Chapter 12: Quadrilaterals

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Exercise 12.1 — Exercise Set 12.1

Q1 Let ABCD be a quadrilateral. (i) List all sides of ABCD adjacent to side AB. List all sides opposite to AB. (ii) List all angles of ABCD adjacent to ∠A. List all angles opposite to ∠A. (iii) Define a pair of opposite sides and a pair of opposite angles without using the names of the vertices.

SOLUTION

(i) Adjacent sides to AB: A side is adjacent to AB if it shares a vertex with AB.
Sides sharing vertex A: DA. Sides sharing vertex B: BC.
So the sides adjacent to AB are BC and DA.

The side opposite to AB is the side that shares no vertex with AB.
The only such side in ABCD is CD.

(ii) Adjacent angles to ∠A: An angle is adjacent to ∠A if its vertex is an adjacent vertex to A.
Adjacent vertices to A are B and D.
So the angles adjacent to ∠A are ∠B and ∠D.

The angle opposite to ∠A is the angle at the vertex not adjacent to A.
That vertex is C, so the angle opposite to ∠A is ∠C.

(iii) Definition — A pair of opposite sides: Two sides of a quadrilateral are opposite sides if they do not share a common vertex (i.e., they are not adjacent).

A pair of opposite angles: Two angles of a quadrilateral are opposite angles if the vertices at which they are formed are not adjacent vertices (i.e., the vertices are not connected by a side of the quadrilateral).

Q2 (*) You have used internal angles of quadrilaterals, but they too require an exact definition. Precisely define the internal angle of a quadrilateral at a given vertex. Your answer should work for a non-convex quadrilateral too. (Hint: use the opposite vertex as well.)

SOLUTION

Let ABCD be a quadrilateral and let V be one of its vertices, say V = A.
The two sides meeting at A are AB and AD.

For a convex quadrilateral, the internal angle at A is simply the angle ∠DAB, measured inside the quadrilateral — the angle swept from ray AB to ray AD through the interior.

For a non-convex quadrilateral (where one internal angle exceeds 180°), we need the opposite vertex to decide which side of angle ∠DAB is the 'interior'.

Definition: The internal angle of quadrilateral ABCD at vertex A is the angle ∠DAB measured on the side that contains the opposite vertex C.

More precisely: among the two angles formed at A by rays AB and AD (which together sum to 360°), the internal angle at A is the one in whose interior the opposite vertex C lies (when we extend the rays to full lines through A).

This definition gives angles less than 180° for convex quadrilaterals and one angle greater than 180° at the dent of a non-convex quadrilateral, which matches our expectation.

Q3 In a quadrilateral ABCD, suppose AB ∥ DC. Can ABCD be non-convex? What if we instead assume AB = CD? What if we instead assume ∠A = ∠C?

SOLUTION

Case 1: AB ∥ DC.
A quadrilateral with AB ∥ DC is a trapezium. A trapezium can be non-convex.
Example: Draw A, B on a lower line and D, C on an upper line with DC ∥ AB, but place D so that the vertex D is positioned to give an internal angle greater than 180°.
However, we must be careful — if AB ∥ DC and the vertices are in convex position, ABCD is a convex trapezium. If one vertex is 'dented in', a non-convex trapezium is possible.
Conclusion: ABCD can be non-convex when AB ∥ DC.

Case 2: AB = CD.
Having two sides equal does not force the quadrilateral to be convex.
A quadrilateral with AB = CD can certainly be non-convex. For example, take a kite-like shape with a dent. Equality of two sides places no restriction on the internal angles.
Conclusion: ABCD can be non-convex when AB = CD.

Case 3: ∠A = ∠C.
By Theorem 3, if BOTH pairs of opposite angles are equal (∠A = ∠C and ∠B = ∠D), then ABCD is a parallelogram, which is convex.
But here only ∠A = ∠C is given. This alone does not force ABCD to be a parallelogram or convex.

A non-convex quadrilateral can have ∠A = ∠C (with one angle being a reflex angle >180°).
Conclusion: ABCD can be non-convex when ∠A = ∠C.

Q4 Consider three non-collinear points A, B, C and draw the lines AB, BC, CA. For every possible location of point D in the plane outside these lines, decide if ABCD is self-intersecting, non-convex, or convex. (Hint: the three lines divide the plane into 7 regions.)

SOLUTION

Three lines through non-collinear points A, B, C divide the plane into 7 regions.
These regions are: 1 central (triangular) region, 3 regions adjacent to each side, and 3 unbounded corner regions.

For D in the central region (inside triangle ABC):
ABCD will be non-convex (the vertex D is inside triangle ABC, creating a dent in the quadrilateral).

For D in one of the three regions adjacent to the sides (the three 'wedge' regions between the extensions of the sides):
The quadrilateral formed may be self-intersecting. Specifically, if D falls in the region beyond side BC (opposite to A), the sides AD and BC will intersect, making ABCD self-intersecting.
Similarly for the regions beyond AB and beyond CA.

For D in one of the three unbounded corner regions (beyond a vertex of triangle ABC):
ABCD will be a convex quadrilateral.
For example, if D is in the unbounded region beyond vertex C (on the far side of both lines AC and BC from B and A respectively), then ABCD is convex.

Summary: The 7 regions give three types of positions for D:
— 1 region (interior of triangle ABC): ABCD is non-convex.
— 3 regions (adjacent to the sides of triangle ABC, between the lines): ABCD is self-intersecting.
— 3 regions (corner/unbounded regions beyond each vertex): ABCD is convex.

Q5 Can a quadrilateral be both self-intersecting and non-planar?

SOLUTION

Yes, a quadrilateral can be both self-intersecting and non-planar.

A non-planar quadrilateral (like BENT in the text) has vertices not all in the same plane, so its four sides do not lie in one plane.

Such a non-planar figure can still have two of its sides (when projected onto some plane, or in three-dimensional space) crossing or intersecting each other at a point that is not a vertex.

For example: Consider four points A, B, C, D in three-dimensional space arranged so that the segments AB and CD, although not in the same plane, intersect at a point E in space.
Then ABCD would be both non-planar and self-intersecting.

Therefore, yes, a quadrilateral can be simultaneously self-intersecting and non-planar.

Exercise 12.2 — Exercise Set 12.2

Q1 True or false? (i) A parallelogram with a right angle is a rectangle. (ii) A rhombus with perpendicular diagonals is a square. (iii) If the diagonals of a parallelogram are equal, then it is a rectangle.

SOLUTION

(i) TRUE.
In a parallelogram, adjacent angles are supplementary (add to 180°). If one angle is 90°, then the adjacent angle is also 90°. So all four angles are 90°, making it a rectangle.

(ii) FALSE.
The diagonals of every rhombus are perpendicular to each other — this is a property of all rhombuses, not just squares. A rhombus that is not a square (sides equal, but angles ≠ 90°) also has perpendicular diagonals.
The extra condition needed to make it a square is that the diagonals are also equal in length (or equivalently, that one angle is 90°).

(iii) TRUE.
In parallelogram ABCD with diagonals AC = BD, consider triangles ABD and BAC. AB = BA (common), AD = BC (opposite sides of parallelogram), and diagonal BD = diagonal AC (given). By SSS, $\triangle ABD \cong \triangle BAC$, so $\angle DAB = \angle CBA$. But $\angle DAB + \angle CBA = 180°$ (co-interior angles, since AD ∥ BC). Therefore $\angle DAB = \angle CBA = 90°$. So ABCD is a rectangle.

Q2 The diagonal AC of a parallelogram ABCD bisects ∠A. Show that it also bisects ∠C and that ABCD is a rhombus.

SOLUTION

Given: ABCD is a parallelogram and diagonal AC bisects ∠A.
So $\angle DAC = \angle BAC$. Let each of these angles equal α.

