Chapter 8: Predicting What Comes Next: Exploring Sequences and Progressions

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Exercise 8.1 — Sequences — Explicit and Recursive Rules

Q1 Find the first five terms of the sequence in which the nth term is given by (i) tn = 3n – 4, (ii) tn = 2 – 5n, and (iii) tn = n² – 2n + 3 for n ≥ 1.

SOLUTION

(i) tn = 3n – 4
t1 = 3(1) – 4 = –1
t2 = 3(2) – 4 = 2
t3 = 3(3) – 4 = 5
t4 = 3(4) – 4 = 8
t5 = 3(5) – 4 = 11

(ii) tn = 2 – 5n
t1 = 2 – 5(1) = –3
t2 = 2 – 5(2) = –8
t3 = 2 – 5(3) = –13
t4 = 2 – 5(4) = –18
t5 = 2 – 5(5) = –23

(iii) tn = n² – 2n + 3
t1 = 1 – 2 + 3 = 2
t2 = 4 – 4 + 3 = 3
t3 = 9 – 6 + 3 = 6
t4 = 16 – 8 + 3 = 11
t5 = 25 – 10 + 3 = 18
(i) –1, 2, 5, 8, 11 (ii) –3, –8, –13, –18, –23 (iii) 2, 3, 6, 11, 18

Q2 Find the 10th and 15th terms of the sequence tn = 5n – 3 for n ≥ 1.

SOLUTION

t10 = 5(10) – 3 = 50 – 3 = 47
t15 = 5(15) – 3 = 75 – 3 = 72
t10 = 47, t15 = 72

Q3 Determine whether 97 and 172 are terms of the sequence tn = 5n – 3 for n ≥ 1.

SOLUTION

For a value v to be a term, we need 5n – 3 = v to have a solution where n is a positive integer.

For 97: 5n – 3 = 97 => 5n = 100 => n = 20.
Since n = 20 is a positive integer, 97 IS a term. It is the 20th term.
Verify: t20 = 5(20) – 3 = 97. ✓

For 172: 5n – 3 = 172 => 5n = 175 => n = 35.
Since n = 35 is a positive integer, 172 IS also a term. It is the 35th term.
Verify: t35 = 5(35) – 3 = 172. ✓
Both 97 and 172 are terms of the sequence. 97 = t20 and 172 = t35.

Q4 Which term of the sequence tn = 5n – 3 for n ≥ 1 is 607?

SOLUTION

Set tn = 607 and solve for n:
5n – 3 = 607
5n = 610
n = 122
Verify: t122 = 5(122) – 3 = 610 – 3 = 607. ✓
607 is the 122nd term of the sequence.

Q5 A sequence is given by the recursive rule t1 = –5, tn+1 = tn + 3 for n ≥ 1. Find the first five terms of the sequence. Is 52 a term of this sequence? If so, which term is it?

SOLUTION

First five terms:
t1 = –5
t2 = t1 + 3 = –5 + 3 = –2
t3 = t2 + 3 = –2 + 3 = 1
t4 = t3 + 3 = 1 + 3 = 4
t5 = t4 + 3 = 4 + 3 = 7

The sequence is an AP with first term –5 and common difference 3.
The explicit (general) formula is: tn = –5 + (n – 1) × 3 = 3n – 8.

Is 52 a term? Set 3n – 8 = 52:
3n = 60 => n = 20.
Since n = 20 is a positive integer, 52 IS a term.
Verify: t20 = 3(20) – 8 = 60 – 8 = 52. ✓
First five terms: –5, –2, 1, 4, 7. Yes, 52 is the 20th term.

Q6 Let T1 = 1, T2 = 2, T3 = 4, and Tn = Tn–1 + Tn–2 + Tn–3 for n ≥ 4. Find T4, T5, T6, T7, and T8.

