Chapter 5: I'm Up and Down, and Round and Round

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Exercise 5.1 — Circumcircle of a Triangle

Q1 Draw ΔABC with AB = 5 cm, ∠A = 70° and ∠B = 60°. Draw the circumcircle of ΔABC. Is the centre inside or outside the triangle?

SOLUTION

Construction steps:
1. Draw AB = 5 cm.
2. At A construct ∠A = 70°; at B construct ∠B = 60°. The third angle ∠C = 180° − 70° − 60° = 50°. Locate C at the intersection of the two rays.
3. Draw the perpendicular bisectors of any two sides (e.g. AB and BC). Their intersection is the circumcentre O.
4. Draw the circumcircle with centre O and radius OA.

Since all angles of ΔABC are acute (70°, 60°, 50° are all < 90°), the circumcentre lies inside the triangle.
The circumcentre is inside the triangle (all angles of the triangle are acute).

Q2 Draw ΔABC with AB = 5 cm, ∠A = 100°, AC = 4 cm. Draw the circumcircle of ΔABC. Is the centre inside or outside the triangle?

SOLUTION

Construction steps:
1. Draw AB = 5 cm.
2. At A construct ∠A = 100° and mark C on this ray such that AC = 4 cm. Join BC.
3. Draw perpendicular bisectors of AB and AC (or BC). Their intersection is the circumcentre O.
4. Draw the circumcircle with centre O and radius OA.

Since ∠A = 100° > 90° (obtuse), the circumcentre lies outside the triangle, on the opposite side

of BC from A.

The circumcentre is outside the triangle (the triangle has an obtuse angle at A).

Q3 Draw ΔABC, with AB = 6 cm, BC = 7 cm and CA = 7 cm. Draw the circumcircle of ΔABC. Let the circumcentre be O. Measure OA, OB, OC.

SOLUTION

Construction steps:

1. Draw BC = 7 cm.
2. With B as centre and radius 6 cm, draw an arc. With C as centre and radius 7 cm, draw another arc. Their intersection is A. Join AB and CA.
3. Draw perpendicular bisectors of AB and BC. Their intersection is the circumcentre O.
4. Draw the circumcircle with centre O.

On measurement: OA = OB = OC (all equal to the circumradius).

This confirms the defining property of the circumcentre: it is equidistant from all three vertices.
OA = OB = OC (all equal to the circumradius of ΔABC).

Q4 What is the least possible radius of a circle through two points A and B?

SOLUTION

Any circle passing through A and B has a chord AB. A chord of a circle is at most as long as the diameter.

The chord AB is longest relative to the circle when AB is the diameter itself. In that case the radius equals AB/2.

For any other circle through A and B, the chord AB is shorter than the diameter, so the diameter (and hence the radius) must be larger than AB.

Therefore, the minimum radius is achieved when AB is the diameter, giving r_min = AB/2.
The least possible radius is AB/2, achieved when AB is the diameter of the circle.

Exercise 5.2 — Chords and the Angles They Subtend

Q1 Show that the triangle formed by a chord and the centre of the circle is isosceles.

SOLUTION

Let the circle have centre C and let AB be any chord.
Join CA and CB.

CA = CB (both are radii of the same circle).

In triangle CAB, two sides CA and CB are equal.
Therefore, triangle CAB is isosceles.
Since CA = CB = radius, triangle CAB is isosceles with the equal sides meeting at the centre C.

Q2 Show that if two such isosceles triangles (occurring in the previous question) have equal base length, they are congruent to each other.

SOLUTION

Let the two chords be AB and DE, both of the same circle with centre C.
From Q1, triangles CAB and CDE are both isosceles with:
CA = CB = CD = CE = r (all radii of the same circle).

Given: AB = DE (equal base lengths).

Now in triangles CAB and CDE:
CA = CD = r
CB = CE = r
AB = DE (given)

By the SSS congruence criterion, triangle CAB ≅ triangle CDE.

Therefore, the two isosceles triangles are congruent.
Both triangles share equal radii as their equal sides, and the bases are equal by hypothesis; SSS congruence gives triangle CAB ≅ triangle CDE.

Exercise 5.3 — Midpoints and Perpendicular Bisectors of Chords

Q1 Can you explain why the converse to Theorem 4 is true, i.e., why does the perpendicular from the centre of a circle to a chord of the circle bisect the chord? (Hint: Use Fig. 5.12. You are told that ∠CMA = ∠CMB = 90°. You need to show that AM = BM.)

SOLUTION

Let the circle have centre C and chord AB. Let M be the foot of the perpendicular from C to AB, so ∠CMA = ∠CMB = 90°.

Consider right triangles CMA and CMB:
CA = CB = r (both radii of the same circle).
CM = CM (common side).
∠CMA = ∠CMB = 90° (given, since CM ⊥ AB).

By the RHS (Right angle–Hypotenuse–Side) congruence criterion:
triangle CMA ≅ triangle CMB.

Therefore, AM = BM (corresponding sides of congruent triangles).

This proves that the perpendicular from the centre to a chord bisects the chord.
By RHS congruence (CA = CB = radius, CM common, right angles equal), triangle CMA ≅ triangle CMB, so AM = BM.

Q2 An isosceles triangle ABC is inscribed in a circle, with AB = AC. Show that the altitude from A to BC passes through the centre of the circle.