Step 1: Show AC bisects ∠C.
Since AB ∥ DC and AC is a transversal:
$\angle BAC = \angle DCA$ (alternate interior angles).
Since AD ∥ BC and AC is a transversal:
$\angle DAC = \angle BCA$ (alternate interior angles).
Since $\angle DAC = \angle BAC = \alpha$, we get $\angle DCA = \angle BCA = \alpha$.
Therefore AC bisects ∠C. ✓

Step 2: Show ABCD is a rhombus.
In triangle ABC: $\angle BAC = \alpha$ and $\angle BCA = \alpha$.
Therefore BC = AB (sides opposite equal angles in a triangle).
But in a parallelogram, AB = DC and BC = AD (opposite sides are equal).
So AB = BC = CD = DA.
Hence ABCD is a rhombus. ✓

Q3 The following questions examine converses of true properties. Answer them with Yes or No. If your answer is No, what extra condition can you add so that the answer becomes Yes? (i) If the diagonals of a quadrilateral ABCD bisect its angles, must ABCD be a rhombus? (ii) If the diagonals of a quadrilateral ABCD bisect each other at right angles, must ABCD be a rhombus? (iii) If the diagonals of a quadrilateral ABCD are of equal length, must ABCD be a rectangle?

SOLUTION

(i) YES, ABCD must be a rhombus.
Suppose diagonals AC and BD bisect all four angles of ABCD. Then in triangle ABC, ∠BAC = ∠DAC and ∠DBC = ∠ABC/2, etc.
Since the diagonal bisects ∠A: ∠DAC = ∠BAC. Since AD ∥ BC (using alternate angles with transversal AC), these equal angles confirm AB ∥ DC similarly. So ABCD is a parallelogram.
In parallelogram ABCD where diagonal AC bisects ∠A, by the reasoning in Q2, ABCD is a rhombus.

(ii) YES, ABCD must be a rhombus.
If both diagonals bisect each other, then by Theorem 4 ABCD is a parallelogram. If, in addition, the diagonals are perpendicular, then the parallelogram has perpendicular diagonals, and a parallelogram with perpendicular diagonals is a rhombus.
So 'diagonals bisect each other AND are perpendicular' is already enough to force a rhombus — no extra condition is needed.
(Note the contrast: perpendicular diagonals ALONE are not enough — a kite also has perpendicular diagonals but is not a rhombus, because in a kite only one diagonal bisects the other. The bisect-each-other condition is what makes the difference.)

(iii) NO, ABCD need not be a rectangle.
A quadrilateral with equal diagonals is not necessarily a rectangle.
Counter-example: An isosceles trapezium (trapezoid with equal legs) has equal diagonals but is not a rectangle.
Extra condition needed: ABCD must also be a parallelogram. A parallelogram with equal diagonals is a rectangle.

Q4 Let ABCD be a parallelogram with AB ≠ BC. Show that the pairwise intersection points of the four angle bisectors form the vertices of a rectangle (see Fig. 12.11). Why did we assume AB ≠ BC?

SOLUTION

In parallelogram ABCD with AB ≠ BC, let the angle bisectors of ∠A, ∠B, ∠C, ∠D be drawn. Let P, Q, R, S be the intersection points of adjacent bisectors as in Fig. 12.11.

Step 1: Adjacent angles in a parallelogram are supplementary.
∠A + ∠B = 180°. The bisectors of ∠A and ∠B make angles of ∠A/2 and ∠B/2 with AB.
In triangle formed by the bisectors from A and B (meeting at a point, say R on side CD side or inside):
The angle at their intersection = 180° − (∠A/2 + ∠B/2) = 180° − (∠A + ∠B)/2 = 180° − 90° = 90°.

Step 2: Each intersection point of two adjacent angle bisectors gives a right angle.
The bisectors from ∠A and ∠B meet at point R: angle at R = 90°.
The bisectors from ∠B and ∠C meet at point Q: since ∠B + ∠C = 180°, angle at Q = 90°.
Similarly for the other two intersection points.

Step 3: The four intersection points P, Q, R, S form a quadrilateral with all right angles — hence a rectangle.

Why AB ≠ BC? If AB = BC, then ABCD is a rhombus. In a rhombus, all angles are bisected by the diagonals. The four bisectors become the two diagonals, which intersect at a single point — so the four intersection points coincide and no rectangle is formed. The condition AB ≠ BC ensures the four bisectors meet at four distinct points.

Exercise 12.3 — Exercise Set 12.3

Q1 In ∆ABC, P, Q, R are the midpoints of sides AB, AC, BC respectively. (i) Show that ∆PQR is congruent to ∆QPA and to two other triangles which you should identify. (ii) Suppose someone erases ∆ABC, leaving only ∆PQR on the paper. Can you reconstruct ∆ABC from ∆PQR?

SOLUTION

(i) P = midpoint of AB, Q = midpoint of AC, R = midpoint of BC.

Midpoint Theorem in ∆ABC:
PQ ∥ BC and PQ = BC/2. Since R is midpoint of BC, PQ = BR = RC = BC/2.
QR ∥ AB and QR = AB/2. Since P is midpoint of AB, QR = AP = PB = AB/2.
PR ∥ AC and PR = AC/2. Since Q is midpoint of AC, PR = AQ = QC = AC/2.

Now consider the four triangles: ∆PQR, ∆QPA (= ∆APQ), ∆PBR, ∆RQC.

∆PQR has sides: PQ = BC/2, QR = AB/2, PR = AC/2.
∆APQ (= ∆QPA) has sides: AP = AB/2, AQ = AC/2, PQ = BC/2.
∆PBR has sides: PB = AB/2, BR = BC/2, PR = AC/2.
∆QCR has sides: QC = AC/2, CR = BC/2, QR = AB/2.

All four triangles have the same three side lengths (AB/2, AC/2, BC/2), so by SSS:
∆PQR ≅ ∆QPA ≅ ∆PBR ≅ ∆RQC.

The two other triangles are ∆PBR and ∆RQC.

(ii) Yes, ∆ABC can be reconstructed from ∆PQR.
P, Q, R are the midpoints of the sides of ∆ABC, so ∆PQR is the medial triangle of ∆ABC. Each side of ∆PQR is parallel to a side of ∆ABC and half its length.
Reconstruction: through each vertex of ∆PQR, draw the line parallel to the opposite side of ∆PQR. These three lines meet pairwise in three points, and those three points are exactly A, B, C.
Equivalently, A is the reflection of R across the midpoint of PQ, B is the reflection of P across the midpoint of QR, and C is the reflection of Q across the midpoint of PR (because, for instance, ARQP is a parallelogram, so A, R are symmetric about the midpoint of the diagonal PQ).
Either construction recovers ∆ABC uniquely from ∆PQR.

Q2 In ∆ABC, let M and N be midpoints of AB and AC respectively. Let D be any point on BC. Show that MN bisects AD.

SOLUTION

Given: In ∆ABC, M = midpoint of AB, N = midpoint of AC, D is any point on BC.
To show: MN bisects segment AD, i.e., MN passes through the midpoint of AD.

Let E be the midpoint of AD. We will show E lies on MN.

Consider triangle ABD. M = midpoint of AB and E = midpoint of AD.

By the Midpoint Theorem applied to ∆ABD:
The segment ME (joining midpoints of AB and AD) is parallel to BD and ME = BD/2.

Consider triangle ACD. N = midpoint of AC and E = midpoint of AD.
By the Midpoint Theorem applied to ∆ACD:
The segment NE (joining midpoints of AC and AD) is parallel to CD and NE = CD/2.

Now, ME ∥ BD and NE ∥ CD. But BD and CD are parts of the same line BC (since D is on BC), so BD and CD are along the same line.
Since ME ∥ BC and NE ∥ BC, both ME and NE are parallel to BC.

Through E, there is a unique line parallel to BC. Since both ME and NE pass through E and are parallel to BC, the points M, E, N are collinear — they all lie on the unique line through E parallel to BC.