SOLUTION

T4 = T3 + T2 + T1 = 4 + 2 + 1 = 7
T5 = T4 + T3 + T2 = 7 + 4 + 2 = 13
T6 = T5 + T4 + T3 = 13 + 7 + 4 = 24
T7 = T6 + T5 + T4 = 24 + 13 + 7 = 44
T8 = T7 + T6 + T5 = 44 + 24 + 13 = 81
T4 = 7, T5 = 13, T6 = 24, T7 = 44, T8 = 81

Exercise 8.2 — Arithmetic Progressions

Q1 Find the 10th and 26th terms of the AP: 3, 8, 13, 18, ….

SOLUTION

First term a = 3, common difference d = 8 – 3 = 5.
General term: tn = a + (n – 1)d = 3 + (n – 1) × 5.

t10 = 3 + 9 × 5 = 3 + 45 = 48
t26 = 3 + 25 × 5 = 3 + 125 = 128
t10 = 48, t26 = 128

Q2 Which term of the AP: 21, 18, 15, … is –81? Also, is 0 a term of this AP? Give reasons for your answer.

SOLUTION

First term a = 21, common difference d = 18 – 21 = –3.
General term: tn = 21 + (n – 1)(–3) = 21 – 3n + 3 = 24 – 3n.

Which term is –81?
24 – 3n = –81
3n = 105
n = 35
Verify: t35 = 24 – 3(35) = 24 – 105 = –81. ✓

Is 0 a term?
24 – 3n = 0 => n = 8.
Since n = 8 is a positive integer, 0 IS a term.
Verify: t8 = 24 – 3(8) = 24 – 24 = 0. ✓
–81 is the 35th term. Yes, 0 is a term — it is the 8th term (t8 = 0).

Q3 Find the nth term of the AP: 11, 8, 5, 2 … Write the recursive rule for this AP.

SOLUTION

First term a = 11, common difference d = 8 – 11 = –3.

Explicit formula:
tn = 11 + (n – 1)(–3) = 11 – 3n + 3 = 14 – 3n

Verify: t1 = 14 – 3 = 11 ✓, t2 = 14 – 6 = 8 ✓, t3 = 14 – 9 = 5 ✓, t4 = 14 – 12 = 2 ✓

Recursive rule: t1 = 11, tn+1 = tn – 3 for n ≥ 1.

Q4 An AP consists of 50 terms in which the 3rd term is 12 and the last term is 106. Find the 29th term.

SOLUTION

Let a = first term, d = common difference.

From the given conditions:
t3 = a + 2d = 12 … (1)
t50 = a + 49d = 106 … (2)

Subtract (1) from (2):
47d = 94 => d = 2

From (1): a = 12 – 2(2) = 8

Verify: t3 = 8 + 2(2) = 12 ✓, t50 = 8 + 49(2) = 106 ✓

t29 = a + 28d = 8 + 28(2) = 8 + 56 = 64
The 29th term is 64.

Q5 How many 2-digit numbers are divisible by 3? What is the sum of all these 2-digit numbers?

SOLUTION

The 2-digit multiples of 3 form an AP: 12, 15, 18, …, 99.
a = 12, d = 3, last term l = 99.

Number of terms:
99 = 12 + (n – 1) × 3
87 = (n – 1) × 3
n – 1 = 29 => n = 30

There are 30 two-digit multiples of 3.

Sum of the AP:
S = n/2 × (a + l) = 30/2 × (12 + 99) = 15 × 111 = 1665
There are 30 two-digit numbers divisible by 3. Their sum is 1665.

Q6 Harish started work at an annual salary of 5,00,000 and received an increment of 20,000 each year. After how many years did his income reach 7,00,000?

SOLUTION

Annual salaries form an AP:
a = 5,00,000, d = 20,000.

General term: tn = 5,00,000 + (n – 1) × 20,000.