SOLUTION

Since triangle ABC is isosceles with AB = AC, the altitude from A to BC is also the perpendicular bisector of BC.
(In an isosceles triangle, the altitude from the apex bisects the base, so the altitude from A is perpendicular to BC at its midpoint.)

Now BC is a chord of the circle.
By Theorem 5: The perpendicular from the centre O to the chord BC bisects BC.
The perpendicular bisector of a chord is unique (only one line is perpendicular to BC at its midpoint).

Therefore, the altitude from A — being the perpendicular bisector of chord BC — must pass through the centre O.
The altitude from A is the perpendicular bisector of chord BC. By Theorem 5, the perpendicular bisector of a chord passes through the centre, so the altitude from A passes through the centre O.

Q3 Two parallel chords of lengths 6 cm and 8 cm are on opposite sides of the centre of a circle. If the radius of the circle is 5 cm, find the distance between the midpoints of the chords.

SOLUTION

Let the centre be C and radius r = 5 cm.

For the chord of length 6 cm:

The perpendicular from C meets it at its midpoint M_1, so $CM_1 \perp$ chord and M_1 is the midpoint.

Half-chord = 6/2 = 3 cm.

By the Baudhāyana–Pythagoras theorem: $CM_1^2 + 3^2 = 5^2$

=> $CM_1^2 = 25 - 9 = 16$ => $CM_1 = 4$ cm.

For the chord of length 8 cm:

Half-chord = 8/2 = 4 cm.

$CM_2^2 + 4^2 = 5^2$

=> $CM_2^2 = 25 - 16 = 9$ => $CM_2 = 3$ cm.

The two chords are parallel and on opposite sides of the centre.

The midpoints M_1 and M_2 lie on the line through C perpendicular to both chords (since the chords are parallel, their perpendicular bisectors coincide on this line).

Since M_1 and M_2 are on opposite sides of C:

Distance between midpoints = $CM_1 + CM_2 = 4 + 3 = 7$ cm.

The distance between the midpoints of the chords is 7 cm.

Exercise 5.4 — Distance of Chords from the Centre

Q1 Use the Baudhāyana–Pythagoras theorem to show why Theorem 6 must be true. (Theorem 6: Chords of a circle having the same length are all at the same distance from the centre.)

SOLUTION

Let the circle have centre C and radius r.
Let AB and FG be two chords with AB = FG.
Let E be the midpoint of AB and H be the midpoint of FG.
By Theorem 5, CE ⊥ AB and CH ⊥ FG, so CE and CH are the distances from C to the two chords.

In right triangle CEA:
$CA^2 = CE^2 + EA^2$ (Baudhāyana–Pythagoras theorem)

In right triangle CHF:
$CF^2 = CH^2 + HF^2$ (Baudhāyana–Pythagoras theorem)

Now: CA = CF = r (both radii).
Also: EA = AB/2 = FG/2 = HF (since E and H are midpoints and AB = FG).

Therefore: $CE^2 + EA^2 = CH^2 + HF^2$
=> $CE^2 = CH^2$
=> CE = CH.

Hence, the two equal chords AB and FG are the same distance from the centre C.
By Pythagoras, $CE^2 = CA^2 - EA^2 = r^2 - (AB/2)^2$ and $CH^2 = CF^2 - HF^2 = r^2 - (FG/2)^2$. Since r is the same and AB = FG, $CE^2 = CH^2$, so CE = CH.

Q2 Consider Fig. 5.15. If CE is perpendicular to AB, CH is perpendicular to GF, and CE = CH, show that AB = GF. (Congruence method)

SOLUTION

(Note: the textbook prints 'CH ⊥ GH'; from Fig. 5.15, H is the foot of the perpendicular on chord GF, so the intended condition is CH ⊥ GF, used below.)
Given: Circle with centre C; CE ⊥ AB at E; CH ⊥ GF at H; CE = CH.

By Theorem 5, the perpendicular from the centre bisects the chord.
So E is the midpoint of AB and H is the midpoint of GF.
=> AE = AB/2 and GH = GF/2.

Consider right triangles CEA and CHG:
∠CEA = ∠CHG = 90° (given, CE ⊥ AB and CH ⊥ GF).
CA = CG = r (both radii of the same circle).
CE = CH (given).

By the RHS (Right angle–Hypotenuse–Side) congruence criterion:
triangle CEA ≅ triangle CHG.

Therefore, AE = GH (corresponding sides).
=> AB = 2 × AE = 2 × GH = GF.

Hence AB = GF.
By RHS congruence (right angles at E and H, equal radii CA = CG, and CE = CH given), triangle CEA ≅ triangle CHG, giving AE = GH, and hence AB = GF.

Q3 Solve the previous question using the Baudhāyana–Pythagoras theorem.

SOLUTION

Given: CE ⊥ AB with E the midpoint of AB; CH ⊥ GF with H the midpoint of GF; CE = CH; CA = CG = r.

Apply the Baudhāyana–Pythagoras theorem to right triangle CEA:
$CA^2 = CE^2 + AE^2$
$\Rightarrow AE^2 = CA^2 - CE^2 = r^2 - CE^2$.

Apply it to right triangle CHG:
$CG^2 = CH^2 + GH^2$
$\Rightarrow GH^2 = CG^2 - CH^2 = r^2 - CH^2$.

Since CE = CH and r is the same for both:
$AE^2 = r^2 - CE^2 = r^2 - CH^2 = GH^2$
$\Rightarrow AE = GH$.

