Chapter 9: Propositions and their Converses

Exercise 9.1 — Exercise Set 9.1

Q1 If two lines are parallel, then the corresponding angles formed by a transversal are equal. Frame the converse and determine if each of the two statements is true or not. Justify the true statements and give a counterexample for each false statement.

SOLUTION

Proposition P: If two lines are parallel, then the corresponding angles formed by a transversal are equal.
Converse Q: If the corresponding angles formed by a transversal are equal, then the two lines are parallel.

Proposition P is TRUE.
Justification: This is the Corresponding Angles Theorem. When a transversal crosses two parallel lines, each pair of corresponding angles occupies the same position at each intersection. The two parallel lines can be thought of as translates of each other along the transversal, so every corresponding angle is identical in measure.

Converse Q is also TRUE.
Justification: This is the converse of the Corresponding Angles Theorem (also called the Corresponding Angles Converse). If corresponding angles are equal, it can be proved (using angle-sum properties and the parallel postulate) that the lines cannot meet — hence they are parallel.

Both the proposition and its converse are true.

Q2 If a quadrilateral is a square, then all its angles are equal. Frame the converse and determine if each of the two statements is true or not. Justify the true statements and give a counterexample for each false statement.

SOLUTION

Proposition P: If a quadrilateral is a square, then all its angles are equal.
Converse Q: If all the angles of a quadrilateral are equal, then it is a square.

Proposition P is TRUE.
Justification: Every square has four right angles (each 90°), so all four angles are equal.

Converse Q is FALSE.
Counterexample: Consider a rectangle that is not a square, for example, a rectangle with sides 2 cm and 5 cm. All four angles are 90°, so all angles are equal. However, it is not a square because its sides are not all equal.

The proposition is true but its converse is false.

Q3 Given any △ABC, bisect the angles at B and C. The bisectors meet at the incentre I of the triangle. Extend the bisectors beyond I till they meet the opposite sides at E and F respectively (BE extended meets AC at E; CF extended meets AB at F). Proposition: If AB = AC, then IE = IF. Frame the converse and determine if each statement is true or not.

SOLUTION

Proposition P: If AB = AC (i.e., △ABC is isosceles with AB = AC), then IE = IF.
Converse Q: If IE = IF, then AB = AC.

Proposition P is TRUE.
Justification: Since AB = AC, triangle ABC is isosceles. Therefore ∠ABC = ∠ACB (base angles of isosceles triangle are equal). Let each equal angle be 2β, so the bisectors from B and C each make an angle β with BC.
Because the triangle is symmetric about the perpendicular from A to BC, the angle bisectors from B and C are symmetric about this axis of symmetry. The incentre I lies on this axis. The extensions IE (from B's bisector to side AC) and IF (from C's bisector to side AB) are therefore symmetric about the same axis, giving IE = IF.

Converse Q is also TRUE.
Justification: The converse can be proved by a similar symmetry/angle-chasing argument. If IE = IF, then by the properties of the angle bisectors and the sides they meet, one can show ∠IEB = ∠IFC (using congruent triangles △IEB and △IFC under the SAS criterion with IE = IF and the equal angles at I formed by the bisectors). This forces AB = AC.

Both the proposition and its converse are true.

Q4 If x = y, then a + x = a + y, where x, y and a are any three numbers. This proposition and its converse are routinely used while solving equations. Frame the converse and determine if each statement is true or not.

SOLUTION

Proposition P: If x = y, then a + x = a + y (where x, y, a are any numbers).
Converse Q: If a + x = a + y, then x = y (where x, y, a are any numbers).

Proposition P is TRUE.
Justification: If x = y, then replacing x with y on the left side of a + x gives a + y. This is the addition property of equality: adding the same number to both sides of an equation preserves equality.

Converse Q is also TRUE.
Justification: If a + x = a + y, subtract a from both sides: (a + x) − a = (a + y) − a, which gives x = y. This is the subtraction property of equality.

Both the proposition and its converse are true. These two together form the basis for 'transposing' terms when solving equations.

Q5 If a and b are perfect squares, then ab is a perfect square. Frame the converse and determine if each statement is true or not.

SOLUTION

Proposition P: If a and b are perfect squares, then ab is a perfect square.
Converse Q: If ab is a perfect square, then a and b are perfect squares.

Proposition P is TRUE.
Justification: If a and b are perfect squares, then $a = m^2$ and $b = n^2$ for some positive integers m and n. Therefore $ab = m^2n^2 = (mn)^2$, which is a perfect square.

Converse Q is FALSE.
Counterexample: Let a = 2 and b = 8. Then $ab = 16 = 4^2$, which is a perfect square. However, a = 2 is not a perfect square (since $\sqrt{2}$ is not an integer), so Q fails.

The proposition is true but its converse is false.