Therefore E lies on line MN, which means MN bisects AD at point E. ✓

Q3 In a quadrilateral ABCD, suppose AB ∥ DC and AB ≠ CD. Suppose G and H are the midpoints of AC and BD respectively. Prove that GH ∥ AB. (Why did we assume AB ≠ CD?)

SOLUTION

Given: Quadrilateral ABCD with AB ∥ DC, AB ≠ CD. G = midpoint of diagonal AC, H = midpoint of diagonal BD.
To prove: GH ∥ AB.

Construction: Draw diagonal BC and let M be the midpoint of BC.

In ∆ABC: G = midpoint of AC, M = midpoint of BC.
By the Midpoint Theorem: GM ∥ AB and GM = AB/2.

In ∆BCD: H = midpoint of BD, M = midpoint of BC.
By the Midpoint Theorem: HM ∥ DC and HM = DC/2.

Since AB ∥ DC, and GM ∥ AB, we get GM ∥ DC.
Also HM ∥ DC.
So GM ∥ HM — both pass through M. Therefore G, M, H are collinear (both on the unique line through M parallel to DC).

Since G, M, H are collinear and GM ∥ AB, the line GH is the line GM, which is parallel to AB.
Therefore GH ∥ AB. ✓

Why did we assume AB ≠ CD?
If AB = CD as well as AB ∥ CD, then ABCD is a parallelogram. In a parallelogram the diagonals bisect each other, so their midpoints G and H coincide at the intersection point of the diagonals. Then GH is a single point and 'GH ∥ AB' would be vacuous. The condition AB ≠ CD ensures G ≠ H so that the line GH is well-defined.

Q4 Suppose the midpoints of sides AB, BC, CD and DA of a quadrilateral ABCD are P, Q, R and S respectively. (i) Show that PR and QS bisect each other. (ii) Show that if AC = BD, then PR and QS are perpendicular. Is the converse true?

SOLUTION

Let P, Q, R, S be midpoints of AB, BC, CD, DA respectively. By Theorem 9 (Varignon), PQRS is a parallelogram.

(i) In any parallelogram, the diagonals bisect each other.
PQRS is a parallelogram with diagonals PR and QS.
Therefore PR and QS bisect each other. ✓

(ii) Show that if AC = BD, then PR ⊥ QS.

By the Midpoint Theorem applied to triangles within ABCD:
In ∆ABC: P = midpoint of AB, Q = midpoint of BC → PQ ∥ AC and PQ = AC/2.
In ∆ACD: S = midpoint of DA, R = midpoint of CD → SR ∥ AC and SR = AC/2.
In ∆BCD: Q = midpoint of BC, R = midpoint of CD → QR ∥ BD and QR = BD/2.
In ∆ABD: P = midpoint of AB, S = midpoint of DA → PS ∥ BD and PS = BD/2.

So in parallelogram PQRS: PQ = SR = AC/2 and QR = PS = BD/2.

If AC = BD, then PQ = QR (both equal to half of AC = BD). So PQRS is a rhombus (all sides equal).
The diagonals of a rhombus are perpendicular. Therefore PR ⊥ QS. ✓

Is the converse true? If PR ⊥ QS, is AC = BD?
If PR ⊥ QS, the diagonals of parallelogram PQRS are perpendicular, which means PQRS is a rhombus, so all four sides of PQRS are equal: PQ = QR.
But PQ = AC/2 and QR = BD/2, so AC/2 = BD/2, giving AC = BD.
Yes, the converse is true.

Q5 Suppose PQRS is the Varignon parallelogram of ABCD. (i) Copy only PQRS on another paper. Show how you will recreate a congruent copy A′B′C′D′ of ABCD from PQRS. (ii) Justify why the quadrilateral A′B′C′D′ you constructed is congruent to ABCD. You may need to show why S is collinear with points A′ and D′ that you constructed, and similarly for P, Q and R. (Hint: Use congruence of triangles, for example ∆SDR ≅ ∆SD′R.) (iii) Show that if PQRS is a square, then AC and BD are perpendicular and equal. Prove the converse.

SOLUTION

(i) Reconstruction of A′B′C′D′ from PQRS:
P, Q, R, S are the midpoints of AB, BC, CD, DA respectively, so each vertex of ABCD is a midpoint-reflection of the vertices of PQRS. Pick any starting point A′, then build the rest so that P, Q, R, S become the side-midpoints:
— choose A′ so that S is the midpoint of D′A′, i.e. set A′ = 2S − R and D′ = R (then S = (A′ + D′)/2 as required),
— B′ = 2P − A′ (so P is the midpoint of A′B′),

— C′ = 2Q − B′ (so Q is the midpoint of B′C′).
A short check confirms R is then automatically the midpoint of C′D′: (C′ + D′)/2 = (2Q − B′ + R)/2 = (2Q − 2P + 2S)/2 = Q − P + S, and in the Varignon relationship Q − P + S = R. So all four of P, Q, R, S are side-midpoints of A′B′C′D′.

(ii) Proof of congruence A′B′C′D′ ≅ ABCD:
By construction P, Q, R, S are the midpoints of the sides of A′B′C′D′, exactly as they are for ABCD.
Take the diagonal AC of ABCD and A′C′ of the new figure. In △ACD, S and R are midpoints of DA and CD, so SR ∥ AC and SR = AC/2; the same midpoint relation in A′B′C′D′ gives SR ∥ A′C′ and SR = A′C′/2. Since SR is one fixed segment, AC = A′C′ and they point the same way.
Using the hint, △SDR ≅ △SD′R (SS R: SR common, SD = SD′ and RD = RD′ from the equal midpoints), which pins D′ to the position of D. The identical argument at the other three vertices places A′, B′, C′ on A, B, C.
Hence A′B′C′D′ ≅ ABCD.

(iii) If PQRS is a square, then AC ⊥ BD and AC = BD.
From the analysis in Q4(ii): PQ = SR = AC/2 and QR = PS = BD/2.
If PQRS is a square: all sides equal → PQ = QR → AC/2 = BD/2 → AC = BD.
Also, in a square, adjacent sides are perpendicular: PQ ⊥ QR.
PQ ∥ AC and QR ∥ BD (from Midpoint Theorem). Since PQ ⊥ QR, we get AC ⊥ BD.
Therefore if PQRS is a square, AC = BD and AC ⊥ BD. ✓

Converse: If AC = BD and AC ⊥ BD, then PQRS is a square.
AC = BD implies PQ = AC/2 = BD/2 = QR, so all sides of PQRS are equal → PQRS is a rhombus.
AC ⊥ BD implies PQ ⊥ QR (since PQ ∥ AC and QR ∥ BD), so PQRS has a right angle → PQRS is a rectangle.
A rhombus that is also a rectangle is a square. Therefore PQRS is a square. ✓

Exercise 12.4 — Exercise Set 12.4

Q1 Justify why the plane cannot be tiled with a regular pentagon. (Hint: Read the first 3 sentences of 'Think and Reflect' in the section on tiling.)

SOLUTION

At any vertex of a tiling, the angles of all the shapes meeting at that vertex must add up to exactly 360°.

Each interior angle of a regular pentagon = (5−2)×180°/5 = 3×180°/5 = 540°/5 = 108°.

For the plane to be tiled with regular pentagons, we would need some whole number k of pentagons to meet at each vertex, with k × 108° = 360°.

Check: 360° ÷ 108° = 360/108 = 10/3, which is not a whole number.

Since 10/3 is not an integer, no whole number of regular pentagons can fit around a vertex without leaving a gap or causing an overlap.

Therefore the plane cannot be tiled using only regular pentagons.

Q2 Draw a non-convex 4-gon DART. Show how we can tile the plane with copies of DART. Both methods that we discussed earlier will work. Which do you prefer?

SOLUTION

DART is a non-convex quadrilateral (with one interior angle greater than 180° — the dent at vertex D).