Therefore: AB = 2 × AE = 2 × GH = GF.
Pythagoras gives $AE^2 = r^2 - CE^2$ and $GH^2 = r^2 - CH^2$. Since CE = CH, AE = GH, so AB = 2·AE = 2·GH = GF.

Exercise 5.5 — Distance of Chords from the Centre — Further Results

Q1 Find the length of the chord of a circle where the radius is 7 cm and perpendicular distance is 6 cm.

SOLUTION

Let the chord be AB and let the perpendicular from centre C to AB meet it at M.
Then CM = 6 cm (perpendicular distance) and CA = 7 cm (radius).
By Theorem 5, M is the midpoint of AB, so AM = AB/2.

Applying the Baudhāyana–Pythagoras theorem to right triangle CMA:

$CA^2 = CM^2 + AM^2$

$7^2 = 6^2 + AM^2$

$49 = 36 + AM^2$

$AM^2 = 13$

$AM = \sqrt{13}$ cm.

Therefore: $AB = 2 \times AM = 2\sqrt{13}$ cm.
The chord length is $2\sqrt{13}$ cm (≈ 7.21 cm).

Q2 Explain why the following statement is true: If the perpendicular distance of a chord from the centre is d and the radius is r, then the chord length is $2\sqrt{(r^2 - d^2)}$.

SOLUTION

Let the chord be AB and let M be the foot of the perpendicular from the centre C to AB.
Then CM = d (given) and CA = r (radius).

By Theorem 5, M is the midpoint of AB, so AM = AB/2.

Triangle CMA is right-angled at M.
By the Baudhāyana–Pythagoras theorem:

$CA^2 = CM^2 + AM^2$

$r^2 = d^2 + AM^2$

$AM^2 = r^2 - d^2$

$AM = \sqrt{(r^2 - d^2)}$.

Therefore: chord length $AB = 2 \times AM = 2\sqrt{(r^2 - d^2)}$.
By Pythagoras on the right triangle formed by the radius, the perpendicular distance, and half the chord: $(\text{half-chord})^2 = r^2 - d^2$, so chord $= 2\sqrt{(r^2 - d^2)}$.

Q3 *In a circle, if the distance of chord AB from the centre is twice the distance of another chord CD from the centre, then can we conclude that CD = 2 AB? Give reasons for your answer.

SOLUTION

Let the distance of CD from the centre be h, so the distance of AB from the centre is 2h.
Let the radius be r.

Using the formula from Q2:
AB = $2\sqrt{(r^2 - (2h)^2)} = 2\sqrt{(r^2 - 4h^2)}$
CD = $2\sqrt{(r^2 - h^2)}$

For CD = 2·AB we would need:
$2\sqrt{(r^2 - h^2)} = 2 \times 2\sqrt{(r^2 - 4h^2)}$
$\sqrt{(r^2 - h^2)} = 2\sqrt{(r^2 - 4h^2)}$
$r^2 - h^2 = 4(r^2 - 4h^2)$
$r^2 - h^2 = 4r^2 - 16h^2$
$15h^2 = 3r^2$
$h^2 = r^2/5$ => $h = r/\sqrt{5}$.

This equality holds only when $h = r/\sqrt{5}$, i.e. only for one specific ratio of h to r.

For example, with r = 5 and h = 1:
AB = $2\sqrt{(25 - 4)} = 2\sqrt{21} \approx 9.17$ cm
CD = $2\sqrt{(25 - 1)} = 2\sqrt{24} \approx 9.80$ cm
2·AB ≈ 18.33 cm ≠ CD.

Therefore, we cannot conclude in general that CD = 2·AB.
No. CD = 2·AB only when $h = r/\sqrt{5}$ (a special case). In general, doubling the distance does not halve the chord in a simple ratio — the relationship is non-linear through $\sqrt{(r^2 - d^2)}$.

Exercise 5.6 — Angles Subtended by an Arc

Q1 In a circle with centre O, the central angle AOB is 60°. If the radius of the circle is 12 cm, what is the length of the chord AB?

SOLUTION

Given: OA = OB = 12 cm (radii), ∠AOB = 60°.

Triangle OAB is isosceles with OA = OB = 12 cm.
The base angles are equal: ∠OAB = ∠OBA = (180° − 60°)/2 = 60°.

Since all three angles equal 60°, triangle OAB is equilateral.
Therefore AB = OA = OB = 12 cm.

Verification using the cosine rule:
$AB^2 = OA^2 + OB^2 - 2 \cdot OA \cdot OB \cdot \cos(\angle AOB)$
$AB^2 = 144 + 144 - 2 \cdot 144 \cdot \cos(60°)$
$AB^2 = 288 - 288 \cdot (1/2) = 288 - 144 = 144$
AB = 12 cm. ✓

Q2 Let A and B be two points on a circle with centre O. (i) Are there points X, Y on the circle, on the same side of AB, such that ∠AXB is different from ∠AYB? (ii) Is it true that if ∠AXB = ∠AYB, then X and Y lie on the same side of the circle? (iii) If ∠AXB = ∠AYB, and X and Y do not lie on the circle, does the circle through A, B and X also pass through Y?

SOLUTION

(i) No.
By Theorem 9 (and its consequence), all points on the circle that lie on the same side of chord AB (i.e. on the same arc) subtend the same angle to AB.
So if X and Y are both on the same arc, ∠AXB = ∠AYB always.
There are no two points on the same arc that give different angles.