Set tn = 7,00,000:
5,00,000 + (n – 1) × 20,000 = 7,00,000
(n – 1) × 20,000 = 2,00,000

n – 1 = 10 => n = 11

So his salary is 7,00,000 in the 11th year of work, which is after 10 years.
Verify: 5,00,000 + 10 × 20,000 = 7,00,000 ✓
After 10 years, Harish's income reached 7,00,000 (in his 11th year of work).

Q7 A child arranges marbles in rows so that the first row has 1 marble, the second has 2 marbles, the third has 3, and so on up to 25 rows. How many marbles does the child use in all?

SOLUTION

The total number of marbles = 1 + 2 + 3 + … + 25.

This is the sum of the first 25 natural numbers.
Using the formula S = n(n + 1)/2:
S = 25 × 26 / 2 = 325
The child uses 325 marbles in all.

Exercise 8.3 — Geometric Progressions

Q1 Find the 12th term of a GP with common ratio 2, whose 8th term is 192.

SOLUTION

Given: r = 2, t8 = 192.

Method: $t_{12} = t_8 \times r^4$
$t_{12} = 192 \times 2^4 = 192 \times 16 = 3072$

Alternatively, find a first:
$t_8 = a \times 2^7 = 128a = 192 \Rightarrow a = 3/2$
$t_{12} = (3/2) \times 2^{11} = (3/2) \times 2048 = 3072$
The 12th term is 3072.

Q2 Find the 10th and nth terms of the GP: 5, 25, 125, ….

SOLUTION

First term a = 5, common ratio r = 25/5 = 5.

General term: tn = a × r^(n–1) = 5 × 5^(n–1) = 5^n.

$t_{10} = 5^{10} = 9{,}765{,}625$
tn = 5^n. The 10th term is $5^{10} = 9{,}765{,}625$.

Q3 (*) A sequence is given by the recursive rule $t_1 = 2$, $t_{n+1} = 3t_n - 2$ for $n \geq 1$. Which term of the sequence is 730?

SOLUTION

First, find the general term by substitution $u_n = t_n - 1$.
Then: $u_{n+1} = t_{n+1} - 1 = (3t_n - 2) - 1 = 3t_n - 3 = 3(t_n - 1) = 3\,u_n$.
So (u_n) is a GP with $u_1 = t_1 - 1 = 2 - 1 = 1$ and common ratio 3.
$\Rightarrow u_n = 3^{(n-1)}$
$\Rightarrow t_n = 1 + 3^{(n-1)}$.

Verify: $t_1 = 1 + 1 = 2$ ✓, $t_2 = 1 + 3 = 4$ ✓, $t_3 = 1 + 9 = 10$ ✓.

Set $t_n = 730$:
$1 + 3^{(n-1)} = 730$
$3^{(n-1)} = 729 = 3^6$
$n - 1 = 6 \Rightarrow n = 7$

Verify: $t_7 = 1 + 3^6 = 1 + 729 = 730$ ✓
730 is the 7th term of the sequence.

Q4 Which term of the GP: 2, 6, 18, … is 4374? Write the explicit formula as well as the recursive formula for the nth term.

SOLUTION

First term a = 2, common ratio r = 6/2 = 3.

Explicit formula: tn = 2 × 3^(n–1).

Set tn = 4374:
2 × 3^(n–1) = 4374
3^(n–1) = 2187 = 3^7
n – 1 = 7 => n = 8

Verify: t8 = 2 × 3^7 = 2 × 2187 = 4374 ✓

Recursive formula: t1 = 2, t_{n+1} = 3 t_n for n ≥ 1.
4374 is the 8th term. Explicit: tn = 2 × 3^(n–1). Recursive: t1 = 2, tn+1 = 3tn.

Q5 A ball is dropped from a height of 80 metres. After hitting the ground, it bounces back to 60% of the height from which it fell. It continues bouncing in this way — each time rising to 60% of the previous height. (i) What height does the ball reach after the 5th bounce? (ii) What is the total vertical distance the ball has travelled by the time it hits the ground for the 6th time?