Method 1 (Rotation by 180° about edge midpoints):
Place one copy of DART. Rotate a second copy by 180° about the midpoint of one edge (say edge AR). The rotated copy fits perfectly along the shared edge AR without gap or overlap (since a 180° rotation maps one copy to the adjacent copy sharing that edge).
Repeat this rotation at every edge of every copy placed. The four angles of DART at any common vertex sum to 360° (since all four angles of ANY quadrilateral add to 360°), so the copies fill the plane completely.

Method 2 (Varignon parallelogram grid):
Construct the Varignon parallelogram of DART (join midpoints of its sides). Draw the tiling of the plane by this parallelogram (a standard parallelogram grid). Then use the reconstruction procedure from Exercise Set 12.3 Q5 to place copies of DART within each cell of the grid. The copies automatically fill the plane.

Preferred method: Method 1 is more direct and easier to carry out physically — just rotate each new copy 180° around the midpoint of a shared edge and continue outward.

Exercises — End-of-Chapter Exercises

Q1 Using a fact about parallelograms, show how to tile the plane using any given triangle. (Hint: Can you use the parallelogram tiling in the introduction?)

SOLUTION

Given any triangle T, we will construct a parallelogram from two copies of T, then use the parallelogram tiling.

Step 1: Take two congruent copies of triangle T. Place the second copy as a 180° rotation of the first about the midpoint of one side (say, the midpoint of side BC).
The two copies together form a parallelogram (since the rotated triangle fits perfectly alongside the original, and opposite sides of the combined figure are equal and parallel).

Step 2: The plane can be tiled by any parallelogram (as shown in the introduction: a rectangular grid of sticks, when sheared, gives a parallelogram tiling).

Step 3: In this parallelogram tiling, each parallelogram is made of two copies of T (one original, one rotated by 180°). Therefore the tiling of the plane by these parallelograms gives a tiling of the plane by copies of triangle T.

Hence any given triangle tiles the plane.

Q2 Mark the midpoint of the line drawn on the paper (see Fig. 12.33), given that the horizontal lines are equally spaced. Justify your answer. [The figure shows a diagonal line segment crossing several equally spaced horizontal lines.]

SOLUTION

From Fig. 12.33: A line segment is drawn across a set of equally spaced horizontal parallel lines. The segment starts at one horizontal line and ends at another.

Let the line segment go from point P (on the bottom horizontal line) to point Q (on the top horizontal line), crossing several equally spaced horizontal lines in between.

Since the horizontal lines are equally spaced, they divide the vertical distance between P and Q into equal parts.

By the Converse of the Midpoint Theorem (Theorem 7), applied repeatedly:
A line parallel to the base of a triangle through the midpoint of one side bisects the other side.

More directly: if there are n intervals between equally spaced horizontal lines, each horizontal line cuts the segment PQ at equal fractions of its length. The midpoint of PQ lies exactly halfway along PQ.

If the horizontal lines are the lines y = 0, 1, 2, …, n (equally spaced), and the segment PQ starts on y = 0 and ends on y = n, then the midpoint of PQ lies on the horizontal line y = n/2.

Therefore: mark the point where the segment PQ crosses the MIDDLE horizontal line (the line midway between P's line and Q's line). That point is the midpoint of PQ.

Justification using midpoint theorem: Consider the triangle formed by P, Q, and the foot of the vertical from Q to P's horizontal line. The equally spaced horizontals define equal divisions. By iterating the midpoint theorem (or by the Basic Proportionality Theorem approach), the middle horizontal line intersects PQ at its midpoint.

Q3 You know that the sum of angles of a quadrilateral is 360°, even for a non-convex quadrilateral. Now consider a self-intersecting quadrilateral ABCD, where AB and CD intersect at point E. Show that $\angle A + \angle B + \angle C + \angle D < 360°$. Can you construct ABCD such that $\angle A + \angle B + \angle C + \angle D = 2°$?

SOLUTION

In a self-intersecting quadrilateral ABCD, sides AB and CD cross at point E, creating two triangles: $\triangle ABE$ and $\triangle CDE$ (roughly — the intersection point E is internal to both sides).

Consider triangle $\triangle AED$ (using vertices A, E, D). The angles of this triangle are $\angle A$ (= $\angle DAE$), $\angle AED$, and $\angle D$ (= $\angle ADE$), summing to 180°.
Consider triangle $\triangle BEC$. Its angles $\angle B$ (= $\angle CBE$), $\angle BEC$, and $\angle C$ (= $\angle BCE$) sum to 180°.

So $(\angle A + \angle AED + \angle D) + (\angle B + \angle BEC + \angle C) = 180° + 180° = 360°$.
Therefore $\angle A + \angle B + \angle C + \angle D = 360° - \angle AED - \angle BEC$.

Now, $\angle AED$ and $\angle BEC$ are vertically opposite angles at E, so $\angle AED = \angle BEC$.
Also $\angle AED > 0°$ (since E is a proper intersection point, not a vertex).
Therefore $\angle AED + \angle BEC = 2\angle AED > 0°$.

Hence $\angle A + \angle B + \angle C + \angle D = 360° - 2\angle AED < 360°$. ✓

Can we construct ABCD with $\angle A + \angle B + \angle C + \angle D = 2°$?
We need $360° - 2\angle AED = 2°$, so $\angle AED = 179°$.
This means the two sides AB and CD intersect at E at a very obtuse angle of 179°.
Yes, this is possible: make AB and CD nearly parallel (almost parallel lines intersecting at a very shallow angle of 1° between them, giving the supplementary angle 179° at E).
Place A, B far apart along one nearly-horizontal line, and C, D far apart along another nearly-horizontal line, arranged so that the sides AB and CD intersect at a very shallow angle. The resulting quadrilateral ABCD is a very 'flat' self-intersecting figure with $\angle A + \angle B + \angle C + \angle D \approx 2°$.

Q4 In a parallelogram ABCD, two points P and Q are taken on diagonal BD such that DP = BQ (see Fig. 12.34). Show that APCQ is a parallelogram.

SOLUTION

Given: ABCD is a parallelogram with diagonal BD. Points P and Q lie on BD such that DP = BQ.
To show: APCQ is a parallelogram.

We will show that the diagonals of APCQ bisect each other.

The diagonals of quadrilateral APCQ are AC and PQ.

In parallelogram ABCD, diagonals AC and BD bisect each other.
Let O be the intersection of AC and BD. Then OA = OC and OB = OD.

Since P is on BD with DP = BQ and Q is on BD with BQ = DP:
O is the midpoint of BD, so OD = OB.
DP = BQ (given). Since P is on segment OD (between D and O) and Q is on segment OB (between O and B) with DP = BQ and OD = OB:
OP = OD − DP and OQ = OB − BQ = OD − DP (since OB = OD and BQ = DP).
Therefore OP = OQ.

So O is the midpoint of PQ (since OP = OQ), and O is the midpoint of AC (since OA = OC).
The diagonals AC and PQ of quadrilateral APCQ both have the same midpoint O.
Therefore the diagonals of APCQ bisect each other, which means APCQ is a parallelogram. ✓

Q5 A right-triangle shaped cutout of a paper is folded such that point A touches point B (Fig. 12.35). Show that the crease line can be used to find the midpoint of not only AB but also that of AC.

SOLUTION

From Fig. 12.35: the right angle is at B, with vertex A at the top-left, B at the bottom-left and C at the bottom-right. So AB is the vertical side, BC is the horizontal side (AB $\perp$ BC at B), and AC is the hypotenuse.

When the paper is folded so that A lands on B, the crease line is the perpendicular bisector of segment AB.

Step 1: The crease gives the midpoint of AB.
The crease is the perpendicular bisector of AB, so it passes through the midpoint M of AB. Hence M (where the crease meets AB) is the midpoint of AB.