(ii) Yes.
If ∠AXB = ∠AYB and both X and Y are on the circle, then by the result that angles in the same arc are equal and angles in opposite arcs are supplementary (not equal in general), X and Y must lie on the same arc — i.e. the same side of AB.
(More precisely, if they were on opposite sides, Theorem 10 would give ∠AXB + ∠AYB = 180°, which contradicts ∠AXB = ∠AYB unless both are 90°. So in general, equal angles force X and Y to be on the same side.)

(iii) Yes.
If ∠AXB = ∠AYB, then by Theorem 10, the four points A, B, X, Y are concyclic.
Therefore, the unique circle through A, B, and X also passes through Y.
(i) No — all points on the same arc subtend equal angles. (ii) Yes — equal inscribed angles imply same arc (same side of AB). (iii) Yes — by Theorem 10 (concyclicity), A, B, X, Y all lie on the same circle.

Q3 Find x in Fig. 5.26. (Figure shows points A, B, C, D on a circle; angle at D is 100°; angle at B is x.)

SOLUTION

From Fig. 5.26: A, B, C, D are four points on the same circle.

A is on the left, C is on the right, D is at the top with ∠ADC = 100°, and B is at the bottom with ∠ABC = x.

D and B lie on opposite sides of chord AC.

By the consequence of Theorem 9, angles subtended by the same chord AC at points on opposite arcs are supplementary:

∠ADC + ∠ABC = 180°

100° + x = 180°

x = 80°.

Exercises — I'm Up and Down, and Round and Round

Q1 In a circle, a chord is 5 cm away from the centre. If the radius of the circle is 13 cm, what is the length of the chord?

SOLUTION

Let O be the centre and AB the chord. Drop the perpendicular OM from O to AB. The foot M is the midpoint of AB, so AM = MB = half the chord.
The distance of the chord from the centre is OM = 5 cm, and the radius is OA = 13 cm.
In the right triangle OMA, by the Baudhayana-Pythagoras theorem:
$OA^2 = OM^2 + AM^2 \Rightarrow AM^2 = 13^2 - 5^2 = 169 - 25 = 144$, so AM = 12 cm.
Therefore the full chord AB = 2 × AM = 2 × 12 = 24 cm.
The chord is 24 cm long.

Q2 An arc of a circle subtends an angle of 70° at the centre. What is the measure of the angle subtended by the arc at a point on the circle?

SOLUTION

By Theorem 9, the angle subtended by an arc at the centre of the circle is double the angle subtended by the same arc at any point on the circle outside the arc.
So the angle at a point on the circle = (1/2) × (angle at the centre) = (1/2) × 70° = 35°.

Q3 The diameter of a circle is 26 cm. A chord of length 24 cm is drawn in the circle. Find the distance from the centre of the circle to the chord.

SOLUTION

The diameter is 26 cm, so the radius r = 26 / 2 = 13 cm.
Let O be the centre and AB the chord of length 24 cm. Drop the perpendicular OM from O to AB; M is the midpoint of AB, so AM = 24 / 2 = 12 cm.
In right triangle OMA: $OA^2 = OM^2 + AM^2 \Rightarrow OM^2 = 13^2 - 12^2 = 169 - 144 = 25$, so OM = 5 cm.
The distance from the centre to the chord is 5 cm.

Q4 A circle has a radius of 15 cm. A chord is drawn. The distance from the centre of the circle to the chord is 9 cm. What is the length of the chord?

SOLUTION

Let O be the centre and AB the chord. Drop the perpendicular OM from O to AB; M is the midpoint of AB.
Distance OM = 9 cm and radius OA = 15 cm.
In right triangle OMA: $AM^2 = OA^2 - OM^2 = 15^2 - 9^2 = 225 - 81 = 144$, so AM = 12 cm.
The chord AB = 2 × AM = 2 × 12 = 24 cm.
The chord is 24 cm long.

Q5 Prove that the perpendicular bisector of a chord passes through the centre of the circle.

SOLUTION

Let AB be a chord of a circle with centre O.
OA and OB are both radii of the circle, so OA = OB. Thus O is equidistant from A and B.
The perpendicular bisector of a segment is exactly the locus of all points equidistant from its two end points (every point equidistant from A and B lies on the perpendicular bisector of AB, and conversely).
Since OA = OB, the centre O is equidistant from A and B, so O lies on the perpendicular bisector of AB.
Therefore the perpendicular bisector of the chord AB passes through the centre O.
Since the centre is equidistant from the chord's end points, it lies on the chord's perpendicular bisector; hence that bisector passes through the centre.

Q6 The diameter of a circle is AB. Point C is on the circumference. What is the measure of the ∠ACB? Explain your reasoning.

SOLUTION

AB is a diameter, and C is a point on the circle. The angle ∠ACB is the angle subtended at C by the arc AB not containing C.
That arc, as seen from the centre O, subtends the straight angle ∠AOB = 180° (A, O, B are collinear along the diameter).
By Theorem 9, the angle at a point on the circle is half the angle at the centre: ∠ACB = (1/2) × ∠AOB = (1/2) × 180° = 90°.
This is the corollary: the angle in a semicircle (the angle subtended by a diameter at any point on the circle) is 90°.
∠ACB = 90° (the angle in a semicircle is a right angle).

Q7 ABCD is a cyclic quadrilateral inscribed in a circle. If ∠A measures 75°, what is the measure of ∠C? If ∠B measures 110°, what is the measure of ∠D?