Q6 If $x = y$, then $x^2 = y^2$, where x and y are real numbers. Frame the converse and determine if each statement is true or not.

SOLUTION

Proposition P: If $x = y$, then $x^2 = y^2$ (x, y real numbers).
Converse Q: If $x^2 = y^2$, then $x = y$ (x, y real numbers).

Proposition P is TRUE.
Justification: If $x = y$, squaring both sides gives $x^2 = y^2$. Squaring is a valid operation that preserves equality.

Converse Q is FALSE.
Counterexample: Let $x = 1$ and $y = -1$. Then $x^2 = 1^2 = 1$ and $y^2 = (-1)^2 = 1$, so $x^2 = y^2$. However, $x = 1 \neq -1 = y$.

The proposition is true but its converse is false.

Q7 If $x = y$, then $x^3 = y^3$, where x and y are real numbers. Frame the converse and determine if each statement is true or not.

SOLUTION

Proposition P: If $x = y$, then $x^3 = y^3$ (x, y real numbers).
Converse Q: If $x^3 = y^3$, then $x = y$ (x, y real numbers).

Proposition P is TRUE.
Justification: If $x = y$, cubing both sides gives $x^3 = y^3$.

Converse Q is also TRUE.
Justification: Suppose $x^3 = y^3$. Then $x^3 - y^3 = 0$, which factors as $(x - y)(x^2 + xy + y^2) = 0$.
We show $x^2 + xy + y^2 > 0$ for real x, y unless $x = y = 0$. Rewrite: $x^2 + xy + y^2 = (x + y/2)^2 + 3y^2/4 \geq 0$, and equals 0 only when $y = 0$ and $x = 0$.

If $x = y = 0$ then $x = y$ trivially. Otherwise $x^2 + xy + y^2 > 0$, so $(x - y) = 0$, giving $x = y$.
Hence $x^3 = y^3$ implies $x = y$ for all real numbers.

Both the proposition and its converse are true.

Q8 If n is divisible by 24, then it is divisible by both 4 and 6. (n is a positive integer.) Frame the converse and determine if each statement is true or not.

SOLUTION

Proposition P: If n is divisible by 24, then n is divisible by both 4 and 6.
Converse Q: If n is divisible by both 4 and 6, then n is divisible by 24.

Proposition P is TRUE.
Justification: $24 = 4 \times 6$ and $4 \mid 24$ and $6 \mid 24$. If $24 \mid n$, then there exists an integer k such that $n = 24k$. Then $n = 4(6k)$, so $4 \mid n$; and $n = 6(4k)$, so $6 \mid n$. Thus n is divisible by both 4 and 6.

Converse Q is FALSE.
Counterexample: $n = 12$. We have $12 \div 4 = 3$ (so $4 \mid 12$) and $12 \div 6 = 2$ (so $6 \mid 12$). However, $12 \div 24 = 0.5$, so 24 does not divide 12.
This happens because $\text{lcm}(4, 6) = 12 \neq 24$. Divisibility by both 4 and 6 only guarantees divisibility by their LCM, which is 12, not 24.

The proposition is true but its converse is false.

Q9 If n is divisible by 60, then it is divisible by both 5 and 12. (n is a positive integer.) Frame the converse and determine if each statement is true or not.

SOLUTION

Proposition P: If n is divisible by 60, then n is divisible by both 5 and 12.
Converse Q: If n is divisible by both 5 and 12, then n is divisible by 60.

Proposition P is TRUE.
Justification: Since $5 \mid 60$ and $12 \mid 60$, if $60 \mid n$ then $n = 60k$ for some integer k. Then $n = 5(12k)$, so $5 \mid n$; and $n = 12(5k)$, so $12 \mid n$.

Converse Q is also TRUE.
Justification: $\text{lcm}(5, 12) = 60$ (since $\gcd(5, 12) = 1$, so lcm $= 5 \times 12 = 60$). If n is divisible by both 5 and 12, then n is divisible by $\text{lcm}(5, 12) = 60$.
Proof that $\gcd(5, 12) = 1$: 5 is prime and does not divide 12 (since $12 = 2^2 \times 3$), so $\gcd(5, 12) = 1$.

Both the proposition and its converse are true.

Q10 If n is the square of a prime number, then it has exactly 3 factors. (n is a positive integer.) Frame the converse and determine if each statement is true or not.

SOLUTION

Proposition P: If n is the square of a prime number, then it has exactly 3 factors.
Converse Q: If n has exactly 3 factors, then n is the square of a prime number.

Proposition P is TRUE.
Justification: Let $n = p^2$ where p is prime. The divisors of p^2 are: 1, p, and p^2. These are exactly 3 divisors, and there are no others (any divisor d of p^2 satisfies $d \mid p^2$, so $d \in \{1, p, p^2\}$ since p is prime). Examples: $4 = 2^2$ has factors {1, 2, 4}; $9 = 3^2$ has factors {1, 3, 9}.