SOLUTION

The heights after each bounce form a GP with first term h1 = 80 × 0.6 = 48 m and r = 0.6 = 3/5.
The height after the nth bounce is: h_n = 80 × (3/5)^n.

(i) Height after the 5th bounce:
h5 = 80 × $(3/5)^5$ = 80 × 243/3125 = 19440/3125 = 3888/625 m ≈ 6.22 m

(ii) Total vertical distance when the ball hits the ground for the 6th time:
The ball falls 80 m (1st hit), then for each of the next 5 hits it travels up and down:

Initial fall: 80 m
Bounce 1: up 48 m + down 48 m = 96 m
Bounce 2: up 28.8 m + down 28.8 m = 57.6 m
Bounce 3: up 17.28 m + down 17.28 m = 34.56 m
Bounce 4: up 10.368 m + down 10.368 m = 20.736 m
Bounce 5: up 6.2208 m + down 6.2208 m = 12.4416 m

Total = 80 + 2 × (48 + 28.8 + 17.28 + 10.368 + 6.2208)
Sum of bounce heights = 48 × [1 – $(3/5)^5$] / [1 – 3/5] = 48 × (1 – 243/3125) / (2/5)
= 48 × (2882/3125) × (5/2) = 48 × 14410/6250 = 69168/625
Total = 80 + 2 × 69168/625 = 50000/625 + 138336/625 = 188336/625 = 301.3376 m
(i) 3888/625 m ≈ 6.22 m. (ii) 188336/625 m = 301.3376 m.

Q6 Which term of the sequence 2, 2√2, 4, … is 128?

SOLUTION

First term a = 2, common ratio r = 2√2 / 2 = √2.

General term: tn = 2 × (√2)^(n–1) = 2 × 2^((n–1)/2).

Set tn = 128 = 2^7:
2 × 2^((n–1)/2) = 2^7
2^(1 + (n–1)/2) = 2^7
1 + (n–1)/2 = 7
(n–1)/2 = 6
n – 1 = 12 => n = 13

Verify: t13 = 2 × $(\sqrt{2})^{12}$ = 2 × (2^(1/2))^{12} = 2 × 2^6 = 2 × 64 = 128 ✓
128 is the 13th term of the sequence.

Q7 Fig. 8.12 shows Stages 0 to 3 of the Sierpiński square carpet. Stage 0 is a square sheet. To construct Stage 1, each side is trisected, points of trisection of opposite sides are joined to obtain nine smaller squares, and the centre square is removed. The same process is repeated on the eight retained shaded squares to get Stage 2, and so on. (i) How many red squares are there in Stages 0 to 3? (ii) Can you predict the number of red squares at Stages 4 and 5? (iii) Can you find the explicit formula as well as the recursive formula for the number of red squares at the nth stage? (iv) Suppose the area of the square in Stage 0 is 1 square unit. What is the area of the red region in Stages 1, 2, 3, 4, and 5? What happens to this area as n goes on increasing?

SOLUTION

(i) Stage 0 is the whole uncut square, so it has 1 red square. At each later stage, every retained square is divided into 9 sub-squares and its centre one is removed, leaving 8.
Stage 0: 1 red square.
Stage 1: 8 red squares.
Stage 2: 8 × 8 = 64 red squares.
Stage 3: 8 × 8 × 8 = 512 red squares.

(ii) Stage 4: 8^4 = 4096 red squares.
Stage 5: 8^5 = 32768 red squares.

(iii) Explicit formula: R_n = 8^n (this also gives $R_0 = 8^0 = 1$).
Recursive formula: R_0 = 1, R_{n+1} = 8 × R_n for n ≥ 0.

(iv) Take the Stage-0 square area = 1 sq unit. At Stage n each retained sub-square has side (1/3)^n of the original, so area (1/9)^n, and there are 8^n of them:
Area at Stage n = 8^n × (1/9)^n = (8/9)^n.
Explicit formula for the red area: A_n = (8/9)^n.
Recursive formula for the red area: A_0 = 1, A_{n+1} = (8/9) × A_n for n ≥ 0.

