Chapter 6: Measuring Space: Perimeter and Area

CONTENTS — 4 exercises · 56 questions

6.1	Perimeter of a Circle and Arc Length	8 Q
6.2	Area of Triangles, Parallelograms, and Quadrilaterals	11 Q
6.3	Area of a Circle, Sector, and Segment	10 Q
Exercises	Measuring Space: Perimeter and Area	27 Q

Exercise 6.1 — Perimeter of a Circle and Arc Length

Q1 The perimeter of a circle is 44 cm. What is its radius?

SOLUTION

Circumference C = 2πr = 44 cm, using π = 22/7.
r = 44 / (2 × 22/7) = 44 × 7/44 = 7 cm.
r = 7 cm

Q2 Calculate, correct to 3 significant figures, the circumference of a circle with: (i) radius 7 cm, (ii) radius 10 cm, (iii) radius 12 cm.

SOLUTION

(i) C = 2πr = 2 × (22/7) × 7 = 44 cm (exact).
(ii) C = 2 × (22/7) × 10 = 440/7 = 62.857... ≈ 62.9 cm.
(iii) C = 2 × (22/7) × 12 = 528/7 = 75.428... ≈ 75.4 cm.
(i) 44 cm, (ii) 62.9 cm, (iii) 75.4 cm

Q3 Calculate the length of the arc of a circle if: (i) the radius is 3.5 cm and the angle at the centre is 60°, and (ii) the radius is 6.3 m and the angle at the centre is 120°.

SOLUTION

Arc length = (θ/360) × 2πr.
(i) r = 3.5 = 7/2 cm, θ = 60°:
Arc = (60/360) × 2 × (22/7) × (7/2) = (1/6) × 22 = 11/3 cm.
(ii) r = 6.3 = 63/10 m, θ = 120°:
Arc = (120/360) × 2 × (22/7) × (63/10) = (1/3) × (2 × 22 × 63)/(7 × 10) = (1/3) × (2772/70) = (1/3) × (198/5) = 66/5 = 13.2 m.
(i) 11/3 cm ≈ 3.67 cm; (ii) 13.2 m

Q4 Find the perimeter of a sector (i.e., the curved portion as well as the two straight portions) of a circle of radius 14 cm and sector angle 75°.

SOLUTION

Arc length = (θ/360) × 2πr = (75/360) × 2 × (22/7) × 14.
= (5/24) × (44 × 14/7) = (5/24) × 88 = 440/24 = 55/3 cm.
Perimeter of sector = arc + 2 radii = 55/3 + 2 × 14 = 55/3 + 28 = 55/3 + 84/3 = 139/3 cm.
139/3 cm ≈ 46.33 cm

Q5 Find the perimeters of the following shapes (taking the arcs to be quarter or half or three-quarters of a circle, as appropriate) (Fig. 6.14i to 6.14ix).

SOLUTION

Take pi = 22/7 throughout.

(i) Stadium shape: a rectangle 80 m long and 60 m wide, with a semicircular cap on each of the two short (60 m) ends. The two 80 m long sides are on the boundary; the two 60 m ends are replaced by semicircles of diameter 60 m (radius 30 m). The two semicircles together make one full circle of diameter 60 m.
Perimeter = 80 + 80 + pi × 60 = 160 + (22/7)(60) = 160 + 1320/7 = 1120/7 + 1320/7 = 2440/7 m ≈ 348.57 m.

(ii) Semicircular annulus: an outer semicircle of diameter 12 cm (radius 6 cm) and an inner semicircle of diameter 8 cm (radius 4 cm), sharing the same base line, joined at the base by two short straight pieces. Each straight piece = (12 − 8)/2 = 2 cm.
Perimeter = (outer arc) + (inner arc) + (two straight bits) = pi×6 + pi×4 + 2 + 2 = 10pi + 4 = (22/7)(10) + 4 = 220/7 + 28/7 = 248/7 cm ≈ 35.43 cm.

(iii) A (dashed) square of side 10 cm with a semicircle drawn outward on each of the four sides. Each semicircle has diameter 10 cm (radius 5 cm); the four square sides are dashed, so only the four arcs form the boundary.
Each semicircular arc = pi×5 = 5pi. Perimeter = 4 × 5pi = 20pi = 20 × (22/7) = 440/7 cm ≈ 62.86 cm.

(iv) Trefoil: three equal circles whose centres form an equilateral triangle. The tick marks show the side of that triangle = the distance between centres = 12 cm, and the circles touch one another, so 2r = 12 => r = 6 cm.
The boundary is the outer arc of each circle. At each centre the two points of contact with the neighbouring circles subtend the interior angle of the equilateral triangle = 60°, so the hidden (inner) arc is 60° and the outer boundary arc of each circle is 360° − 60° = 300°.
Each outer arc = (300/360) × 2pi × 6 = (5/6) × 12pi = 10pi = 220/7 cm.
Perimeter = 3 × 10pi = 30pi = 30 × (22/7) = 660/7 cm ≈ 94.29 cm.

(v) Four-petal flower: a (dashed) square of side 14 cm with a semicircular petal on each side, each petal a semicircle of diameter 14 cm (radius 7 cm). The four square sides are dashed; only the four arcs form the boundary.

Each arc = pi×7 = 7pi. Perimeter = 4 × 7pi = 28pi = 28 × (22/7) = 88 cm.

(vi) One large semicircle spanning the whole 28 cm base (radius 14 cm) on top, and four small semicircular bumps below the base, each of diameter 7 cm (radius 3.5 cm). The 28 cm base line is dashed, so it is NOT part of the perimeter.
Perimeter = (large arc) + (four small arcs) = pi×14 + 4 × (pi×3.5) = 14pi + 14pi = 28pi = 28 × (22/7) = 88 cm.

(vii) Three semicircles built on the three sides of a right triangle with legs 8 cm and 6 cm (right angle between them). The hypotenuse = √(8² + 6²) = √(64+36) = √100 = 10 cm. The three triangle sides are dashed; the boundary is the three semicircular arcs, of diameters 8, 6 and 10 cm, i.e. radii 4, 3 and 5 cm.
Perimeter = pi×4 + pi×3 + pi×5 = 12pi = 12 × (22/7) = 264/7 cm ≈ 37.71 cm.

(viii) One large semicircle spanning the whole 12 cm base (three 4 cm segments; radius 6 cm) on top, and three small semicircular bumps below, each of diameter 4 cm (radius 2 cm). The base line is dashed, so it is NOT part of the perimeter.
Perimeter = pi×6 + 3 × (pi×2) = 6pi + 6pi = 12pi = 12 × (22/7) = 264/7 cm ≈ 37.71 cm.

(ix) S-curve: a dashed base line 20 cm long, marked as two 10 cm segments; a semicircle of diameter 10 cm (radius 5 cm) is drawn above the left segment and another of diameter 10 cm below the right segment.
Each arc = pi×5 = 5pi. Perimeter = 2 × 5pi = 10pi = 10 × (22/7) = 220/7 cm ≈ 31.43 cm.
(i) 2440/7 m ≈ 348.57 m; (ii) 248/7 cm ≈ 35.43 cm; (iii) 440/7 cm ≈ 62.86 cm; (iv) 660/7 cm ≈ 94.29 cm; (v) 88 cm; (vi) 88 cm; (vii) 264/7 cm ≈ 37.71 cm; (viii) 264/7 cm ≈ 37.71 cm; (ix) 220/7 cm ≈ 31.43 cm

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Q6 If the diameter of a car tyre is 56 cm, then: (i) How far does the car need to travel for the tyre to complete one revolution? (ii) How many revolutions does the tyre make if the car travels 10 km?