Step 2: The crease gives the midpoint of AC.
Let ℓ be the crease (the perpendicular bisector of AB) and let N be the point where ℓ meets AC.
Since the right angle is at B, AB $\perp$ BC. The crease ℓ is also perpendicular to AB (it is the perpendicular bisector of AB).
Two lines perpendicular to the same line AB are parallel, so $\ell \parallel$ BC.
In $\triangle$ABC, the line ℓ passes through M (the midpoint of AB) and is parallel to BC.
By Theorem 7 (Converse of the Midpoint Theorem), the line through the midpoint of AB parallel to BC bisects AC.
Therefore N is the midpoint of AC.

Hence the single crease locates the midpoint of AB (at M) and the midpoint of AC (at N).

Q6 You saw how to use the Midpoint Theorem to divide a given triangle into 4 congruent triangles. Can we divide a triangle into 3 congruent triangles? (i) Draw medians AP, BQ and CR of ∆ABC, meeting in a common point M. Cut along each median to get 6 triangles. Show with justification how to assemble the 6 pieces into 3 congruent triangles. Prove that you get a triangle. What are its side lengths? *(ii) Repeat the procedure with each of the 3 assembled triangles to show each of the resulting 9 triangles has sides AB/3, BC/3, AC/3. (iii) Can we cut a triangle and reassemble it into 2 congruent triangles?

SOLUTION

(i) Let P, Q, R be the midpoints of BC, CA, AB respectively. The medians AP, BQ, CR meet at the centroid M. Cutting along the three medians produces 6 small triangles: ∆AMR, ∆RMB, ∆BMP, ∆PMC, ∆CMQ, ∆QMA.

All 6 triangles have equal area. Each median divides ∆ABC into two equal-area halves, and the centroid divides each median in the ratio 2 : 1, so every one of the 6 pieces has area = (1/6)·area(∆ABC).

Assembling 3 congruent triangles (using the hint): take the two pieces that meet along a median segment from M, for example ∆BMP and ∆CMP, which share the common side MP and have PB = PC (P is the midpoint of BC). Flip ∆CMP over so that side PC is laid onto side PB (match P↔P, C↔B). The two pieces now join to form one triangle whose vertices are the two images of M together with B(=C), i.e. a triangle with sides MB, MC and the base formed by the two copies of MP.

Doing the same at the other two medians gives 3 triangles. Each assembled triangle has as its three sides the three centroid-to-vertex distances MA = (2/3)·(median from A), MB = (2/3)·(median from B), MC = (2/3)·(median from C). Since all three assembled triangles are built from the same three lengths MA, MB, MC, they are congruent to one another (SSS).

Why it is a genuine triangle: the three vertices of each assembled piece are not collinear (M, B, C are not collinear), so the pieces form a proper triangle, not a degenerate segment.

Side lengths of each assembled triangle: MA, MB, MC, i.e. two-thirds of each of the three medians of ∆ABC.

*(ii) Repeating the same cut-and-assemble procedure on each of the 3 congruent triangles subdivides each into 3 smaller congruent triangles, giving 9 triangles in all. Carrying the construction through (tracking the median lengths at each stage), each of the final 9 triangles turns out to have sides AB/3, BC/3 and CA/3 — that is, a one-third scaled copy of ∆ABC. This is the Sallows dissection (discovered 2014).

*(iii) Cutting ∆ABC into 2 congruent triangles:
The median AP alone (P = midpoint of BC) splits ∆ABC into ∆ABP and ∆ACP, which have equal area but are not congruent in general (they are congruent only when AB = AC).
To obtain 2 congruent triangles, follow the hint: first cut along median AP, then in ∆ABP draw the median from P to the midpoint R of AB and cut along PR. This yields three pieces: ∆ACP, ∆APR and ∆BPR. The pieces ∆APR and ∆BPR are congruent (PR is a median of ∆ABP, and AR = RB), and reassembling ∆ACP with one of them produces two congruent triangles. So a

triangle can be cut and reassembled into 2 congruent triangles.

Q7 A more general midpoint theorem and its converse. In a quadrilateral ABCD, suppose AB ∥ DC. (ABCD is called a trapezium.) Let E be the midpoint of AD. A line drawn through E intersects side BC at F. (i) If EF ∥ AB, then show that F is the midpoint of BC and conclude that EF = (AB + CD)/2. (ii) If F is the midpoint of BC, then show that EF ∥ AB.

SOLUTION

(i) Given: Trapezium ABCD with AB ∥ DC, E = midpoint of AD, EF ∥ AB where F is on BC. Show F is midpoint of BC and EF = (AB + DC)/2.

Construction: Extend line EF and draw diagonal AC, meeting line EF at point G.

In ∆ACD: E = midpoint of AD, and G is on AC. Since EG is part of line EF, and EF ∥ AB ∥ DC, we have EG ∥ DC.
By the Converse of the Midpoint Theorem (Theorem 7) applied to ∆ACD:
Line through E (midpoint of AD) parallel to DC meets AC at G, where G is the midpoint of AC. ✓

In ∆ABC: G = midpoint of AC (just established), and GF ∥ AB (since EF ∥ AB and G is on EF).
By the Converse of the Midpoint Theorem applied to ∆ABC:
Line through G (midpoint of AC) parallel to AB meets BC at F, where F is the midpoint of BC. ✓

For EF = (AB + DC)/2:
By the Midpoint Theorem in ∆ACD: GE = DC/2.
By the Midpoint Theorem in ∆ABC: GF = AB/2.
Therefore EF = EG + GF = DC/2 + AB/2 = (AB + DC)/2. ✓

(ii) Given: Trapezium ABCD with AB ∥ DC, E = midpoint of AD, F = midpoint of BC. Show EF ∥ AB.

Method 1 (using midpoint M of BD):
Let M = midpoint of BD.
In ∆ABD: E = midpoint of AD, M = midpoint of BD. By Midpoint Theorem: EM ∥ AB and EM = AB/2.
In ∆BCD: M = midpoint of BD, F = midpoint of BC. By Midpoint Theorem: FM ∥ DC and FM = DC/2.
Since AB ∥ DC, we get EM ∥ DC, i.e., EM ∥ FM. Both pass through M, so E, M, F are collinear.
The line EF is the same as line EM, which is parallel to AB.
Therefore EF ∥ AB. ✓

Method 2 (using part (i)): Draw the line through E parallel to AB. By part (i), this line meets BC at its midpoint. But F is also the midpoint of BC. So the line through E parallel to AB passes through F, i.e., EF ∥ AB. ✓

Q8 The diagonals AC and BD of a parallelogram ABCD intersect at O. A line through O meets AB and CD at points P and Q respectively. Show that O is the midpoint of PQ. (Multiple proofs are possible. Which is the simplest?)

SOLUTION

Given: Parallelogram ABCD with diagonals AC and BD intersecting at O. A line through O meets AB at P and CD at Q.
To show: O is the midpoint of PQ, i.e., OP = OQ.

Proof using congruent triangles (simplest):

Consider triangles ∆AOP and ∆COQ.
— AO = CO (diagonals of a parallelogram bisect each other, so O is the midpoint of AC).
— ∠AOP = ∠COQ (vertically opposite angles at O).
— ∠OAP = ∠OCQ (AB ∥ DC with transversal AC, alternate interior angles).

By AAS (or ASA): ∆AOP ≅ ∆COQ.
Therefore OP = OQ. ✓

Hence O is the midpoint of PQ.

Q9 (*) ABCD is a trapezium with parallel sides AD = 3 cm and BC = 5 cm. E and F are the midpoints of the non-parallel sides. Find the ratio of the areas of the 4-gons AEFD and EBCF.

SOLUTION

Given: Trapezium ABCD with AD ∥ BC, AD = 3 cm, BC = 5 cm (from Fig. 12.38: AD on top, BC on bottom). E = midpoint of AB (non-parallel side), F = midpoint of DC (non-parallel side).

Step 1: Find EF (the midsegment).
By Exercise 7(i) (the general midsegment result): EF ∥ AD ∥ BC and EF = (AD + BC)/2 = (3 + 5)/2 = 4 cm.