SOLUTION

By Theorem 11, in a cyclic quadrilateral the sum of each pair of opposite angles is 180°.
∠A and ∠C are opposite, so ∠A + ∠C = 180° => ∠C = 180° - 75° = 105°.
∠B and ∠D are opposite, so ∠B + ∠D = 180° => ∠D = 180° - 110° = 70°.
∠C = 105° and ∠D = 70°.

Q8 Quadrilateral PQRS is inscribed in a circle. If ∠P = (2x + 10)° and ∠R = (3x − 20)°, find the value of x and the measures of ∠P and ∠R.

SOLUTION

PQRS is a cyclic quadrilateral, so opposite angles ∠P and ∠R sum to 180° (Theorem 11).
(2x + 10) + (3x − 20) = 180
5x − 10 = 180 => 5x = 190 => x = 38.
∠P = 2x + 10 = 2(38) + 10 = 86°.
∠R = 3x − 20 = 3(38) − 20 = 114 − 20 = 94°.
Check: 86° + 94° = 180°, as required.
x = 38, ∠P = 86° and ∠R = 94°.

Q9 The distance of a chord of length 16 cm from the centre of a circle is 6 cm. Find the radius of the circle.

SOLUTION

Let O be the centre and AB the chord of length 16 cm. Drop the perpendicular OM from O to AB; M is the midpoint, so AM = 16 / 2 = 8 cm.
The distance OM = 6 cm.
In right triangle OMA: $OA^2 = OM^2 + AM^2 = 6^2 + 8^2 = 36 + 64 = 100$, so OA = 10 cm.
The radius is 10 cm.

Q10 A cyclic quadrilateral has sides 5, 5, 12, 12 units. Find its area.

SOLUTION

Take the vertices in order A, B, C, D with AB = 5, BC = 5, CD = 12, DA = 12. The two 5-sides meet at B and the two 12-sides meet at D, so the quadrilateral is a kite symmetric about the diagonal BD.
The diagonal BD splits the kite into triangle ABD (AB = 5, AD = 12) and triangle CBD (CB = 5, CD = 12); by the symmetry these two triangles are congruent.
Because the quadrilateral is cyclic, opposite angles sum to 180 degrees. By the symmetry angle A = angle C, and angle A + angle C = 180 degrees, so angle A = angle C = 90 degrees.
So the right angles are at A and C. Then BD is a diameter, and in right triangle ABD, $BD^2 = AB^2 + AD^2 = 5^2 + 12^2 = 25 + 144 = 169$, giving BD = 13.
Area = area(ABD) + area(CBD) = (1/2)(5)(12) + (1/2)(5)(12) = 30 + 30 = 60 square units.
The area is 60 square units. (Kite symmetric about BD; right angles at A and C make BD = 13 a diameter, and the two right triangles each have area (1/2)(5)(12) = 30.)

Q11 Consider a cyclic quadrilateral. Without drawing its circumcircle, how can we find out whether the centre of the circumcircle lies inside the quadrilateral or outside? What is the best way of finding out?

SOLUTION

The circumcentre lies INSIDE the quadrilateral when the quadrilateral 'wraps around' the centre, and OUTSIDE when the whole quadrilateral is pushed to one side of the centre.

The clean test uses the angle-in-a-semicircle idea. In a cyclic quadrilateral, each interior angle stands on the opposite side: that angle is exactly 90 degrees when the opposite side is a diameter (passes through the centre), less than 90 degrees when the opposite side is a shorter chord, and more than 90 degrees when the opposite side subtends a major arc (so the centre lies on the far side of that side).

Best way - check the interior angles: if ANY interior angle is obtuse (greater than 90 degrees), the side opposite it has the centre on its far side, so the centre lies OUTSIDE the quadrilateral. If ALL four interior angles are 90 degrees or less, the centre lies INSIDE (on the boundary when an angle is exactly 90 degrees, since then the opposite side is a diameter through the centre).

Examine the interior angles: if all four angles are ≤ 90° the circumcentre lies inside (on the boundary when an angle is exactly 90°); if any angle is obtuse (> 90°) the centre lies outside the quadrilateral. So the best test is to check whether any angle exceeds 90°.

Q12 When two chords intersect, each of them is divided into two line segments. Show that if the intersecting chords are of equal length, then the line segments of one chord are equal to the corresponding line segments of the other chord.

SOLUTION

Let two equal chords AB and CD of a circle with centre O intersect at a point P. So AB = CD.

Equal chords are equidistant from the centre (Theorem 6). So the perpendicular distances from O to AB and to CD are equal; call this common distance ON = OM, where M and N are the feet of the perpendiculars (the midpoints of AB and CD respectively).

Since M is the midpoint of AB and N is the midpoint of CD, we have AM = MB = AB/2 and CN = ND = CD/2, and AM = CN because AB = CD.

Now OM = ON and OP is common; in right triangles OMP and ONP, ∠OMP = ∠ONP = 90°, OM = ON, OP = OP. By RHS congruence, △OMP ≅ △ONP, so MP = NP.

The two segments of AB are AP = AM − MP (or AM + MP) and PB = MB + MP (or MB − MP); similarly for CD using CN, ND and NP. Since AM = CN, MB = ND, and MP = NP, the segments of AB equal the corresponding segments of CD.

Hence for the chord AB the two pieces are equal to the two pieces of CD: {AP, PB} = {CP, PD} as a pair.

Equal chords are equidistant from the centre, so their midpoints are equidistant from the intersection point; the RHS-congruent right triangles give MP = NP, making the segments of one chord equal to the corresponding segments of the other.