Converse Q is also TRUE.
Justification: Suppose n has exactly 3 factors: 1, d, and n (where d is the 'middle' factor with $1 < d < n$). Since 1 and n are always factors, the third factor d must satisfy $d^2 = n$ (the factor pairs are (1, n) and (d, d), and with 3 factors there is exactly one self-paired factor). So $n = d^2$.
We claim d must be prime. If d were composite, say $d = ab$ with $1 < a, b < d$, then a and b would also be factors of $n = d^2$ distinct from 1, d, n — giving more than 3 factors, a contradiction.
Therefore d is prime and $n = d^2$ is the square of a prime.

Both the proposition and its converse are true.

Q11 If n is a product of two unequal prime numbers, then it has exactly 4 divisors. (n is a positive integer.) Frame the converse and determine if each statement is true or not.

SOLUTION

Proposition P: If n is a product of two unequal prime numbers, then it has exactly 4 divisors.
Converse Q: If n has exactly 4 divisors, then n is a product of two unequal prime numbers.

Proposition P is TRUE.
Justification: Let $n = p \times q$ where p and q are distinct primes ($p \neq q$). The divisors of n are: 1, p, q, and $pq = n$. These are exactly 4, and there are no others (any divisor must be built from p and q, giving only these four).

Converse Q is FALSE.
Counterexample: $n = 8 = 2^3$. The divisors of 8 are: 1, 2, 4, 8 — exactly 4 divisors. However, 8 is not a product of two unequal primes; it equals 2^3, the cube of a single prime.
More generally, $n = p^3$ for any prime p has exactly 4 divisors: 1, p, p^2, p^3. So having exactly 4 divisors does not force n to be a product of two distinct primes.

The proposition is true but its converse is false.

Q12 If n and n + 3 have no factors in common, then n is not a multiple of 3. (n is a positive integer.) Frame the converse and determine if each statement is true or not.

SOLUTION

Proposition P: If n and n + 3 have no factors in common (i.e., gcd(n, n+3) = 1), then n is not a multiple of 3.
Converse Q: If n is not a multiple of 3, then n and n + 3 have no factors in common.

Key observation: Any common factor d of n and n + 3 must also divide (n + 3) − n = 3. Therefore d | 3, so d ∈ {1, 3}. This means gcd(n, n+3) is either 1 or 3.

Proposition P is TRUE.
Justification: Suppose gcd(n, n+3) = 1 (no common factor other than 1). From the key observation, the only possible common factors are 1 and 3. Since gcd = 1, the factor 3 is not a common factor, meaning 3 does not divide n. Hence n is not a multiple of 3.

Converse Q is also TRUE.
Justification: Suppose 3 does not divide n. From the key observation, gcd(n, n+3) ∈ {1, 3}. Since 3 ∤ n, the value 3 cannot be a common factor, so gcd(n, n+3) = 1. Thus n and n+3 have no factors in common.

Both the proposition and its converse are true.

Q13 There are no known 'neat' expressions that generate only primes! Find counterexamples to the following claims. (i) All numbers of the form $4n^2 + 1$ are prime. (ii) All numbers of the form $n^2 + n + 11$ are prime. (iii) All numbers of the form $4^n + 3$ are prime.

SOLUTION

(i) Claim: All numbers of the form $4n^2 + 1$ are prime.
Counterexample: n = 4. Then $4n^2 + 1 = 4(16) + 1 = 64 + 1 = 65 = 5 \times 13$.
Since 65 is composite, the claim is false.

(ii) Claim: All numbers of the form $n^2 + n + 11$ are prime.
Counterexample: n = 10. Then $n^2 + n + 11 = 100 + 10 + 11 = 121 = 11 \times 11 = 11^2$.
Since 121 is composite, the claim is false. (Note: for n = 0 through n = 9, the expression does yield primes: 11, 13, 17, 23, 31, 41, 53, 67, 83, 101 — but n = 10 breaks the pattern.)

(iii) Claim: All numbers of the form $4^n + 3$ are prime.
Counterexample: n = 4. Then $4^4 + 3 = 256 + 3 = 259 = 7 \times 37$.
Since 259 is composite, the claim is false.

Q14 Find counterexamples to the following statements. (i) If n is a prime number, then $2^n - 1$ is a prime number. (ii) If n is an even number, then $2^n + 1$ is a prime number.

SOLUTION

(i) Claim: If n is a prime number, then $2^n - 1$ is a prime number.
Counterexample: n = 11. n = 11 is prime. Then $2^{11} - 1 = 2048 - 1 = 2047 = 23 \times 89$.
Since 2047 is composite, the claim is false.
(Numbers of the form $2^n - 1$ are called Mersenne numbers; not all of them are prime even when n is prime.)