Step 2: Let h be the height of the trapezium ABCD (perpendicular distance between the parallel sides AD and BC).
Since E and F are midpoints of the non-parallel sides, the height from AD to EF equals h/2, and the height from EF to BC equals h/2.

Step 3: Area of AEFD (a trapezoid with parallel sides AD = 3 and EF = 4, height = h/2):
Area(AEFD) = (1/2)(AD + EF)(h/2) = (1/2)(3 + 4)(h/2) = 7h/4.

Step 4: Area of EBCF (a trapezoid with parallel sides EF = 4 and BC = 5, height = h/2):
Area(EBCF) = (1/2)(EF + BC)(h/2) = (1/2)(4 + 5)(h/2) = 9h/4.

Step 5: Ratio:
Area(AEFD) : Area(EBCF) = 7h/4 : 9h/4 = 7 : 9.

Verification: 7h/4 + 9h/4 = 16h/4 = 4h = (1/2)(3+5)h = Area(ABCD). ✓

The ratio of the areas of AEFD and EBCF is 7 : 9.

Q10 Consider 4 points A, B, C, D in the plane with no three collinear. (i) How many different quadrilaterals do they form if the quadrilateral is allowed to be self-intersecting or non-convex? (ii) How many of these quadrilaterals are self-intersecting? How many are convex?

SOLUTION

(i) The number of different quadrilaterals formed by 4 labeled points A, B, C, D:
A quadrilateral is determined by an ordering of the 4 points as vertices (going around the boundary). The number of distinct cyclic orderings of 4 labeled points = (4−1)!/2 = 3!/2 = 3.
(Dividing by 2 because going clockwise vs anticlockwise gives the same quadrilateral.)

The 3 distinct quadrilaterals are: ABCD, ABDC, and ACBD.
(ABCD means sides AB, BC, CD, DA; ABDC means sides AB, BD, DC, CA; ACBD means sides AC, CB, BD, DA.)

Answer: 3 different quadrilaterals.

(ii) The answer depends on the configuration of the 4 points.

Case 1: The 4 points are in convex position (all four are vertices of their convex hull).
Then exactly 1 of the 3 quadrilaterals is convex (the one that goes around the convex hull in order), and the other 2 are self-intersecting ('butterfly' shapes).
Self-intersecting: 2. Convex: 1. Non-convex (but not self-intersecting): 0.

Case 2: One point (say D) lies inside the triangle formed by the other three (A, B, C).
Then all 3 quadrilaterals involving D as an 'interior' vertex are non-convex or self-intersecting.
Specifically: 2 are self-intersecting and 1 is non-convex (with the dent at D).
Self-intersecting: 2. Convex: 0. Non-convex: 1.

Q11 Suppose P is a point on side AB of ∆ABC and the line through P parallel to BC meets AC in point Q. For parts (i) to (iii), assume AP = 1, AQ = √5, and see Fig. 12.39. (i) If PB = 2, find QC. (ii)* If PB = 1/3, find QC. (iii)* If PB = 2/3, find QC. (iv)* Show that if AP/PB is a rational number, then AP/PB = AQ/QC.

SOLUTION

Setup: P on AB, Q on AC, PQ ∥ BC, AP = 1, AQ = √5.

(i) PB = 2. Find QC.
AB = AP + PB = 1 + 2 = 3.
Since PQ ∥ BC, triangles APQ and ABC are similar (AA: shared angle A, and ∠APQ = ∠ABC as corresponding angles).
Therefore AP/AB = AQ/AC → 1/3 = √5/AC → AC = 3√5.
QC = AC − AQ = 3√5 − √5 = 2√5.

Using only the Midpoint Theorem (the dotted line in the left figure): let M be the midpoint of PB, so PM = MB = 1. The line through M parallel to PQ (and hence to BC) and the line through P parallel to BC cut off equal successive parts on AC. Since AP = PM = MB = 1 are three equal steps along AB, the parallels divide AC into three equal steps too, each of length AQ = √5.

Thus QC = 2√5.

(ii)* PB = 1/3. Find QC.
AB = AP + PB = 1 + 1/3 = 4/3.
AP/PB = 1/(1/3) = 3.
By the ratio AP/AB = 1/(4/3) = 3/4, so AQ/AC = 3/4, AC = 4AQ/3 = 4√5/3.
QC = AC − AQ = 4√5/3 − √5 = 4√5/3 − 3√5/3 = √5/3.

Using the Midpoint Theorem (dotted lines in the right figure): here PB = 1/3, so AP = 3·PB. Divide AP into three equal parts of length 1/3 (division points P_1 at 1/3 and P_2 at 2/3 from A) and draw parallels to BC through P_1, P_2, P and B. These equally spaced parallels cut AC into four equal steps each of length AQ/3 = √5/3, and QC is the single step beyond Q, so QC = √5/3.

(iii)* PB = 2/3. Find QC.
AB = AP + PB = 1 + 2/3 = 5/3.
AP/PB = 1/(2/3) = 3/2.
AQ/AC = AP/AB = 1/(5/3) = 3/5, so AC = 5√5/3.
QC = AC − AQ = 5√5/3 − √5 = 5√5/3 − 3√5/3 = 2√5/3.

(iv)* Show that if AP/PB is rational, then AP/PB = AQ/QC.
Let AP/PB = p/q where p, q are positive integers.
Then AP = p·u and PB = q·u for some unit length u, so AB = (p+q)u.

Divide AB into (p+q) equal segments (each of length u). The division points create (p+q−1) points on AB.
Point P is at position p from A (after p segments).

Draw lines through each division point on AB parallel to BC. By iterating the Midpoint Theorem (or its generalization proved using midpoint theorem p+q−1 times):
These parallel lines divide AC into (p+q) equal segments as well.
Point Q (intersection with the line through P ∥ BC) is at position p from A on AC.
So AQ/AC = p/(p+q), meaning QC/AC = q/(p+q).
Therefore AQ/QC = p/q = AP/PB. ✓

Q12 (i) Suppose ABCD is a parallelogram and M, N are midpoints of AB and CD respectively. Show that segments DM and BN trisect segment AC. (ii) Use part (i) to find a procedure to trisect any given segment PQ. Find two other ways to trisect PQ, one using Exercise 11 above and a third using the Centroid Theorem.

SOLUTION

(i) Let M = midpoint of AB and N = midpoint of CD. The diagonals of ABCD intersect at O (midpoint of both diagonals AC and BD).

In parallelogram ABCD: AB ∥ DC and AB = DC. M is midpoint of AB, N is midpoint of DC.
So DM and BN are segments, not the full diagonals.

We find where DM and BN meet AC using coordinates.

Let A = (0,0), B = (2,0), C = (3,2), D = (1,2) (a parallelogram since AB = DC = (2,0) and AD = BC = (1,2)).
M = midpoint of AB = (1,0). N = midpoint of CD = (2,2).
AC goes from (0,0) to (3,2). Parametrize: (3t, 2t) for t ∈ [0,1].
DM goes from D=(1,2) to M=(1,0). This is the vertical line x=1.
Intersection of AC with x=1: 3t=1 → t=1/3. Point = (1, 2/3). This is at t=1/3 of AC from A.
BN goes from B=(2,0) to N=(2,2). Vertical line x=2.
Intersection of AC with x=2: 3t=2 → t=2/3. Point = (2, 4/3). This is at t=2/3 of AC from A.

So DM meets AC at 1/3 of the way from A, and BN meets AC at 2/3 of the way from A.
Therefore DM and BN trisect AC. ✓

General proof: Let ABCD be any parallelogram with A=(0,0), B=(a,0), C=(a+c,d), D=(c,d). M=(a/2,0), N=((a+2c)/2, d). By parametrizing AC and finding intersections, one gets t=1/3 and t=2/3, confirming trisection. The key: by the Midpoint Theorem in suitable triangles, the trisection points are exactly where DM and BN cross AC.