SOLUTION

(i) Distance per revolution = circumference = πd = (22/7) × 56 = 22 × 8 = 176 cm = 1.76 m.
(ii) 10 km = 10 × 1000 × 100 = 1 000 000 cm.
Revolutions = 1 000 000 / 176 = 62 500/11 ≈ 5681.82.
(i) 176 cm (1.76 m); (ii) 62 500/11 ≈ 5682 revolutions

Q7 Find the total perimeter of all the petals in each of the given flowers. (i) Fig. 6.15A: 4-petal flower, centres of arcs at midpoints of sides of a square of side 14 cm. (ii) Fig. 6.15B: 6-petal flower, centres of arcs at vertices of a regular hexagon of side 42 cm.

SOLUTION

(i) Centre at each side-midpoint → each arc is a semicircle of radius 14/2 = 7 cm.
4 semicircle arcs: Perimeter = 4 × π × 7 = 28 × (22/7) = 88 cm.

(ii) Centre at each vertex of the regular hexagon. The arc from vertex V passes through both

adjacent vertices, and the angle subtended at V = interior angle of hexagon = 120°.
Each petal consists of two 60° arcs (since when two adjacent vertex-circles intersect, each arc subtends 60° at its own centre — equilateral triangle formed by two adjacent vertices and the intersection).
Each arc length = (60/360) × 2π × 42 = (1/6) × 84π = 14π.
6 petals × 2 arcs each = 12 arcs total.
Total = 12 × 14π = 12 × 14 × (22/7) = 12 × 44 = 528 cm.
(i) 88 cm; (ii) 528 cm

Q8 The ratio of the perimeters of two circles is 5:4. What is the ratio of their radii?

SOLUTION

Perimeter (circumference) of a circle = 2πr.
If the two circles have radii r_1 and r_2:
$2\pi r_1 / 2\pi r_2 = r_1/r_2 = 5/4$.
So the ratio of their radii equals the ratio of their perimeters.
Ratio of radii = 5:4

Exercise 6.2 — Area of Triangles, Parallelograms, and Quadrilaterals

Q1 Find the area of triangle ADE in Fig. 6.31, where ABCD is a rectangle with DC = 10 cm, and E is a point on BC such that EC = 8 cm; A is at the same height as E (so AE is horizontal and AD is vertical).

SOLUTION

Place D at the origin. DC = 10 cm (horizontal base), so C = (10, 0).
E is on BC with EC = 8 cm, so E = (10, 8).
A is at the same height as E: A = (0, 8).
Triangle ADE: vertices A(0, 8), D(0, 0), E(10, 8).
AD is vertical (length 8 cm); AE is horizontal (length 10 cm); angle DAE = 90°.
Area = (1/2) × AD × AE = (1/2) × 8 × 10 = 40 cm².

Q2 The parallel sides of a trapezium are 40 cm and 20 cm. If its non-parallel sides are both equal, each being 26 cm, find the area of the trapezium.

SOLUTION

The trapezium is isosceles. Difference of parallel sides = 40 − 20 = 20 cm, shared equally on each end, so each extension = 10 cm.
Each non-parallel side forms a right triangle with hypotenuse 26 cm and base 10 cm.
Height h = $\sqrt{(26^2 - 10^2)} = \sqrt{(676 - 100)} = \sqrt{576} = 24$ cm.
Area = (1/2)(sum of parallel sides) × h = (1/2)(40 + 20) × 24 = (1/2)(60)(24) = 720 cm².

Q3 Find the area of a triangle, given that its sides are 8 cm and 11 cm long, and its perimeter is 32 cm.

SOLUTION

Third side = 32 − 8 − 11 = 13 cm.
Semi-perimeter s = 32/2 = 16 cm.
By Heron's formula: Area = $\sqrt{[s(s-a)(s-b)(s-c)]}$
= $\sqrt{[16 \times (16-8) \times (16-11) \times (16-13)]}$
= $\sqrt{[16 \times 8 \times 5 \times 3]} = \sqrt{1920} = \sqrt{(64 \times 30)} = 8\sqrt{30}$ cm².
$8\sqrt{30} \approx 8 \times 5.477 \approx 43.82$ cm².
$8\sqrt{30}$ cm² ≈ 43.82 cm²

Q4 The sides of a triangular plot are in the ratio 3:5:7; its perimeter is 300 m. Find its area.

SOLUTION

Let sides be 3k, 5k, 7k. Then 15k = 300, so k = 20.
Sides: a = 60 m, b = 100 m, c = 140 m.
s = 300/2 = 150 m.

Area = √[s(s−a)(s−b)(s−c)] = √[150 × 90 × 50 × 10]
= √6 750 000 = √(2 250 000 × 3) = 1500√3 m².
1500√3 ≈ 1500 × 1.732 ≈ 2598 m².
1500√3 m² ≈ 2598 m²

Q5 One diagonal of a rhombus is twice as long as the other diagonal. If the rhombus has area 128 cm², find the length of the shorter diagonal.

SOLUTION

Let the shorter diagonal be d_2 and longer diagonal $d_1 = 2d_2$.
Area of rhombus = $(1/2)(d_1)(d_2) = (1/2)(2d_2)(d_2) = d_2^2 = 128$.
$d_2 = \sqrt{128} = \sqrt{(64 \times 2)} = 8\sqrt{2}$ cm.
$8\sqrt{2} \approx 11.31$ cm.
Verification: $d_1 = 16\sqrt{2}$, area = $(1/2)(16\sqrt{2})(8\sqrt{2}) = (1/2)(256) = 128$ ✓
Shorter diagonal = $8\sqrt{2}$ cm ≈ 11.31 cm

Q6 ABCD is a parallelogram. P and Q are any two points on side AB. What can you say about the ratio area(△PCD) : area(△QCD)?

SOLUTION

Both △PCD and △QCD have the same base CD (a side of the parallelogram).
P and Q both lie on AB, which is parallel to CD.
The perpendicular distance from AB to CD equals the height h of the parallelogram.
So the height from P to CD = height from Q to CD = h.
Area(△PCD) = (1/2) × CD × h = Area(△QCD).
Therefore the ratio = 1:1 regardless of where P and Q are on AB.
area(△PCD) : area(△QCD) = 1:1

Q7 O is any point on the diagonal PR of a parallelogram PQRS. Prove that the areas of triangles PSO and PQO are equal.

SOLUTION

Set up coordinates: P = (0,0), Q = (a, 0), R = (a+b, c), S = (b, c).
O lies on PR, so O = P + t(R − P) = (t(a+b), tc) for some $t \in [0,1]$.
Area(△PSO): vertices P(0,0), S(b,c), O(t(a+b), tc).

= (1/2)	det([S−P, O−P])	= (1/2)	b·tc − c·t(a+b)	
= (1/2)	t(bc − ca − bc)	= (1/2)	−tca	= tca/2.

Area(△PQO): vertices P(0,0), Q(a,0), O(t(a+b), tc).

= (1/2)	det([Q−P, O−P])	= (1/2)	a·tc − 0·t(a+b)	
= (1/2)	atc	= tca/2.		

Area(△PSO) = Area(△PQO) = tca/2. ∎
Area(△PSO) = Area(△PQO) (both equal tca/2 in coordinates). ∎

Q8 If the mid-points of the sides of a 4-gon are joined in order, prove that the area of the parallelogram thus formed will be half of the area of the given 4-gon.

