q13 Find the value of: (i) $x^3 + y^3 - 12xy + 64$, when $x + y = -4$ (ii) $x^3 - 8y^3 - 36xy - 216$, when $x = 2y + 6$

SOLUTION

Both parts use $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$.

(i) Write $x^3 + y^3 - 12xy + 64 = x^3 + y^3 + 4^3 - 3(x)(y)(4)$, so $a = x$, $b = y$, $c = 4$.

Then it equals $(x + y + 4)(x^2 + y^2 + 16 - xy - 4y - 4x)$.

Given $x + y = -4$, we have $x + y + 4 = -4 + 4 = 0$.

So the whole product $= 0 \times (\ldots) = 0$.

(ii) Write $x^3 - 8y^3 - 216 - 36xy = x^3 + (-2y)^3 + (-6)^3 - 3(x)(-2y)(-6)$, so $a = x$, $b = -2y$, $c = -6$.

[Check: $-3 \cdot x \cdot (-2y) \cdot (-6) = -36xy$, and $(-2y)^3 = -8y^3$, $(-6)^3 = -216$. ✓]

Then it equals $(x - 2y - 6)(a^2 + b^2 + c^2 - ab - bc - ca)$.

Given $x = 2y + 6$, we have $x - 2y - 6 = 0$.

So the whole product $= 0 \times (\ldots) = 0$.

(i) 0 (ii) 0