Q13 Draw a circle in which a chord of 6 cm length stands at a distance of 3 cm from the centre. (Hint: Is it a circumcircle of a suitable triangle?)

SOLUTION

We first find the radius so the chord fits the condition. Let the chord AB = 6 cm, so its half is AM = 3 cm, and the distance of the chord from the centre is OM = 3 cm.

By the Baudhayana-Pythagoras theorem, radius r = OA = $\sqrt{(OM^2 + AM^2)} = \sqrt{(3^2 + 3^2)} = \sqrt{(9 + 9)} = \sqrt{18} = 3\sqrt{2} \approx 4.24$ cm.

Construction: (1) Draw a line segment AB = 6 cm and mark its midpoint M. (2) At M draw a line perpendicular to AB. (3) On this perpendicular mark O with OM = 3 cm (the centre). (4) With O as centre and radius OA = $3\sqrt{2}$ cm (≈ 4.24 cm), draw the circle; it passes through A and B, and AB is the required chord at distance 3 cm from O.

Regarding the hint: yes — such a circle is the circumcircle of the isosceles triangle OAB reflected, or more simply the circle can be obtained as the circumcircle of the triangle formed by A, B and a third point on it; the centre O sits where the perpendicular bisectors meet.

Radius = $3\sqrt{2}$ cm ≈ 4.24 cm. Construct AB = 6 cm, erect the perpendicular at its midpoint M, mark O at 3 cm along it, then draw the circle of radius $3\sqrt{2}$ cm centred at O.

Q14 Show that a rectangle is the only parallelogram that can be inscribed in a circle.

SOLUTION

Suppose a parallelogram ABCD is inscribed in a circle, so ABCD is a cyclic quadrilateral.

In any cyclic quadrilateral, opposite angles are supplementary (Theorem 11): $\angle A + \angle C = 180°$.

In a parallelogram, opposite angles are equal: $\angle A = \angle C$.

Combining: $\angle A + \angle A = 180° \Rightarrow 2\angle A = 180° \Rightarrow \angle A = 90°$. Hence $\angle C = 90°$ too.

Similarly $\angle B = \angle D = 90°$. A parallelogram with all angles 90° is a rectangle.

Therefore the only parallelogram that can be inscribed in a circle is a rectangle.

For a cyclic parallelogram, equal opposite angles (parallelogram) must also be supplementary (cyclic), forcing each to be 90°; so it must be a rectangle.

Q15 Show that if a rectangle is inscribed in a circle, then the point of intersection of its diagonals must lie at the centre of the circle.

SOLUTION

Let rectangle ABCD be inscribed in a circle with centre O. Its diagonals AC and BD meet at a point P.

In a rectangle the diagonals are equal and bisect each other, so P is the midpoint of both AC and BD, and PA = PB = PC = PD (each equals half a diagonal).

Each angle of a rectangle is 90°. So, for instance, $\angle ABC = 90°$, meaning the diagonal AC subtends a right angle at B; by the semicircle corollary, AC must be a diameter of the circle. Likewise BD is a diameter.

The two diameters AC and BD of the same circle both pass through the centre, so they

intersect at the centre O. But AC and BD intersect at P.
Therefore P = O; the intersection of the diagonals is the centre of the circle.
Each rectangle angle is 90°, so both diagonals are diameters; two diameters meet only at the centre, so the diagonals' intersection point is the centre O.

Q16 Consider all chords of a circle of a fixed length. What is the shape formed by the midpoints of all these chords?

SOLUTION

Let the given circle have centre O and radius r, and consider all chords of a fixed length L.
For a chord of length L, its midpoint M is the foot of the perpendicular from O, and the distance OM = d satisfies $(L/2)^2 + d^2 = r^2$, so $d = \sqrt{(r^2 - (L/2)^2)}$, a fixed number independent of the particular chord.
Thus every such midpoint M lies at the same distance d from the fixed point O.
The locus of points at a fixed distance d from a fixed point O is a circle centred at O with radius d.
Therefore the midpoints of all chords of the fixed length form a circle concentric with the given circle, of radius $\sqrt{(r^2 - (L/2)^2)}$.
They form a circle concentric with the given one, of radius $\sqrt{(r^2 - (L/2)^2)}$.

Q17 In a circle with centre O, chords AB and AC are congruent. Explain why this statement is true: "The centre of the circle lies on the angle bisector of ∠BAC".

SOLUTION

We are given two congruent chords AB and AC (AB = AC) sharing the common point A on the circle.
Equal chords are equidistant from the centre (Theorem 6). Let OM ⊥ AB with foot M, and ON ⊥ AC with foot N. Then OM = ON.
Consider triangles AMO and ANO. OM = ON (just shown), OA = OA (common), and ∠AMO = ∠ANO = 90°. By RHS congruence, ΔAMO ≅ ΔANO.
Hence the corresponding angles ∠OAM = ∠OAN, i.e. OA makes equal angles with AB and AC.
So the line AO bisects ∠BAC, which means the centre O lies on the angle bisector of ∠BAC.
Because AB = AC are equidistant from O, the right triangles AMO and ANO are RHS-congruent, giving ∠OAB = ∠OAC; hence AO bisects ∠BAC and O lies on that bisector.

Q18 Two parallel chords of lengths 10 cm and 24 cm are on the same side of the centre of a circle. The distance between the chords is 7 cm. Find the radius of the circle.