SOLUTION

Let the 4-gon have vertices A, B, C, D (with A at the origin). Let B = b, C = c, D = d (position vectors).
Mid-points: P = b/2 (mid of AB), Q = (b+c)/2 (mid of BC), R = (c+d)/2 (mid of CD), S = d/2 (mid of DA).
PQ = Q − P = c/2.
SR = R − S = (c+d)/2 − d/2 = c/2.
Since PQ = SR (equal vectors), PQRS is a parallelogram.

Area(PQRS) =	PQ × PS	=	c/2 × (d/2 − b/2)	= (1/4)	c × (d − b)	.
For the 4-gon ABCD: Area = (1/2)	AC × BD	where AC = c (from A) and BD = d − b.				
Area(ABCD) = (1/2)	c × (d − b)	.				
Therefore Area(PQRS) = (1/4)	c × (d−b)	= (1/2) × Area(ABCD). ∎				

Area(PQRS) = (1/2) × Area(ABCD). ∎

Q9 In △ABC, the midpoint of BC is D (Fig. 6.32). Median AD is drawn. P is any point on AD. Show that area(△ABP) = area(△ACP).

SOLUTION

Since D is the midpoint of BC, the median AD divides △ABC into two triangles of equal area:
Area(△ABD) = Area(△ACD) = (1/2) × Area(△ABC).

Now consider P on AD. In triangle ABD, the point P lies on median AD from A.
△PBD and △PCD share the same base BD = DC (since D is midpoint of BC),
and have the same height (perpendicular distance from P to BC).
So Area(△PBD) = Area(△PCD).

Area(△ABP) = Area(△ABD) − Area(△PBD).
Area(△ACP) = Area(△ACD) − Area(△PCD).
Since Area(△ABD) = Area(△ACD) and Area(△PBD) = Area(△PCD),
it follows that Area(△ABP) = Area(△ACP). ∎

Q10 Given a square ABCD, let P be a point within it. Join PA, PB, PC, PD (Fig. 6.33). What is the ratio of the areas of the red region (△PAB and △PCD) and the green region (△PBC and △PDA)?

SOLUTION

Let the square have side a, with A(0,a), B(a,a), C(a,0), D(0,0) and P = (x, y) inside.
Area(△PAB) = (1/2) × AB × (perpendicular from P to AB) = (1/2) × a × (a − y).
Area(△PCD) = (1/2) × CD × (perpendicular from P to CD) = (1/2) × a × y.
Red area = Area(△PAB) + Area(△PCD) = (a/2)(a−y) + (a/2)(y) = a²/2.

Area(△PBC) = (1/2) × a × (a − x).
Area(△PDA) = (1/2) × a × x.
Green area = a²/2.

Red area = Green area = a²/2 regardless of P's position.
Ratio = 1:1.

Q11 In △ABC, D is the midpoint of AB. P is any point on BC, and Q is a point on AB such that CQ ∥ PD. PQ is joined (Fig. 6.34). Prove that Area(△BPQ) = (1/2) × Area(△ABC).

SOLUTION

Set up coordinates: B = (0,0), C = (c, 0), A = (a, b).
D = midpoint of AB = (a/2, b/2).
P is on BC: P = (p, 0) for some 0 < p < c.
Q is on AB with CQ ∥ PD: points on AB have the form Q = (qa, qb).

Direction of PD: D − P = (a/2 − p, b/2).
Direction of CQ: Q − C = (qa − c, qb).
For CQ ∥ PD: (qa − c)·(b/2) = qb·(a/2 − p).
Expanding: qab/2 − cb/2 = qab/2 − qbp => −cb/2 = −qbp => q = c/(2p).

So Q = (ca/(2p), cb/(2p)).
Area(△BPQ): B(0,0), P(p,0), Q(ca/2p, cb/2p).
= (1/2)|det([[p, 0],[ca/2p, cb/2p]])| = (1/2)|p·cb/(2p) − 0| = cb/4.

Area(△ABC) = (1/2)|det([[a,b],[c,0]])| = (1/2)|−bc| = bc/2.
Therefore Area(△BPQ) = cb/4 = (1/2) × (bc/2) = (1/2) × Area(△ABC). ∎
Area(△BPQ) = (1/2) × Area(△ABC). ∎

Exercise 6.3 — Area of a Circle, Sector, and Segment

Q1 Find the area of a sector of a circle with radius 7 cm if the angle of the sector is 60°.

SOLUTION

Area of sector = $(\theta/360) \times \pi r^2$.
$= (60/360) \times (22/7) \times 7^2$
$= (1/6) \times (22/7) \times 49$
$= (1/6) \times 154 = 154/6 = 77/3$ cm².
$77/3$ cm² ≈ 25.67 cm²

Q2 Find the area of a quadrant of a circle whose circumference is 44 cm.

SOLUTION

Circumference $= 2\pi r = 44 \Rightarrow r = 44/(2 \times 22/7) = 44 \times 7/44 = 7$ cm.
Area of quadrant (90° sector) $= (90/360) \times \pi r^2$
$= (1/4) \times (22/7) \times 49$
$= (1/4) \times 154 = 77/2 = 38.5$ cm².

Q3 The length of the minute hand of a clock is 7 cm. Find the area swept by the minute hand in 10 minutes.

SOLUTION

In 60 minutes, the minute hand sweeps 360°.
In 10 minutes, angle swept = (10/60) × 360° = 60°.
Area swept = $(60/360) \times \pi r^2 = (1/6) \times (22/7) \times 49$
$= 154/6 = 77/3$ cm².
$77/3$ cm² ≈ 25.67 cm²

Q4 A chord of a circle of radius 10 cm subtends 90° at the centre. Find the area of the corresponding: (i) minor sector (that subtends 90° at the centre), and (ii) major sector (that subtends 270° at the centre). (Use $\pi \approx 3.14$.)

SOLUTION

(i) Area of minor sector $= (90/360) \times \pi \times r^2$
$= (1/4) \times 3.14 \times 100 = 78.5$ cm².

(ii) Area of major sector $= (270/360) \times \pi \times r^2$
$= (3/4) \times 3.14 \times 100 = 235.5$ cm².

Check: $78.5 + 235.5 = 314 = \pi \times 10^2$ ✓
(i) 78.5 cm²; (ii) 235.5 cm²

Q5 A chord of a circle of radius 15 cm subtends an angle of 60° at the centre of the circle. Find the areas of the corresponding minor and major segments of the circle. (Use π ≈ 3.14 and √3 ≈ 1.73.)

SOLUTION

Step 1 — Triangle OAB (O = centre, A and B on circle, ∠AOB = 60°):
OA = OB = 15 cm. By the cosine rule: $AB^2 = 15^2 + 15^2 - 2(15)(15)\cos 60° = 450 - 225 = 225$, so AB = 15 cm.
Triangle OAB is equilateral (all sides = 15 cm).
Area(△OAB) = (√3/4) × 15^2 = (1.73/4) × 225 = 389.25/4 = 97.3125 cm^2.

Step 2 — Minor sector (60°):
Area = (60/360) × 3.14 × 225 = (1/6) × 706.5 = 117.75 cm^2.

Step 3 — Minor segment:
Area = minor sector − triangle = 117.75 − 97.3125 = 20.4375 ≈ 20.44 cm^2.

Step 4 — Circle area:
= 3.14 × 225 = 706.5 cm^2.

Step 5 — Major segment:
= Circle − minor segment = 706.5 − 20.44 = 686.06 cm^2.
Minor segment ≈ 20.44 cm^2; Major segment ≈ 686.06 cm^2

Q6 A car has two wipers which do not overlap. Each wiper has a blade of length 28 cm and sweeps through an angle of 120°. Find the total area cleaned at each sweep of the blades.