(ii) To trisect a given segment PQ using part (i):
Treat PQ as the diagonal AC of a parallelogram. Pick any two points B and D (not on PQ) so that PBQD is a parallelogram with diagonal PQ (B and D symmetric about the midpoint of PQ). Let M be the midpoint of side PB and N the midpoint of side QD. By part (i), the segments DM and BN cross PQ at its two trisection points.

Using Exercise 11: Place PQ as side AB of a triangle ABC. Choosing AP = 1 and PB = 2 and drawing PQ ∥ BC gives (by Ex 11(i)) AQ : QC = 1 : 2, so the line through P parallel to BC marks the first trisection point of AC; repeating with AP = 2, PB = 1 marks the second. Transferring these back to PQ trisects it.

Using the Centroid Theorem: Draw any triangle having PQ as one side and locate its centroid G, which divides each median in the ratio 2 : 1. Use this 2 : 1 division of a median to transfer a 1 : 2 point onto PQ; two such transfers give both trisection points of PQ.

Q13 Is the Midpoint Theorem for Quadrilaterals (Theorem 9) true when the quadrilateral is non-convex? How about when it is self-intersecting? (i) Can you explain why this is so? See the geometric reasoning in the text. (ii) There is one exception: In a very special case the Varignon parallelogram becomes a single segment. When will this happen? *(iii) Does your reasoning apply even when ABCD is non-planar?

SOLUTION

Theorem 9: The midpoints of the four sides of any quadrilateral ABCD form a parallelogram (the Varignon parallelogram PQRS).

The theorem holds for non-convex and self-intersecting quadrilaterals as well.

(i) Explanation: The proof of Theorem 9 uses only the Midpoint Theorem applied to the two triangles formed by a diagonal of ABCD.
Specifically: draw diagonal AC of ABCD (or BD).

In ∆ABC: P = midpoint of AB, Q = midpoint of BC → PQ ∥ AC and PQ = AC/2.
In ∆ACD: S = midpoint of DA, R = midpoint of CD → SR ∥ AC and SR = AC/2.
Therefore PQ ∥ SR and PQ = SR → one pair of opposite sides of PQRS are equal and parallel.
By Theorem 5, PQRS is a parallelogram.

This reasoning uses only midpoints and the Midpoint Theorem for triangles, which is valid regardless of whether the triangles are 'normal' or have reflex vertices. The diagonal AC always exists (it is just a line segment), so the proof works for non-convex and self-intersecting quadrilaterals.

(ii) The Varignon parallelogram degenerates to a single segment when all four midpoints are collinear.
This happens when PQRS has zero area, i.e., when the 'parallelogram' collapses.
PQ = AC/2 and SR = AC/2 (both parallel to AC); QR = BD/2 and PS = BD/2 (both parallel to BD).
The Varignon parallelogram degenerates (becomes a segment) when AC and BD are parallel — i.e., when the two diagonals of ABCD are parallel to each other.
In that special case, PQ ∥ SR ∥ QR ∥ PS (all four sides of PQRS become parallel to the same direction), and PQRS collapses to a segment along that direction.

*(iii) Yes, the reasoning applies even when ABCD is non-planar.
The Midpoint Theorem for triangles is a statement about triangles, and triangles are always planar (any three points determine a plane). So ∆ABC lies in one plane and ∆ACD lies in another (possibly different) plane. In each triangle, the Midpoint Theorem gives PQ ∥ AC and SR ∥ AC (both parallel to the segment AC in 3D space), so PQ ∥ SR and PQ = SR. Similarly QR ∥ PS and QR = PS. Therefore PQRS is a parallelogram (lying in some plane in 3D space, not necessarily the same plane as ABCD).

Q14 Let P, Q, R, S be four points on sides AB, BC, CD and DA respectively of a quadrilateral ABCD. Suppose PQRS is a parallelogram. Must P, Q, R and S be midpoints of the respective sides?

SOLUTION

No, P, Q, R, S need not be midpoints of the respective sides.

The Varignon theorem (Theorem 9) says that IF P, Q, R, S are midpoints, THEN PQRS is a parallelogram. This is a converse question: if PQRS is a parallelogram, must P, Q, R, S be midpoints?

Counter-example: Consider a square ABCD with A=(0,0), B=(2,0), C=(2,2), D=(0,2).
Let P = (1,0) (midpoint of AB), Q = (2,1) (midpoint of BC), R = (1,2) (midpoint of CD), S = (0,1) (midpoint of DA). This is the standard Varignon parallelogram (a square).

Now take P' = (t, 0) on AB for any 0 < t < 2, Q' = (2, t) on BC, R' = (2−t, 2) on CD, S' = (0, 2−t) on DA.
P'Q': from (t,0) to (2,t) — this has direction (2−t, t).

S'R': from (0, 2−t) to (2−t, 2) — direction (2−t, t). So P'Q' ∥ S'R'.
P'S': from (t,0) to (0, 2−t) — direction (−t, 2−t).
Q'R': from (2,t) to (2−t,2) — direction (−t, 2−t). So P'S' ∥ Q'R'.
Also |P'Q'| = |(2−t, t)| = |S'R'| and |P'S'| = |Q'R'|.
So P'Q'R'S' is a parallelogram for ANY t ∈ (0,2), not just t=1 (midpoint).

Therefore P, Q, R, S need NOT be midpoints for PQRS to be a parallelogram.
There are infinitely many parallelograms inscribed in a quadrilateral (one for each value of the parameter t), and only one of them (t = midpoint) is the Varignon parallelogram.

Q15 (i) Complete the following proof of the Midpoint Theorem. Extend segment PQ beyond Q until point S. By how much should we extend PQ? (Hint: prove ∆APQ ≅ ∆CSQ.) (ii) Give a similar proof of the converse of the Midpoint Theorem (Theorem 7). Start by extending PQ up to a suitable point S.

SOLUTION

(i) In ∆ABC, P = midpoint of AB, Q = midpoint of AC. Extend PQ beyond Q to point S such that QS = PQ.

We want to prove ∆APQ ≅ ∆CSQ.
In ∆APQ and ∆CSQ:
— AQ = CQ (since Q = midpoint of AC).
— ∠AQP = ∠CQS (vertically opposite angles at Q).
— PQ = QS (by construction: we extended PQ by PQ, so QS = PQ).
By SAS: ∆APQ ≅ ∆CSQ. ✓

Therefore AP = CS and ∠APQ = ∠CSQ.
Since ∠APQ = ∠CSQ are alternate interior angles for lines PB and CS with transversal PS, we get PB ∥ CS. Also CS = AP = PB (since P is the midpoint of AB).
So BPSC has one pair of opposite sides, BP and CS, equal and parallel; hence BCSP is a parallelogram.
In parallelogram BCSP: BC ∥ PS and BC = PS.
Therefore PQ ∥ BC (PS ∥ BC and S lies on PQ extended).
Also BC = PS = PQ + QS = 2PQ, so PQ = BC/2.

(ii) Proof of the Converse (Theorem 7): P = midpoint of AB, and the line through P parallel to BC meets AC at Q. We prove Q = midpoint of AC.

Through P, there is exactly one line parallel to BC. We know the line PQ is parallel to BC (given).
Let M be the midpoint of AC. By the forward Midpoint Theorem (part (i)) applied to the midpoints P of AB and M of AC, the segment PM is parallel to BC, so line PM is also a line through P parallel to BC.
By the uniqueness of the parallel through P to BC, line PM and line PQ are the same line.
So Q (on AC, on line PQ) coincides with M (on AC, on line PM). Hence Q = M, the midpoint of

AC.

Q16 (*) Review all the properties of a rhombus/rectangle/square that you proved in Grade 8. Formulate a converse of each. Decide if the converse is true. There are many possibilities here!

SOLUTION

Properties and their converses:

RHOMBUS:
Property: All four sides of a rhombus are equal.
Converse: A quadrilateral with all four sides equal is a rhombus. TRUE (a rhombus is defined as a parallelogram with all sides equal; but a quadrilateral with all 4 sides equal is a rhombus if it's also a parallelogram — and a quadrilateral with all 4 sides equal IS a parallelogram by Theorem 2). TRUE.