SOLUTION

Let O be the centre, radius r. Let the 24 cm chord have half-length 12 cm and the 10 cm chord have half-length 5 cm.

Distance of the 24 cm chord from O: $d_1 = \sqrt{(r^2 - 12^2)} = \sqrt{(r^2 - 144)}$.

Distance of the 10 cm chord from O: $d_2 = \sqrt{(r^2 - 5^2)} = \sqrt{(r^2 - 25)}$.

The longer chord is nearer the centre, so $d_2 > d_1$, and since both chords are on the same side of O, the gap between them is $d_2 - d_1 = 7$.

So $\sqrt{(r^2 - 25)} - \sqrt{(r^2 - 144)} = 7$. Try r = 13: $\sqrt{(169 - 25)} - \sqrt{(169 - 144)} = \sqrt{144} - \sqrt{25} = 12 - 5 = 7$. ✓

Therefore r = 13 cm.

The radius is 13 cm.

Q19 A regular hexagon is inscribed in a circle of radius r. Find the length of the sides of the hexagon and the distance of each side from the centre of the circle.

SOLUTION

A regular hexagon inscribed in a circle of radius r has its 6 vertices on the circle, dividing the circle into 6 equal arcs, so each side subtends a central angle of 360°/6 = 60°.

Consider one side, say the chord joining two adjacent vertices, with the centre O. Triangle O-vertex-vertex has the two radii = r and the included central angle 60°.

Since the two equal sides make a 60° angle, the base angles are each (180° − 60°)/2 = 60°, so the triangle is equilateral. Hence the side of the hexagon equals r.

Distance of a side from the centre = apothem = the perpendicular distance from O to the chord.

Half the side is r/2, so distance = $\sqrt{(r^2 - (r/2)^2)} = \sqrt{(r^2 - r^2/4)} = \sqrt{(3r^2/4)} = (\sqrt{3}/2)\, r$.

So each side has length r and lies at distance $(\sqrt{3}/2)\, r$ from the centre.

Side length = r; distance of each side from the centre = $(\sqrt{3}/2)\, r$.

Q20 A quadrilateral MNOP is inscribed in a circle. If MN is a diameter, what can you say about ∠MOP and ∠MNP? Explain your reasoning.

SOLUTION

MNOP is a cyclic quadrilateral with vertices in order M, N, O, P on the circle, and MN is a diameter.

Key fact (angle in a semicircle): a diameter subtends a right angle at every other point of the circle. Here the diameter is MN.

At vertex O: MN subtends the angle ∠MON = 90°.

At vertex P: MN subtends the angle ∠MPN = 90°.

So the diameter MN makes a right angle at each of the other two vertices O and P.

The angles literally named in the question — ∠MOP (at O, between OM and OP) and ∠MNP (at

N, between NM and NP) — are NOT the semicircle angles and are not, in general, forced to be 90°. The right angles guaranteed by MN being a diameter are ∠MON and ∠MPN.
MN, being a diameter, subtends a right angle at O and at P: ∠MON = 90° and ∠MPN = 90° (angles in a semicircle). ∠MOP and ∠MNP are not, in general, right angles.

Q21 Let ABCD be a cyclic quadrilateral. Explain why the exterior angle at any vertex is equal to the interior opposite angle (e.g., ∠CDE = ∠ABC, where E is a point on the extension of side CD).

SOLUTION

Let ABCD be a cyclic quadrilateral, and extend side CD beyond D to a point E, forming the exterior angle ∠CDE at vertex D.
∠ADC and ∠CDE are angles on the straight line CDE, so they are supplementary: ∠ADC + ∠CDE = 180°.
∠ADC and ∠ABC are opposite angles of the cyclic quadrilateral, so by Theorem 11 they are supplementary: ∠ADC + ∠ABC = 180°.
Comparing the two equations: ∠CDE = 180° − ∠ADC = ∠ABC.
Therefore the exterior angle ∠CDE equals the interior opposite angle ∠ABC. The same argument works at any vertex.
The exterior angle and its adjacent interior angle are supplementary, and that interior angle and its opposite interior angle are also supplementary (cyclic); so the exterior angle equals the interior opposite angle, e.g. ∠CDE = ∠ABC.

Q22 "There is no chord of a circle that is longer than its diameter." How do you justify this statement?

SOLUTION

Let the circle have centre O and radius r; the diameter has length 2r.
Take any chord AB. Join O to A and O to B; both OA and OB are radii, so OA = OB = r.
By the triangle inequality in triangle OAB, AB ≤ OA + OB = r + r = 2r.
Equality AB = 2r holds only when O lies on AB, i.e. when the chord passes through the centre — which is exactly the diameter.
Hence every chord has length at most 2r, so no chord can be longer than the diameter.
(Equivalently: the chord nearest the centre is the longest; the diameter passes through the centre at distance 0, so it is the greatest chord.)
For any chord AB, the triangle inequality gives AB ≤ OA + OB = 2r, with equality only when the chord passes through the centre (the diameter); so no chord exceeds the diameter.

Q23 Let A be any point within a given circle with centre O. Show that the shortest chord of the circle that passes through point A is the one that is perpendicular to OA.

SOLUTION

Let the circle have centre O and radius r, and let A be a fixed interior point. Any chord through A has some perpendicular distance d from the centre O; recall a chord at distance d from the centre has length $2\sqrt{(r^2 - d^2)}$.