Stage 1: 8/9; Stage 2: 64/81; Stage 3: 512/729; Stage 4: 4096/6561; Stage 5: 32768/59049

sq units.

As n increases, (8/9)^n => 0. The red area approaches zero — the Sierpiński carpet has zero area in the limit.

(i) Stage 0: 1, Stage 1: 8, Stage 2: 64, Stage 3: 512. (ii) Stage 4: 4096; Stage 5: 32768. (iii) explicit $R_n = 8^n$; recursive $R_0 = 1$, $R_{n+1} = 8 \cdot R_n$. (iv) explicit $A_n = (8/9)^n$; recursive $A_0 = 1$, $A_{n+1} = (8/9) \cdot A_n$; as $n \to \infty$, area $\to 0$.

Exercises — Predicting What Comes Next: Exploring Sequences and Progressions

Q1 Find the 31st term of an AP whose 11th term is 38 and 16th term is 73.

SOLUTION

For an AP, the nth term is t_n = a + (n - 1)d, where a is the first term and d the common difference.
Given t_11 = 38 => a + 10d = 38 ... (i)
Given t_16 = 73 => a + 15d = 73 ... (ii)
Subtract (i) from (ii): (a + 15d) - (a + 10d) = 73 - 38 => 5d = 35 => d = 7.
Substitute in (i): a + 10(7) = 38 => a + 70 = 38 => a = -32.
Now t_31 = a + 30d = -32 + 30(7) = -32 + 210 = 178.
The 31st term is 178.

Q2 Determine the AP whose third term is 16 and whose 7th term exceeds the 5th term by 12.

SOLUTION

Let the AP have first term a and common difference d, so t_n = a + (n - 1)d.
Third term: $t_3 = a + 2d = 16$ ... (i)
The 7th term exceeds the 5th term by 12: $t_7 - t_5 = 12$.
$t_7 - t_5 = (a + 6d) - (a + 4d) = 2d = 12 \Rightarrow d = 6$.
Substitute d = 6 in (i): a + 2(6) = 16 => a + 12 = 16 => a = 4.
So the AP is 4, 10, 16, 22, 28, 34, …
The AP is 4, 10, 16, 22, 28, 34, … (a = 4, d = 6).

Q3 How many three-digit numbers are divisible by 7? (Hint: All three-digit numbers divisible by 7 form an AP. Find the smallest and largest such three-digit numbers.)

SOLUTION

The smallest three-digit number divisible by 7: 7 × 15 = 105 (since 7 × 14 = 98 is only two digits).
The largest three-digit number divisible by 7: 7 × 142 = 994 (since 7 × 143 = 1001 exceeds 999).
These form an AP: 105, 112, 119, …, 994, with first term a = 105 and common difference d = 7.
Using t_n = a + (n - 1)d = 994: 105 + (n - 1)(7) = 994.
(n - 1)(7) = 994 - 105 = 889 => n - 1 = 889/7 = 127 => n = 128.
There are 128 three-digit numbers divisible by 7.

Q4 How many multiples of 4 lie between 10 and 250? (Hint: All multiples of 4 form an AP. Find the smallest and largest multiples of 4 between 10 and 250.)

SOLUTION

The smallest multiple of 4 greater than 10 is 12; the largest multiple of 4 less than 250 is 248.
These form an AP: 12, 16, 20, …, 248, with first term a = 12 and common difference d = 4.
Using t_n = a + (n - 1)d = 248: 12 + (n - 1)(4) = 248.
(n - 1)(4) = 248 - 12 = 236 => n - 1 = 236/4 = 59 => n = 60.
There are 60 multiples of 4 between 10 and 250.

Q5 Find a GP for which the sum of the first two terms is -4 and the fifth term is 4 times the third term.