SOLUTION

Area swept by one wiper = (θ/360) × π × r^2
= (120/360) × (22/7) × 28^2
= (1/3) × (22/7) × 784
= (1/3) × 22 × 112
= 2464/3 cm^2.

Total area for two wipers = 2 × 2464/3 = 4928/3 cm^2.
4928/3 cm^2 ≈ 1642.67 cm^2

Q7 *A chord of a circle of radius r subtends an angle of 60° at the centre of the circle. Show that the area of the corresponding minor segment of the circle is equal to $\pi r^2(1/6 - \sqrt{3}/4)$.

SOLUTION

Let O be the centre, A and B the ends of the chord, with ∠AOB = 60°.
OA = OB = r. By the cosine rule: $AB^2 = r^2 + r^2 - 2r^2\cos 60° = r^2(2-1) = r^2$, so AB = r.
Triangle OAB is equilateral with side r.
Area(△OAB) = (√3/4)r^2.

Area of minor sector (60°) = (60/360) × πr² = πr²/6.

Area of minor segment = sector − triangle
= πr²/6 − (√3/4)r²
= r²(π/6 − √3/4). ∎

Q8 *An equilateral triangle is inscribed in a circle of radius r. Show that the ratio of the area of the triangle to the area of the circle is equal to 3√3/(4π) ≈ 0.413.

SOLUTION

For an equilateral triangle inscribed in a circle of radius r (circumradius = r):
The circumradius of an equilateral triangle with side s is r = s/√3, so s = r√3.
Area of equilateral triangle = (√3/4)s² = (√3/4)(r√3)² = (√3/4)(3r²) = (3√3/4)r².

Area of circle = πr².

Ratio = (3√3/4)r² / πr² = 3√3/(4π).

3√3/(4π) ≈ 3 × 1.732/(4 × 3.14159) ≈ 5.196/12.566 ≈ 0.413. ∎
Ratio = 3√3/(4π) ≈ 0.413. ∎

Q9 *A square is inscribed in a circle of radius r. Show that the ratio of the area of the square to the area of the circle is equal to 2/π ≈ 0.637.

SOLUTION

For a square inscribed in a circle of radius r, the diagonal of the square = diameter = 2r.
Side of square = 2r/√2 = r√2.
Area of square = (r√2)² = 2r².

Area of circle = πr².

Ratio = 2r²/(πr²) = 2/π.

2/π ≈ 2/3.14159 ≈ 0.637. ∎
Ratio = 2/π ≈ 0.637. ∎

Q10 *A hexagon is inscribed in a circle of radius r. Show that the ratio of the area of the hexagon to the area of the circle is equal to 3√3/(2π) ≈ 0.827. Can you see why the answer is exactly twice the answer to Question 8?

SOLUTION

A regular hexagon inscribed in a circle of radius r has side = r (since each vertex is at distance r from centre, and all sides = r).
The hexagon consists of 6 equilateral triangles each with side r.
Area of one equilateral triangle = (√3/4)r².
Area of hexagon = 6 × (√3/4)r² = (3√3/2)r².

Ratio = $(3\sqrt{3}/2)r^2 / \pi r^2 = 3\sqrt{3}/(2\pi)$.

$3\sqrt{3}/(2\pi) \approx 3 \times 1.732/(2 \times 3.14159) \approx 5.196/6.283 \approx 0.827$. ∎

Why twice Q8? The inscribed equilateral triangle (Q8) uses every other vertex of the hexagon. It covers exactly 3 of the 6 equilateral triangles that make up the hexagon — i.e., half the hexagon. So the hexagon's area = 2 × triangle's area, and dividing both by πr^2 gives ratio(hexagon) = 2 × ratio(triangle). ∎

Ratio = $3\sqrt{3}/(2\pi) \approx 0.827$. This is exactly twice the Q8 ratio ($3\sqrt{3}/(4\pi)$) because the inscribed equilateral triangle covers exactly half of the regular hexagon. ∎

Exercises — Measuring Space: Perimeter and Area

Q1 Identities in algebra can sometimes be shown as area relationships. The area model shown corresponds to the identity $(a + b)^2 = a^2 + 2ab + b^2$. Draw figures corresponding to the identities $(a + b)(a – b) = a^2 – b^2$ and $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$.

SOLUTION

For $(a + b)(a – b) = a^2 – b^2$:
Start with a square of side a (area a^2). Cut off a small square of side b from one corner (area b^2). The remaining L-shaped region has area $a^2 – b^2$.
Rearrange this L-shape into a rectangle: it is made of a strip of size $a \times (a – b)$ and a strip of size $b \times (a – b)$. Placing them side by side gives a rectangle of length $(a + b)$ and width $(a – b)$.
So the L-shaped region $a^2 – b^2$ equals a rectangle of area $(a + b)(a – b)$. This is the area picture of $(a + b)(a – b) = a^2 – b^2$.

For $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$:
Draw a large square of side $(a + b + c)$. Divide each side into three parts of lengths a, b, c (in the same order on both a horizontal and a vertical side).
The grid lines cut the big square into 9 smaller rectangles:
• Three squares on the diagonal: $a \times a = a^2$, $b \times b = b^2$, $c \times c = c^2$.
• Off-diagonal rectangles that occur in pairs: two of $a \times b$ (total $2ab$), two of $b \times c$ (total $2bc$), two of $c \times a$ (total $2ca$).
Adding all nine areas: $a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$, which equals the area $(a + b + c)^2$ of the whole square.
$(a+b)(a–b)$: a square of side a with a corner square of side b removed, rearranged into an $(a+b)$ by $(a–b)$ rectangle. $(a+b+c)^2$: a square of side $(a+b+c)$ split into 3 squares (a^2, b^2, c^2) and 6 rectangles giving $2ab + 2bc + 2ca$.

Q2 An isosceles triangle has perimeter 40 cm; the equal sides are 15 cm each. Find the area of the triangle.

SOLUTION

The two equal sides are 15 cm each, so their total is 30 cm.
Base = perimeter – (sum of equal sides) = 40 – 30 = 10 cm.
Drop a perpendicular (height h) from the apex to the base. It bisects the base, so each half of the base is $10 \div 2 = 5$ cm.
By the Baudhāyana–Pythagoras theorem: $h^2 = 15^2 – 5^2 = 225 – 25 = 200$, so $h = \sqrt{200} = 10\sqrt{2}$ cm.
Area = ½ × base × height = ½ × 10 × $10\sqrt{2}$ = $50\sqrt{2}$ cm².
$50\sqrt{2} \approx 50 \times 1.41421 \approx 70.71$ cm².
Area = $50\sqrt{2} \approx 70.71$ cm².

Q3 An isosceles triangle has base 10 cm, and its area is 60 cm². What are the lengths of the equal sides?

SOLUTION

Area = ½ × base × height, so 60 = ½ × 10 × h.
This gives 60 = 5h, so h = 12 cm (the height to the base).
The height bisects the base, so each half of the base = 10 ÷ 2 = 5 cm.
Each equal side is the hypotenuse of a right triangle with legs 5 cm and 12 cm.
Equal side = √(5² + 12²) = √(25 + 144) = √169 = 13 cm.
Each equal side = 13 cm.

Q4 The area of a right-angled triangle is 54 sq. cm. One of its legs has length 12 cm. Find its perimeter.

SOLUTION

For a right-angled triangle, the two legs are the base and height. Area = ½ × leg_1 × leg_2.
54 = ½ × 12 × (other leg), so 54 = 6 × (other leg), giving other leg = 9 cm.
The hypotenuse = √(12² + 9²) = √(144 + 81) = √225 = 15 cm.
Perimeter = 12 + 9 + 15 = 36 cm.
Perimeter = 36 cm.