Since the chord length $2\sqrt{(r^2 - d^2)}$ decreases as d increases, the shortest chord through A is the one whose distance d from O is as large as possible.

For a chord through A, the foot M of the perpendicular from O to the chord satisfies OM = d, and M lies on the chord. Because A is on the chord, OM ≤ OA (the perpendicular distance from O to the chord is the shortest distance from O to any point of the chord, and A is a point of the chord).

So d = OM ≤ OA, and d attains its maximum value OA exactly when M = A, i.e. when OA itself is perpendicular to the chord.

Thus the largest possible d is OA, achieved by the chord through A perpendicular to OA; this gives the smallest chord length $2\sqrt{(r^2 - OA^2)}$.

Therefore the shortest chord through A is the one perpendicular to OA.

A chord's length is $2\sqrt{(r^2 - d^2)}$, so the shortest chord has the greatest distance d from O; since d ≤ OA with equality when the chord ⊥ OA, that perpendicular chord is the shortest.

Q24 How would you use the given figure (Fig. 5.30) to justify the statement that the angle in a semicircle is 90°? (In Fig. 5.30, A is the apex on the circle at the top, O is the centre on the diameter, the two diameter endpoints are unlabelled, a and b are the base angles at those two ends, and OA is drawn as a radius.)

SOLUTION

In Fig. 5.30, let the diameter be the base with centre O, and let the top vertex A be a point on the circle; join A to O and to the two ends of the diameter. The two marked angles a and b are the base angles of the two triangles formed.

OA is a radius, and the two segments from O to the ends of the diameter are also radii, all equal to r. So each of the two triangles formed (from O to A and to an end of the diameter) is isosceles.

In the left isosceles triangle the two equal sides give equal base angles, both equal to a; in the right isosceles triangle the two equal base angles are both b.

The whole angle at A (the angle in the semicircle) is a + b.

In the big triangle (the two ends of the diameter and A), the three angles are a (at one end), b (at the other end), and (a + b) at A. Their sum is 180°: a + b + (a + b) = 180°, so 2(a + b) = 180°, giving a + b = 90°.

Therefore the angle at A, which is a + b, equals 90°: the angle in a semicircle is a right angle.

The two radii create two isosceles triangles with base angles a and b; the big triangle gives a + b + (a + b) = 180°, so a + b = 90°. The angle at the top (in the semicircle) is a + b = 90°.

Q25 In a circle, two chords CC' and DD' are drawn perpendicular to a diameter AB. Prove that the segment MM' joining the midpoints of the chords CD and C'D' is perpendicular to AB.

SOLUTION

Let AB be a diameter with centre O. Chords CC' and DD' are each perpendicular to AB. Since a chord perpendicular to a diameter is bisected by that diameter, AB bisects CC' and bisects DD'.

So AB is the perpendicular bisector of CC' and of DD'. This means the figure is symmetric about the line AB: reflecting in AB sends C to C' and D to D' (and vice versa).

Under reflection in AB, the segment CD maps to the segment C'D'. Therefore the midpoint M of CD maps to the midpoint M' of C'D'.

A point and its mirror image in a line are placed symmetrically about that line, so the segment joining a point to its reflection is perpendicular to the mirror line. Since M' is the reflection of M in AB, the segment MM' is perpendicular to AB.

Hence MM', joining the midpoints of CD and C'D', is perpendicular to the diameter AB.

AB perpendicularly bisects both CC' and DD', so reflection in AB swaps C↔C' and D↔D', sending midpoint M of CD to midpoint M' of C'D'; a point and its reflection are joined by a segment perpendicular to the mirror line, so MM' $\perp$ AB.

Q26 How would you use the given figure (Fig. 5.31) to justify the statement that the sum of the opposite angles of a cyclic quadrilateral is 180°? (In the figure, ABCD is cyclic with centre O; p, q are the parts of the angle at A/B and u, v are the parts of the angle at C/D formed by joining the vertices to O.)

SOLUTION

Let ABCD be a cyclic quadrilateral with centre O. Join O to each vertex A, B, C, D. This splits the quadrilateral into four triangles: OAB, OBC, OCD, ODA. Each triangle has two sides equal to the radius r, so each is isosceles, and its two base angles are equal.

Label the equal base angles: in the isosceles triangles meeting at each vertex, the interior angle of the quadrilateral at that vertex is the sum of two such base angles. Using the figure's letters, the interior angle at A is (say) p + (a part), and the marked pieces p, q, u, v are these equal base angles paired across the vertices.

Add up the interior angles of the quadrilateral: $\angle A + \angle B + \angle C + \angle D = 360°$ (angle sum of a quadrilateral).

Because of the isosceles pairs, the four interior angles split into eight base angles that pair up equally around O; the two opposite angles $\angle A + \angle C$ together contain exactly one of each equal pair, and likewise $\angle B + \angle D$ contain the matching pair. So $\angle A + \angle C = \angle B + \angle D$.

Since $\angle A + \angle B + \angle C + \angle D = 360°$ and $\angle A + \angle C = \angle B + \angle D$, each of these sums must be half of 360°.

Therefore $\angle A + \angle C = 180°$ and $\angle B + \angle D = 180°$: the opposite angles of the cyclic quadrilateral are supplementary.

Joining the centre O to the four vertices makes four isosceles triangles whose equal base angles (p, q, u, v) pair up so that $\angle A + \angle C = \angle B + \angle D$; since the four angles total 360°, each opposite pair sums to 180°.