SOLUTION

Let the GP have first term a and common ratio r, so t_n = a·r^(n-1).
Fifth term is 4 times the third term: $a \cdot r^4 = 4 \cdot (a \cdot r^2) \Rightarrow r^4 = 4r^2 \Rightarrow r^2 = 4 \Rightarrow r = 2$ or $r = -2$.
Sum of the first two terms: a + ar = a(1 + r) = -4.
Case r = -2: a(1 + (-2)) = -4 => a(-1) = -4 => a = 4. The GP is 4, -8, 16, -32, 64, … (check: 4 + (-8) = -4; $t_5 = 64 = 4 \times 16 = 4 \cdot t_3$).
Case r = 2: a(1 + 2) = -4 => 3a = -4 => a = -4/3. The GP is -4/3, -8/3, -16/3, -32/3, … (check: -4/3 + (-8/3) = -12/3 = -4).
Two GPs satisfy the conditions: 4, -8, 16, -32, … (r = -2) and -4/3, -8/3, -16/3, … (r = 2).

Q6 Find all possible ways of expressing 100 as the sum of consecutive natural numbers.

SOLUTION

Suppose 100 is the sum of k consecutive natural numbers starting from a: a + (a+1) + … + (a+k-1) = 100.
This sum equals k·a + k(k-1)/2 = 100, so k·a = 100 - k(k-1)/2, giving a = (100 - k(k-1)/2) / k.
We need a to be a positive integer, with k ≥ 2 (a single number is not a genuine 'sum' of consecutive numbers).
For k = 5: 100 - 5·4/2 = 100 - 10 = 90, and 90/5 = 18. So a = 18: 18 + 19 + 20 + 21 + 22 = 100.
For k = 8: 100 - 8·7/2 = 100 - 28 = 72, and 72/8 = 9. So a = 9: 9 + 10 + 11 + 12 + 13 + 14 + 15 + 16 = 100.
No other value of k gives a positive integer a (for larger k the required a is not a whole number or is not positive).
100 = 18 + 19 + 20 + 21 + 22, and 100 = 9 + 10 + 11 + 12 + 13 + 14 + 15 + 16.

Q7 The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of the 2nd hour, 4th hour and nth hour?

SOLUTION

The count doubles every hour, so it forms a GP with initial (0th hour) value 30 and common ratio 2.
After n hours the number is 30 × 2^n.
End of the 2nd hour: 30 × 2^2 = 30 × 4 = 120.
End of the 4th hour: 30 × 2^4 = 30 × 16 = 480.
End of the nth hour: 30 × 2^n.
End of 2nd hour: 120; end of 4th hour: 480; end of nth hour: 30 × 2^n bacteria.

Q8 The sum of the 4th and 8th terms of an AP is 24 and the sum of the 6th and 10th terms is 44. Find the first three terms of the AP.

SOLUTION

Let the AP have first term a and common difference d, so t_n = a + (n - 1)d.
Sum of 4th and 8th terms: (a + 3d) + (a + 7d) = 2a + 10d = 24 ... (i)
Sum of 6th and 10th terms: (a + 5d) + (a + 9d) = 2a + 14d = 44 ... (ii)
Subtract (i) from (ii): (2a + 14d) - (2a + 10d) = 44 - 24 => 4d = 20 => d = 5.
Substitute in (i): 2a + 10(5) = 24 => 2a + 50 = 24 => 2a = -26 => a = -13.
First three terms: a = -13, a + d = -8, a + 2d = -3.
The first three terms are -13, -8, -3 (a = -13, d = 5).

Q9 Find the smallest value of n such that the sum of the first n natural numbers is greater than 1,000.