Q5 The sides of a triangle are in the ratio 2 : 3 : 4, and its perimeter is 45 cm. Find its area.

SOLUTION

Let the sides be 2x, 3x and 4x. Then 2x + 3x + 4x = 9x = 45, so x = 5.
The sides are 2×5 = 10 cm, 3×5 = 15 cm and 4×5 = 20 cm.
Semi-perimeter s = 45 ÷ 2 = 22.5 cm.
s – a = 22.5 – 10 = 12.5; s – b = 22.5 – 15 = 7.5; s – c = 22.5 – 20 = 2.5.
By Heron's formula, Area = √(s(s–a)(s–b)(s–c)) = √(22.5 × 12.5 × 7.5 × 2.5).
22.5 × 12.5 × 7.5 × 2.5 = 5273.4375, and √5273.4375 = 75√15 ÷ 4 ≈ 72.62 cm².
Area = 75√15 / 4 ≈ 72.62 cm².

Q6 The sides of a triangle have lengths 7 cm, 24 cm, 25 cm. Find the area of the triangle in two different ways.

SOLUTION

Method 1 — Heron's formula:
Semi-perimeter s = (7 + 24 + 25) ÷ 2 = 56 ÷ 2 = 28 cm.
Area = √(s(s–a)(s–b)(s–c)) = √(28 × (28–7) × (28–24) × (28–25)) = √(28 × 21 × 4 × 3).
28 × 21 × 4 × 3 = 7056, and √7056 = 84 cm².

Method 2 — ½ × base × height (recognising a right triangle):
Check: 7² + 24² = 49 + 576 = 625 = 25². By the converse of the Baudhāyana–Pythagoras

theorem, the triangle is right-angled, with the two legs 7 cm and 24 cm and hypotenuse 25 cm.
Area = ½ × 7 × 24 = ½ × 168 = 84 cm².
Both methods give the same area.
Area = 84 cm² (by Heron's formula and by ½ × 7 × 24).

Q7 If the wheel of a bicycle has a diameter of 60 cm, find how far a cyclist will have travelled after the wheel has rotated 100 times.

SOLUTION

In one rotation the wheel covers a distance equal to its circumference.
Circumference = π × diameter = (22/7) × 60 = 1320/7 cm.
Distance in 100 rotations = 100 × 1320/7 = 132000/7 cm ≈ 18857.14 cm.
Convert to metres: 18857.14 ÷ 100 ≈ 188.57 m.
Distance ≈ 188.57 m (132000/7 cm).

Q8 Find the area of a quadrant of a circle whose circumference is 66 cm.

SOLUTION

Circumference = 2πr = 66, so 2 × (22/7) × r = 66.
(44/7) r = 66, so r = 66 × 7 ÷ 44 = 462 ÷ 44 = 10.5 cm.
Area of the full circle = πr² = (22/7) × (10.5)² = (22/7) × 110.25 = 346.5 cm².
A quadrant is one-fourth of the circle: 346.5 ÷ 4 = 86.625 cm².
Area of the quadrant = 86.625 cm².

Q9 The wheel of a car has an outer radius of 28 cm. Calculate how far the car travels after one complete turn of the wheel, and how many times the wheel turns during a journey of 1 km.

SOLUTION

Distance in one turn = circumference = 2πr = 2 × (22/7) × 28.
= 2 × 22 × 4 = 176 cm = 1.76 m.

Number of turns in 1 km:
1 km = 1000 m = 100000 cm.
Number of turns = total distance ÷ distance per turn = 100000 ÷ 176 = 568.18… ≈ 568 turns (about 568 complete turns, with a little left over).
One turn = 176 cm (1.76 m); the wheel turns 100000/176 ≈ 568 times in 1 km.

Q10 Two rectangles have the same area and the same perimeter. Does this mean that they are congruent to each other?

SOLUTION

Let a rectangle have length l and width w. Its area is lw and its (half-)perimeter is l + w.

If two rectangles have the same area A = lw and the same perimeter, then l + w is fixed and lw is fixed for both.

So l and w are the two roots of the same quadratic equation $t^2 - (l + w)t + lw = 0$. A quadratic has just one pair of roots.

Therefore both rectangles must have the same pair of side lengths {l, w}.

Two rectangles with the same length and the same width are congruent (one is just a copy of the other).

So yes — if two rectangles have the same area and the same perimeter, they must be congruent.

Yes. Same area and same perimeter force the same pair of side lengths, so the rectangles are congruent.

Q11 You know that the area of a parallelogram is base × height. Using this and the figure, show that the area of a trapezium is half the sum of the parallel sides × height, i.e., ½(a + b)h.

SOLUTION

Take the trapezium with parallel sides of lengths a and b and height (distance between the parallel sides) h.

Make an identical copy of the trapezium and rotate it 180°. Fit the copy against the original along a slanted (non-parallel) side.

The two trapezia together form a parallelogram. Its base is a + b (the two parallel sides now lie end to end along one line), and its height is the same h.

Area of the parallelogram = base × height = (a + b) × h.

This parallelogram is made of two identical trapezia, so the area of one trapezium is half of it.

Area of trapezium = ½ × (a + b) × h.

Area of trapezium = ½(a + b)h (two copies form a parallelogram of base a+b and height h).

Q12 By dividing a trapezium into two triangles show that its area is half the sum of the parallel sides multiplied by the height (the same formula as the one given above).

SOLUTION

Let the trapezium be ABCD with AB = a and DC = b as the parallel sides, and let h be the distance between them (the height).

Draw the diagonal AC. This divides the trapezium into two triangles: $\triangle ABC$ and $\triangle ACD$.

In $\triangle ABC$, take AB = a as the base. The perpendicular distance from C to line AB is h (the height of the trapezium). So area($\triangle ABC$) = ½ × a × h.

In $\triangle ACD$, take DC = b as the base. The perpendicular distance from A to line DC is the same h.

So area(△ACD) = ½ × b × h.
Area of trapezium = area(△ABC) + area(△ACD) = ½ah + ½bh = ½(a + b)h.
Area of trapezium = ½ah + ½bh = ½(a + b)h.

Q13 Show how we can use two identical copies of a trapezium to make a parallelogram. How will this give us the formula for the area of a trapezium?

SOLUTION

Take a trapezium with parallel sides a and b and height h.
Make an exact copy of it. Turn the copy upside down (rotate it 180°) and place it next to the original so that a slanted side of the copy meets a slanted side of the original.
The two together form a parallelogram: the long parallel side a of one lines up next to the short parallel side b of the other, so the parallelogram has base (a + b). Its height is still h.
Area of this parallelogram = base × height = (a + b)h.
Since the parallelogram is exactly two copies of the trapezium, one trapezium has half this area.
Area of trapezium = ½(a + b)h.
Two copies (one rotated 180°) form a parallelogram of base (a+b) and height h, area (a+b)h; so one trapezium = ½(a+b)h.

Q14 Show that the area of a kite is half the product of its diagonals. Show this: (i) using algebra, and (ii) using geometry.

SOLUTION

Let the kite have diagonals of lengths d_1 and d_2. In a kite the diagonals are perpendicular, and one diagonal bisects the other. Let d_1 be the diagonal that is cut into two equal halves, each of length $d_1/2$, by the other diagonal d_2.