(ii) Claim: If n is an even number, then $2^n + 1$ is a prime number.
Counterexample: n = 6. n = 6 is even. Then $2^6 + 1 = 64 + 1 = 65 = 5 \times 13$.
Since 65 is composite, the claim is false.

Q15 Consider the statement: 'If a number is divisible by 8, then it is divisible by both 2 and 4'. (i) Justify the statement. (ii) To check whether a given number is divisible by 8, is it enough to check whether it is divisible by 2 and 4? Why or why not?

SOLUTION

(i) Justification of the statement:
Since $8 = 2 \times 4$ and both 2 and 4 are divisors of 8, if a number n is divisible by 8 then $n = 8k$ for some integer k.
Then $n = 2(4k)$, so $2 \mid n$; and $n = 4(2k)$, so $4 \mid n$.
Therefore n is divisible by both 2 and 4. The statement is TRUE.

(ii) No, checking divisibility by 2 and 4 is NOT enough to conclude divisibility by 8.
This is because the converse of the statement is false: being divisible by 2 and 4 does not guarantee divisibility by 8.
Counterexample: n = 12. We have $12 = 2 \times 6$ (so $2 \mid 12$) and $12 = 4 \times 3$ (so $4 \mid 12$). However, $12 \div 8 = 1.5$, which is not an integer, so 8 does not divide 12.
lcm(2, 4) = 4, not 8. So divisibility by 2 and 4 only guarantees divisibility by 4.
To test divisibility by 8, one must check the last three digits of the number (if they form a number divisible by 8, the whole number is divisible by 8).

Q16 Recall that a shortcut to check whether a given number is divisible by 3 is to add the digits of the number and check if the sum is a multiple of 3. Express the relationship between 'a number is divisible by 3' and 'sum of the digits is a multiple of 3' using 'If-then' sentences.

SOLUTION

The relationship between the two conditions can be expressed as two 'If-then' statements:

Proposition P: If a number is divisible by 3, then the sum of its digits is a multiple of 3.

Converse Q: If the sum of the digits of a number is a multiple of 3, then the number is divisible by 3.

Both P and Q are true — each is the converse of the other, and both hold.

Justification: Any number n with digits $dd_{-1}\ldots d_1d_0$ can be written as $n = d \times 10 + \ldots + d_1 \times 10 + d_0$. Since $10 \equiv 1 \pmod 3$, each power $10 \equiv 1 \pmod 3$. Therefore $n \equiv d + d_{-1} + \ldots + d_1 + d_0 \pmod 3$. So $3 \mid n$ if and only if $3 \mid$ (sum of digits of n). This proves both P and Q simultaneously.

Example: 123 has digit sum 1 + 2 + 3 = 6, which is a multiple of 3. And indeed $123 = 3 \times 41$.

Q17 Suppose we have a category of quadrilaterals Q, to be constructed using two thin sticks as diagonals (endpoints joined to form the quadrilateral). (i) Suppose Q satisfies: 'If a quadrilateral is of type Q, then it has equal-length diagonals.' (a) Should the two sticks be of equal length? (b) Will it matter how the two sticks are put together? (ii) Instead, suppose Q satisfies: 'If a quadrilateral has equal diagonals, then it is of type Q.' What are your answers to (a) and (b) now?

SOLUTION

Part (i): Property of Q is: 'If a quadrilateral is of type Q, then it has equal-length diagonals.'

(a) Yes, the two sticks must be of equal length.
Reason: We want to construct a quadrilateral of type Q. The property tells us that every type-Q quadrilateral has equal-length diagonals. So the diagonals of our constructed quadrilateral must be equal in length. Since the sticks become the diagonals, the sticks must be of equal length.

(b) No, it will not matter how the two equal-length sticks are put together (i.e., at what angle they cross or where along each stick the crossing point is).
Reason: The property only requires the diagonals to be of equal length — it places no condition on the angle of intersection or the position of the crossing. Any arrangement of two equal-length sticks as diagonals will produce a quadrilateral with equal-length diagonals, which is what type Q requires.

Part (ii): Property of Q is: 'If a quadrilateral has equal diagonals, then it is of type Q.'
This is the converse of the property in part (i).

(a) The sticks need not be of equal length to obtain a type-Q quadrilateral.
Reason: The property only says equal diagonals imply type Q. It does not say type Q requires equal diagonals. So a quadrilateral with unequal diagonals might still be of type Q. We cannot conclude from this property alone that the diagonals must be equal.
(However, using equal-length sticks is sufficient to guarantee type Q: equal sticks → equal diagonals → type Q by the given property.)

(b) No, it will not matter how the sticks are put together — as long as the sticks are of equal length.
Reason: If the sticks are of equal length, any arrangement gives equal diagonals, which guarantees type Q by the given property. The angle of intersection or position of crossing does not affect diagonal lengths, so the arrangement is irrelevant for ensuring type Q.