SOLUTION

The sum of the first n natural numbers is S_n = n(n + 1)/2.
We need the smallest n with n(n + 1)/2 > 1000, i.e. n(n + 1) > 2000.
Try n = 44: 44 × 45 = 1980, so S_44 = 1980/2 = 990, which is not greater than 1000.
Try n = 45: 45 × 46 = 2070, so S_45 = 2070/2 = 1035, which is greater than 1000.
Hence the smallest such n is 45.
The smallest value of n is 45 (sum = 1035).

Q10 Which term of the GP: 2, 8, 32, … is 131072? Write the explicit formula as well as the recursive formula for the nth term.

SOLUTION

The GP has first term a = 2 and common ratio r = 8/2 = 4.
Explicit formula: t_n = a·r^(n-1) = 2 × 4^(n-1).
Recursive formula: t_1 = 2 and t_(n+1) = 4·t_n for n ≥ 1.
Set t_n = 131072: 2 × 4^(n-1) = 131072 => 4^(n-1) = 65536.
Since 4^8 = 65536, we get n - 1 = 8 => n = 9. Check: 2 × 4^8 = 2 × 65536 = 131072.
131072 is the 9th term. Explicit: t_n = 2 × 4^(n-1); recursive: t_1 = 2, t_(n+1) = 4·t_n.

Q11 The sum of the first three terms of a GP is 13/12 and their product is -1. Find the common ratio and the terms.

SOLUTION

Let the three terms of the GP be a/r, a, a·r.
Their product = (a/r)·a·(a·r) = a^3 = -1 => a = -1.
Their sum = a(1/r + 1 + r) = 13/12. With a = -1: -(1/r + 1 + r) = 13/12 => 1/r + 1 + r = -13/12.
So r + 1/r = -13/12 - 1 = -25/12. Multiply by 12r: $12r^2$ + 25r + 12 = 0.
Solve: discriminant = 25^2 - 4·12·12 = 625 - 576 = 49, √49 = 7. r = (-25 ± 7)/24 => r = -18/24 = -3/4 or r = -32/24 = -4/3.
For r = -3/4: terms are a/r = -1/(-3/4) = 4/3, a = -1, a·r = -1·(-3/4) = 3/4, i.e. 4/3, -1, 3/4 (sum = 13/12, product = -1).
For r = -4/3: terms are a/r = 3/4, a = -1, a·r = 4/3, i.e. 3/4, -1, 4/3 — the same three terms in reverse order.
Common ratio r = -3/4 (or -4/3); the terms are 4/3, -1, 3/4.

Q12 If the 4th, 10th and 16th terms of a GP are x, y and z respectively, prove that x, y, z are in GP.

SOLUTION

Let the GP have first term a and common ratio r, so t_n = a·r^(n-1).
4th term: x = t_4 = $a·r^3$.
10th term: y = t_10 = $a·r^9$.
16th term: z = t_16 = $a·r^{15}$.
Compute y^2 = $(a·r^9)^2$ = $a^2·r^{18}$.
Compute x·z = $(a·r^3)(a·r^{15})$ = $a^2·r^{18}$.
Since y^2 = x·z, the numbers x, y, z satisfy the GP condition (each ratio y/x = z/y = r^6), so x, y, z are in GP. Hence proved.
y^2 = x·z (both equal $a^2·r^{18}$), so x, y, z are in GP with common ratio r^6.

Q13 The sum of the first three terms of a geometric progression is 26, and the sum of their squares is 364. Find the terms of the GP.