(i) Using algebra (dividing into two triangles by the diagonal d_2):
The diagonal d_2 splits the kite into two triangles, each having base d_2 and heights that are the two halves of d_1. Let those two heights be p and q with $p + q = d_1$.
Area = $\frac{1}{2} \times d_2 \times p + \frac{1}{2} \times d_2 \times q = \frac{1}{2} \times d_2 \times (p + q) = \frac{1}{2} \times d_2 \times d_1 = \frac{1}{2} d_1 d_2$.

(ii) Using geometry (enclosing rectangle):
Draw a rectangle whose sides are parallel to the diagonals, of length d_1 and width d_2, so that the kite fits inside with its vertices touching the rectangle's sides.
Because the diagonals are perpendicular and one bisects the other, the four right triangles between the kite and the rectangle are congruent in matching pairs to the four right triangles inside the kite. So the kite covers exactly half of the rectangle.
Area of kite = ½ × (area of rectangle) = ½ × d_1 × d_2.
Area of a kite = ½ × (product of diagonals) = ½ $d_1 d_2$.

Q15 Three problems about fitting congruent shapes together. (i) Rectangle ABCD has sides a, b, and rectangle PQRS has sides 2a, 2b. Show that PQRS has 4 times the area of ABCD. Does this mean that 4 copies of rectangle ABCD will fit into rectangle PQRS? Check and see! (ii) △ABC has sides a, b, c, and △PQR has sides 2a, 2b, 2c. Show that △PQR has 4 times the area of △ABC. Does this mean that 4 copies of △ABC will fit into △PQR? Check and see! (iii) △ABC has sides a, b, c, and △PQR has sides 3a, 3b, 3c. Show that △PQR has 9 times the area of △ABC. Does this mean that 9 copies of △ABC will fit into △PQR? Check and see!

SOLUTION

(i) Area of ABCD = a × b = ab. Area of PQRS = 2a × 2b = 4ab = 4 × ab.
So PQRS has 4 times the area of ABCD. And yes — divide PQRS into halves along each side (each 2a-side into two a-parts, each 2b-side into two b-parts). This makes a 2×2 grid of a×b rectangles, i.e. exactly 4 copies of ABCD fit into PQRS.

(ii) △PQR has each side twice the matching side of △ABC, so it is similar to △ABC with scale factor 2. When lengths scale by 2, areas scale by $2^2 = 4$. So area(△PQR) = 4 × area(△ABC).
And yes — the larger triangle can be tiled by 4 copies of the smaller: join the midpoints of the three sides of △PQR. This makes 4 smaller triangles, each congruent to △ABC (the middle one is turned over), so 4 copies of △ABC fit into △PQR.

(iii) Now the scale factor is 3, so areas scale by $3^2 = 9$. So area(△PQR) = 9 × area(△ABC).
And yes — divide each side of △PQR into 3 equal parts and draw lines parallel to the sides. This cuts △PQR into 9 small triangles, each congruent to △ABC, so 9 copies of △ABC fit into △PQR.
(i) 4× area, 4 copies fit. (ii) 4× area (scale 2 → area ×4), 4 copies fit. (iii) 9× area (scale 3 → area ×9), 9 copies fit.

Q16 Fig. 6.43: What fraction of the triangle is shaded? Fig. 6.44: What fraction of the square is shaded?

SOLUTION

Read the tick marks — they show exactly how each side is divided.

Fig. 6.43 (triangle). Call the triangle T (apex), B (bottom-left) and R (bottom-right).
The tick marks show: the LEFT side TB carries double ticks marking its two equal halves, so its midpoint M has TM = MB; the RIGHT side TR carries double ticks marking THREE equal parts, so two points P and Q trisect it with TP = PQ = QR - P is 1/3 of the way from T to R, and Q is 2/3 of the way.
The shaded region is the quadrilateral B–M–P–Q. Its two unshaded companions are the top triangle TMP and the bottom triangle RQB, so:
shaded = area(TBR) − area(TMP) − area(RQB).

Triangle TMP shares the angle at T with triangle TBR, with TM = (1/2)TB and TP = (1/3)TR, so its area scales by the product of those ratios:
area(TMP) = (1/2) × (1/3) × area(TBR) = (1/6) area(TBR).

Triangles RQB and RTB share the vertex B and have bases RQ and RT on the same line (the right side); since Q is 2/3 of the way from T to R, RQ = (1/3)RT, so
area(RQB) = (1/3) area(RTB) = (1/3) area(TBR).

Therefore shaded = area(TBR) × (1 − 1/6 − 1/3) = area(TBR) × (1 − 1/2) = (1/2) area(TBR).
So 1/2 of the triangle is shaded.

Fig. 6.44 (square). The tick marks show every side is bisected. Each corner is joined to the midpoint of a non-adjacent side, the same way round — a pinwheel of four congruent segments that encloses the small tilted shaded square.
Put the square at (0,0), (1,0), (1,1), (0,1) and draw the four segments (0,0)→(1/2,1), (1,0)→(0,1/2), (1,1)→(1/2,0), (0,1)→(1,1/2).
The points where consecutive segments cross are (0.2, 0.4), (0.6, 0.2), (0.8, 0.6), (0.4, 0.8). These are the corners of the inner square; its side² = 0.2, so its area = 1/5 of the unit square.
So 1/5 of the square is shaded.
Fig. 6.43: 1/2 of the triangle is shaded (left side bisected at M, right side trisected at P and Q; unshaded top triangle = 1/6, unshaded bottom triangle = 1/3, leaving 1/2). Fig. 6.44: 1/5 of the square is shaded (the central square of the corner-to-midpoint pinwheel).

Q17 Fig. 6.45: What fraction of the rectangle is covered by the circles? Fig. 6.46: What fraction of the rectangle is covered by the circles?

SOLUTION

Both figures show equal circles packed in a row inside a rectangle, each circle touching the top and bottom edges and its neighbours. Fig. 6.45 has 3 such circles; Fig. 6.46 has 4.
Each circle of radius r sits exactly inside a square cell of side 2r (it touches all four sides of the cell). In one cell:
- area of the circle = pi r²,
- area of the cell = 2r x 2r = 4r².
So each circle covers pi r² / 4r² = pi/4 of its cell.
The whole rectangle is just these equal square cells placed side by side (3 cells in Fig. 6.45, 4 in Fig. 6.46), so the covered fraction is the same for the whole rectangle, independent of the number of circles:
fraction covered = pi/4 ~ (22/7)/4 = 11/14 ~ 0.79.
In both Fig. 6.45 (3 circles) and Fig. 6.46 (4 circles) each circle is inscribed in a 2r x 2r square cell, so the covered fraction is pi/4 ~ 11/14 ~ 0.79 - the same regardless of the number of circles.

Q18 Use the above to make a conjecture about the area occupied by circles fitted into a rectangle in the manner shown. Test your conjecture for particular cases: 10 circles; 20 circles; 50 circles. Then prove your conjecture!

SOLUTION

Conjecture: When equal circles are packed into a rectangle so that each circle just fits inside a square cell (each cell of side equal to the circle's diameter, cells filling the rectangle exactly), the fraction of the rectangle covered by the circles is always π/4 — no matter how many circles there are.

Testing for particular cases:
Let each circle have radius r, so diameter 2r; each square cell has area $(2r)^2 = 4r^2$ and each circle has area πr^2.
• 10 circles: circles' area = $10\pi r^2$; rectangle's area = $10 \times 4r^2 = 40r^2$. Fraction = $10\pi r^2/40r^2 = \pi/4$.
• 20 circles: $20\pi r^2/(20 \times 4r^2) = 20\pi r^2/80r^2 = \pi/4$.
• 50 circles: $50\pi r^2/(50 \times 4r^2) = 50\pi r^2/200r^2 = \pi/4$.
In every case the fraction is π/4.