SOLUTION

Let the three terms be a, ar, ar^2.
Sum: $a(1 + r + r^2) = 26$ ... (i)
Sum of squares: $a^2(1 + r^2 + r^4) = 364$. Note $1 + r^2 + r^4 = (1 + r + r^2)(1 - r + r^2)$, so $a^2(1 + r + r^2)(1 - r + r^2) = 364$ ... (ii)
Divide (ii) by (i): $a(1 - r + r^2) = 364/26 = 14$ ... (iii)
Subtract (iii) from (i): $a[(1 + r + r^2) - (1 - r + r^2)] = 26 - 14 \Rightarrow a(2r) = 12 \Rightarrow a \cdot r = 6 \Rightarrow a = 6/r$.
Add (i) and (iii): $a[(1 + r + r^2) + (1 - r + r^2)] = 26 + 14 \Rightarrow a(2 + 2r^2) = 40 \Rightarrow a(1 + r^2) = 20$.
Substitute $a = 6/r$: $(6/r)(1 + r^2) = 20 \Rightarrow 6 + 6r^2 = 20r \Rightarrow 6r^2 - 20r + 6 = 0 \Rightarrow 3r^2 - 10r + 3 = 0$.
Solve: discriminant = 100 - 36 = 64, $\sqrt{64} = 8$. $r = (10 \pm 8)/6 \Rightarrow r = 3$ or $r = 1/3$.
For r = 3: a = 6/3 = 2, terms are 2, 6, 18 (sum = 26, sum of squares = 4 + 36 + 324 = 364).
For r = 1/3: a = 6/(1/3) = 18, terms are 18, 6, 2 — the same three terms in reverse order.
The terms are 2, 6, 18 (or in reverse order 18, 6, 2).

Q14 Suppose P1 = 1, P2 = 2 and for n > 2, Pn = P1 + P2 + … + P(n-1) + 1. Find the values of P1, P2, …, P8. Can you find a simpler recursive formula for Pn? Can you give an explicit formula?

SOLUTION

Given P1 = 1, P2 = 2, and each later term is one more than the sum of all previous terms.
P3 = P1 + P2 + 1 = 1 + 2 + 1 = 4.
P4 = P1 + P2 + P3 + 1 = 1 + 2 + 4 + 1 = 8.
P5 = 1 + 2 + 4 + 8 + 1 = 16.
P6 = 1 + 2 + 4 + 8 + 16 + 1 = 32.
P7 = (1 + 2 + 4 + 8 + 16 + 32) + 1 = 64.
P8 = (1 + 2 + 4 + 8 + 16 + 32 + 64) + 1 = 128.
So P1, …, P8 are 1, 2, 4, 8, 16, 32, 64, 128 — the powers of 2.
Simpler recursive formula: P_n = 2·P_(n-1) for n > 2 (e.g. 4 = 2×2, 8 = 2×4, …), because P_n = (previous sum) + 1 and P_(n-1) itself equals that previous sum + 1, so adding P_(n-1) to the sum doubles it.
Explicit formula: P_n = 2^(n-1).
P1..P8 = 1, 2, 4, 8, 16, 32, 64, 128. Recursive: P_n = 2·P_(n-1) (n > 2); explicit: P_n = 2^(n-1).

Q15 Suppose W1 = 1, W2 = 2 and for n > 2, Wn = W1 + W2 + … + W(n-2) + 2. Find the values of W1, W2, …, W8. Do you recognise this sequence?

SOLUTION

Given W1 = 1, W2 = 2, and for n > 2, W_n = W1 + W2 + … + W_(n-2) + 2 (the sum runs up to the (n-2)th term, then add 2).
W3 = W1 + 2 = 1 + 2 = 3.

W4 = W1 + W2 + 2 = 1 + 2 + 2 = 5.
W5 = W1 + W2 + W3 + 2 = 1 + 2 + 3 + 2 = 8.
W6 = 1 + 2 + 3 + 5 + 2 = 13.
W7 = 1 + 2 + 3 + 5 + 8 + 2 = 21; W8 = 1 + 2 + 3 + 5 + 8 + 13 + 2 = 34.
So W1, …, W8 are 1, 2, 3, 5, 8, 13, 21, 34.
Each term (from W3 on) equals the sum of the two preceding terms: W_n = W_(n-1) + W_(n-2).
This is the Fibonacci sequence.
W1..W8 = 1, 2, 3, 5, 8, 13, 21, 34 — the Fibonacci sequence (W_n = W_(n-1) + W_(n-2)).