Proof: With n circles, total circle area = $n\pi r^2$ and total rectangle area = $n \times 4r^2 = 4nr^2$. Fraction = $n\pi r^2 \div 4nr^2 = \pi/4$. The n cancels, so the fraction is π/4 for every n. (Using $\pi \approx 22/7$, this is about $11/14 \approx 0.79$.)
The circles always cover the fraction π/4 ($\approx 11/14 \approx 0.79$) of the rectangle, independent of the number of circles; proof: $n\pi r^2/(4nr^2) = \pi/4$.

Q19 The figure (Fig. 6.47) shows nine identical rectangles fitted together to make a large rectangle whose area is 72 cm². Find the perimeter of each small rectangle.

SOLUTION

Let each small rectangle have length L (long side) and width w (short side).
Reading Fig. 6.47: the nine rectangles form two rows. The top row is 4 rectangles laid with their long side L horizontal (each contributes width L); the bottom row is 5 rectangles laid with their short side w horizontal (each contributes width w).
The big rectangle has one width, measured either along the top or the bottom, so the two must be equal:
$4L = 5w$. ...(1)

All nine identical rectangles tile the big rectangle of area 72 cm², so one small rectangle has area:
$L \times w = 72 \div 9 = 8$ cm². ...(2)

From (1), $L = 5w/4$. Substitute into (2):
$(5w/4) \times w = 8 \Rightarrow 5w^2/4 = 8 \Rightarrow w^2 = 32/5 = 6.4 \Rightarrow w = \sqrt{6.4} = (4\sqrt{10})/5 \approx 2.53$ cm.
Then $L = 5w/4 = \sqrt{10} \approx 3.16$ cm. Check: $L \times w = \sqrt{10} \times (4\sqrt{10})/5 = 40/5 = 8$ ✓.

Perimeter of one small rectangle = $2(L + w) = 2(\sqrt{10} + (4\sqrt{10})/5) = 2 \times (9\sqrt{10})/5 = (18\sqrt{10})/5 \approx 11.38$ cm.

Each small rectangle is L = √10 ≈ 3.16 cm by w = (4√10)/5 ≈ 2.53 cm (area 8 cm²); perimeter = (18√10)/5 ≈ 11.38 cm.

Q20 Fig. 6.48 shows lines drawn from a vertex of a triangle to the points of trisection of the opposite side. Show that the areas of the shaded blue triangle and the shaded red triangle are equal.

SOLUTION

Let the triangle be ABC with the opposite side BC trisected at points P and Q, so BP = PQ = QC = (1/3)BC. Lines AP and AQ are drawn from vertex A. This splits △ABC into three triangles: △ABP, △APQ, △AQC.

All three triangles have equal bases (BP = PQ = QC) along the same line BC, and they all have the same height — the perpendicular distance from A to line BC.

Since area of a triangle = ½ × base × height, and both base and height are the same for all three, the three triangles △ABP, △APQ and △AQC have equal areas (each is one-third of the area of △ABC).

The shaded blue triangle and the shaded red triangle are two of these three equal-area triangles.

Therefore the shaded blue triangle and the shaded red triangle have equal area.

Trisecting BC gives three sub-triangles with equal base and the same height, so each has area (1/3)area(△ABC); the blue and red shaded triangles are two of these, hence equal.

Q21 Fig. 6.49 shows a quarter circle in a square with its centre at one vertex, passing through the two adjacent vertices. There are two semicircles on two adjacent sides as diameters. They create the shaded regions A and B. Show that A and B have equal area.

SOLUTION

Let the square have side length a. Place the centre of the quarter circle at one vertex; it passes through the two adjacent vertices, so the quarter circle has radius a. Two semicircles are drawn on two adjacent sides of the square as diameters, so each semicircle has diameter a and radius a/2.

Area of the quarter circle = ¼ × πa² = πa²/4.

Area of one semicircle (radius a/2) = ½ × π × (a/2)² = ½ × π × a²/4 = πa²/8.

Area of the two semicircles together = 2 × πa²/8 = πa²/4.

So the quarter circle and the two semicircles have exactly the same total area (both equal πa²/4).

Now regions A and B are what is left after the quarter circle and the semicircles overlap inside the square. Because the quarter-circle area equals the combined semicircle area, subtracting the common overlapping parts from each leaves regions of equal area.

In detail: let the overlap between the two semicircles and the quarter circle be the shared pieces. Region A = (part of semicircles not covered by the quarter circle) and Region B = (part of the quarter circle not covered by the semicircles). Since (two semicircles) = (quarter circle) in

area, removing their common overlap from both sides preserves equality, giving Area(A) = Area(B).

The quarter circle has area $\pi a^2/4$ and the two semicircles together also have area $\pi a^2/4$; since their total areas are equal, the leftover shaded regions A and B are equal.

Q22 In Fig. 6.50, four semicircles have been drawn within the given square whose side is 2 units. The centres of these semicircles are the midpoints of the sides. They create a 4-petalled flower (shown in blue). Find the perimeter and the area of this flower.

SOLUTION

The square has side 2 units. Each semicircle has its diameter equal to a side of the square (2 units) with centre at the midpoint of that side, so each semicircle has radius r = 1 unit and is drawn inward.

Perimeter of the flower:

The boundary of the 4-petalled flower is made up of 4 semicircular arcs (one from each semicircle), each of radius 1.

Length of one semicircular arc = $\pi r = \pi \times 1 = \pi$ units.

Perimeter of the flower = $4 \times \pi = 4\pi \approx 12.57$ units.

Area of the flower:

Consider the two semicircles drawn on a pair of opposite sides — together they can be viewed as one full circle of radius 1 (area π) inside the square. The four semicircles form two such 'circles', total area $2 \times \pi(1)^2 = 2\pi$, but they lie inside the square of area $2^2 = 4$ and overlap exactly on the 4 petals.

The overlap of the two circle-areas (which is the flower) equals (sum of the two circle areas) − (area of the square they fill): $2\pi - 4$.

So area of the 4-petalled flower = $2\pi - 4 \approx 2.28$ square units.

Numerical check confirms $2\pi - 4 \approx 2.28$.

Perimeter = $4\pi \approx 12.57$ units; Area = $2\pi - 4 \approx 2.28$ square units.

Q23 In Fig. 6.51 we see two concentric circles with a common centre O. A chord BC of the larger circle is drawn, touching the smaller circle at A. The length of BC is l. Show that the area of the green region enclosed between the two circles is $(1/4)\pi l^2$.

SOLUTION

Let the larger circle have radius R and the smaller circle radius r, both centred at O.

BC is a chord of the larger circle that touches (is tangent to) the smaller circle at A. A tangent is perpendicular to the radius at the point of contact, so OA $\perp$ BC and OA = r.

Since OA is perpendicular to the chord BC from the centre O, it bisects BC. So A is the midpoint of BC, giving BA = AC = l/2.

In right triangle OAB (right-angled at A): OB = R, OA = r, AB = l/2. By the Baudhāyana–Pythagoras theorem, $OB^2 = OA^2 + AB^2$, i.e. $R^2 = r^2 + (l/2)^2$.

So $R^2 - r^2 = (l/2)^2 = l^2/4$.
The green region between the two circles is the annulus (ring). Its area = $\pi R^2 - \pi r^2 = \pi(R^2 - r^2) = \pi \times (l^2/4) = (1/4)\pi l^2$. (Shown.)
Using tangent $\perp$ radius and Pythagoras, $R^2 - r^2 = (l/2)^2 = l^2/4$, so the ring area = $\pi(R^2 - r^2) = \frac{1}{4}\pi l^2$.

Q24 In Fig. 6.52, semicircles have been drawn on all the sides of a right-angled triangle as shown. Show that Area(A) + Area(B) = Area(C).

SOLUTION

Let the right-angled triangle have legs of length a and b and hypotenuse of length c. By the Baudhāyana–Pythagoras theorem, $a^2 + b^2 = c^2$.
On each side a semicircle is drawn with that side as diameter. A semicircle of diameter s has radius s/2, so its area = $\frac{1}{2} \times \pi \times (s/2)^2 = \frac{1}{2} \times \pi \times s^2/4 = \pi s^2/8$.
Semicircle A is on a leg (say length a): Area(A) = $\pi a^2/8$.
Semicircle B is on the other leg (length b): Area(B) = $\pi b^2/8$.
Semicircle C is on the hypotenuse (length c): Area(C) = $\pi c^2/8$.
Now Area(A) + Area(B) = $\pi a^2/8 + \pi b^2/8 = (\pi/8)(a^2 + b^2)$.
Since $a^2 + b^2 = c^2$, this equals $(\pi/8)c^2 = \pi c^2/8$ = Area(C).
Therefore Area(A) + Area(B) = Area(C). (This is a 'lune-style' consequence of the Pythagoras theorem.)
Each semicircle has area $\pi(\text{side})^2/8$, so Area(A)+Area(B) = $(\pi/8)(a^2+b^2) = (\pi/8)c^2$ = Area(C).

Q25 Fig. 6.53 shows two congruent circles (radius r) passing through each other's centres. Find the area of the region enclosed by the two circles in terms of the common radius r.

SOLUTION

Let the two circles have centres A and B, each of radius r. Each circle passes through the other's centre, so the distance between centres is AB = r.
The region enclosed by the two circles is their union. Union area = (area of circle A) + (area of circle B) − (area of the overlapping lens where they intersect).

Area of each circle = πr^2, so the two circles together give $2\pi r^2$.

Overlap (lens) area: the circles meet at two points; join them to the centres. In triangle formed by A, B and an intersection point, all three sides equal r (AB = r, and each intersection point is at distance r from both centres), so it is equilateral. The chord joining the two intersection points subtends 120° at each centre.
The lens is made of two identical circular segments, each from a 120° sector minus an equilateral-related triangle. Computing the standard lens area for centre-distance d = r gives lens area = $(2\pi/3 - \sqrt{3}/2)\, r^2$.

Union area = $2\pi r^2 - (2\pi/3 - \sqrt{3}/2)\, r^2 = (2\pi - 2\pi/3 + \sqrt{3}/2)\, r^2 = (4\pi/3 + \sqrt{3}/2)\, r^2$.
Numerically, $(4\pi/3 + \sqrt{3}/2) \approx 5.05$, so the enclosed area $\approx 5.05\, r^2$.

Enclosed (union) area = $(4\pi/3 + \sqrt{3}/2)\, r^2 \approx 5.05\, r^2$.

Q26 In Fig. 6.54, we see three triangles within a rectangle. The areas of the triangles are A, B, C, as marked. Show that the area of the rectangle is 2(A + C)(B + C)/C.

SOLUTION

What the figure shows: the rectangle has a point P on its TOP edge. Below P is an interior point M (directly under P); from M three segments are drawn - up to P, down to the bottom-left corner, and across to a point Q on the RIGHT edge (Q at the same height as M). The vertical segment PM and the horizontal segment MQ meet at a right angle at M. This creates the three marked triangles: A (blue, on the left), C (pink, the right triangle P-M-Q), and B (green, below).

Let the rectangle have width w and height h, so its area S = w*h. Let a be the horizontal distance from the left edge to P (and M), and let v be the vertical distance PM (from the top edge down to M). Then the horizontal leg of C is u = MQ = w - a, and its vertical leg is v.

Read the three areas off the figure:
C = (1/2) * u * v (right angle at M, legs u = w - a and v). ...(1)
A + C = (1/2) * w * v (these two triangles together have the FULL width w as base and height v). ...(2)
B + C = (1/2) * h * u (these two together have the FULL height h as base and width u). ...(3)
(Check with coordinates A(0,0)-style: A = (1/2)a v, C = (1/2)(w-a)v, B = (1/2)(h-v)(w-a); indeed A+C = (1/2)wv and B+C = (1/2)h(w-a) = (1/2)hu.)

Form the product of (2) and (3), divided by (1):
2(A + C)(B + C) / C = 2 * ((1/2) w v)((1/2) h u) / ((1/2) u v) = 2 * (w v h u / 4) / (u v / 2) = w h.

The u's and v's cancel, leaving w*h = S.
Therefore area of the rectangle = 2(A + C)(B + C)/C. (Shown.) The target is a rational expression, not a square root.
Writing the pink right triangle as C = (1/2)uv (right angle at the interior point M), and the two rectangle-spanning pairs as A+C = (1/2)*w*v and B+C = (1/2)*h*u, gives 2(A+C)(B+C)/C = 2*((1/2)wv)((1/2)hu)/((1/2)uv) = wh = area of the rectangle.

Q27 In Fig. 6.55 we see two shaded regions formed by a quarter circle, a semicircle, and a triangle. Show that the areas of the two shaded regions are equal.

SOLUTION

What the figure shows (labelled points): A, O, C lie on a straight base line with O the midpoint of AC. A semicircle is drawn on AC as diameter, centred at O, curving upward through B. B is the top point of this semicircle, directly above O. A quarter-circle arc is drawn with centre A, from B up to E on the left. The chord AB and segment BO bound the triangle. The two shaded regions are the hatched 'lune' between the quarter-circle arc and the semicircle arc, and the segment/triangle piece the arcs cut out.

Set coordinates A(0,0), O(1,0), C(2,0), so AC = 2 and the semicircle on AC has radius 1, centre O.
B is the top of the semicircle: B(1,1). Then AB = $\sqrt{(1^2 + 1^2)} = \sqrt{2}$.
The quarter circle has centre A and radius AB = $\sqrt{2}$ (it passes through B and reaches E on the left).

Areas of the two curved pieces:
• Semicircle on AC (radius 1): area = $\frac{1}{2} \cdot \pi \cdot 1^2 = \pi/2$.
• Quarter circle centre A (radius $\sqrt{2}$): area = $\frac{1}{4} \cdot \pi \cdot (\sqrt{2})^2 = \frac{1}{4} \cdot \pi \cdot 2 = \pi/2$.
So the semicircle and the quarter circle have EQUAL area, each π/2 — the fact the figure is built on.

Let S1 = the upper lune between the quarter-circle arc and the semicircle arc, and S2 = the other shaded piece (segment/triangle). Let X be the region common to both the quarter-circle sector and the semicircle on that side.
Quarter-circle region = S1 + X => π/2 = S1 + X.
Semicircle region on the same side = S2 + X => π/2 = S2 + X.
Subtracting: (S1 + X) − (S2 + X) = π/2 − π/2 = 0, so S1 = S2.

Therefore the two shaded regions have equal area — because the quarter circle and semicircle have the same area (π/2), removing their common overlap leaves equal shaded pieces. (There is NO square in the figure.)
With A(0,0), O(1,0), C(2,0), B(1,1): the semicircle on AC (radius 1) has area π/2 and the quarter circle centred at A (radius $\sqrt{2}$) also has area π/2. Being equal, subtracting their common overlap leaves the two shaded regions equal.