Property: The diagonals of a rhombus bisect each other at right angles.
Converse: A quadrilateral whose diagonals bisect each other at right angles is a rhombus. TRUE (bisect each other → parallelogram; perpendicular → rhombus).

Property: The diagonals of a rhombus bisect the vertex angles.
Converse: A parallelogram in which the diagonal bisects a vertex angle is a rhombus. TRUE (shown in Ex 12.2 Q2).

RECTANGLE:
Property: All angles of a rectangle are 90°.
Converse: A quadrilateral with all angles 90° is a rectangle. TRUE (also a parallelogram, since opposite angles are equal and all are 90°).

Property: The diagonals of a rectangle are equal.
Converse: A parallelogram with equal diagonals is a rectangle. TRUE (shown in Ex 12.2 Q1(iii)).

Property: The diagonals of a rectangle bisect each other.
Converse: A quadrilateral whose diagonals bisect each other is a rectangle. FALSE. Counter-example: any parallelogram (diagonals bisect each other, but not necessarily 90° angles). Extra condition: add that angles are 90°.

SQUARE:
Property: A square has all sides equal and all angles 90°.
Converse: A quadrilateral with all sides equal and all angles 90° is a square. TRUE.

Property: The diagonals of a square are equal and perpendicular.
Converse: A parallelogram with equal and perpendicular diagonals is a square. TRUE (equal diagonals → rectangle; perpendicular diagonals → rhombus; rectangle + rhombus = square).

Property: The diagonals of a square bisect the vertex angles (each diagonal makes 45° with the sides).
Converse: A parallelogram in which diagonals bisect all vertex angles is a square. TRUE

(diagonal bisecting vertex angles → rhombus; if both diagonals do so, it's a rhombus with equal diagonals → square).

Q17 (*) Show that the sum of angles of a non-planar quadrilateral is always less than 360°. Can you find a non-planar quadrilateral ABCD for which ∠A + ∠B + ∠C + ∠D = 2°? (Hint: Think of a diagonal, say AC, as a hinge around which triangles ABC and ADC can rotate.) What happens to each angle of ABCD as you do this rotation?

SOLUTION

A non-planar quadrilateral ABCD has vertices not all in one plane. The diagonal AC serves as a hinge.

Consider the planar version first: when ABCD is planar (flat), ∠A + ∠B + ∠C + ∠D = 360°.

Now 'fold' triangle ADC about hinge AC so that D lifts out of the plane of triangle ABC. The dihedral angle between the two triangular faces increases from 180° (flat) toward some value.

Effect on angles:
∠B is the angle at B inside triangle ABC; the fold moves only triangle ADC, so triangle ABC and ∠B are unchanged.
∠D is the angle at D inside triangle ADC; folding is a rigid motion of that triangle, so its shape and ∠D are unchanged.
∠A (the angle between edges AB and AD) and ∠C (the angle between CB and CD) are 'split' by the diagonal AC: ∠A = ∠BAC + ∠CAD and ∠C = ∠BCA + ∠ACD. Each of these four part-angles lives inside one of the two rigid triangles, so each part-angle stays fixed. However, when the two triangles are no longer coplanar, the two parts at A no longer lie in one plane, and the true 3D angle between AB and AD is strictly less than the flat sum ∠BAC + ∠CAD. The same holds at C.

So folding strictly decreases ∠A and ∠C while ∠B and ∠D are unchanged.

As the fold angle θ between the two faces increases from 0 to 180°:
The angles ∠A and ∠C in the quadrilateral decrease strictly below their planar values.
So ∠A + ∠B + ∠C + ∠D < ∠A_planar + ∠B + ∠C_planar + ∠D = 360°.
Therefore the angle sum is strictly less than 360° for any non-planar quadrilateral. ✓

Can we get ∠A + ∠B + ∠C + ∠D = 2°?
As we fold more and more (increasing the dihedral angle), ∠A and ∠C each decrease toward their minimum values. If we start with a planar quadrilateral where ∠B and ∠D are very small (close to 0°), and then fold so ∠A and ∠C also approach 0°, the total sum approaches 0°.
Yes, by choosing a very 'flat' quadrilateral (with small ∠B and ∠D ≈ 1° each) and then folding strongly, we can make ∠A + ∠C ≈ 0°, giving total ≈ 2°.
So yes, a non-planar quadrilateral ABCD with ∠A + ∠B + ∠C + ∠D = 2° can be constructed.

Q18 (*) A third method to tile the plane using a 4-gon. Two copies of SOME sharing a vertex appear to be shifted copies of each other. (i) Prove the exact match of SOME with S′O′M′E′ using four parallelograms. (ii) Follow the described procedure along each diagonal to build a 3×3 grid of 9 copies.

SOLUTION

(i) In the third tiling method (Fig. 12.42): Two copies of SOME are placed so that the diagonal EO of the first copy and diagonal E′O′ of the second are collinear with O = E′.
So the shared point is O = E′ — the endpoint of one diagonal meets the start of the other.

We want to prove S′O′M′E′ exactly matches SOME shifted along diagonal EO.

The 'shift' is a translation by vector EO (the diagonal vector from E to O).

Proof using four parallelograms:
Consider any side of SOME, say side SO. The 'shifted' copy has side S′O′ = SO translated by EO.
Since SO and S′O′ are parallel and equal (translation preserves direction and length), SS′O′O is a parallelogram.
Similarly: OM and O′M′ are parallel and equal → OO′M′M is a parallelogram.
ME and M′E′ are parallel and equal → MM′E′E is a parallelogram.
ES and E′S′ are parallel and equal → EE′S′S is a parallelogram.
These four parallelograms (one per side of SOME) confirm that S′O′M′E′ is exactly SOME translated by the vector EO. ✓

(ii) Starting from the original SOME, slide along diagonal EO in both directions to get copies sharing a vertex at O, at E, etc. This gives a row of copies along the EO direction.
Similarly, slide along the other diagonal OM in both directions.
The 3×3 grid of 9 copies is built by:
— 1 original SOME at center.
— 2 copies along the EO direction (one each side).
— 2 copies along the OM direction.
— 4 corner copies (combining shifts in both diagonal directions).
These 9 copies meet only at vertices (where diagonals end/begin), and the gaps between them are also congruent copies of SOME. Continuing this procedure tiles the entire plane.

Q19 (*) Is there a 4-gon with given side lengths? (i) Recall the triangle inequality: for a triangle with sides a, b, c (with $a \le b \le c$), the triangle exists iff $a + b > c$. (ii) Suppose a 4-gon has 2, 5, 11 as three side lengths. Can the fourth side be 100? Can it be 10? Can it be 1? What are the possible lengths of the fourth side? (iii) For given positive numbers a, b, c, d, how will you decide if there is a 4-gon with these side lengths?

SOLUTION

(i) Triangle inequality (recalled): For positive $a \le b \le c$, a triangle with sides a, b, c exists if and only if $a + b > c$ (the sum of the two smaller sides exceeds the largest).

(ii) Given three sides of a quadrilateral: 2, 5, 11. Let the fourth side be $x > 0$.
For a quadrilateral to exist, each side must be less than the sum of the other three sides.
The binding constraints are:
— $x < 2 + 5 + 11 = 18$ (the fourth side is less than the sum of the other three).
— $11 < 2 + 5 + x \rightarrow x > 4$ (the largest known side must be less than the sum of the others including x).
The other constraints ($2 < 5+11+x$ and $5 < 2+11+x$) are automatically satisfied for all $x > 0$.

So the valid range is: $4 < x < 18$.

Can x = 100? No — $100 > 18$. ✗
Can x = 10? Yes — $4 < 10 < 18$. ✓
Can x = 1? No — $1 \le 4$. ✗

Possible lengths of the fourth side: any value x with $4 < x < 18$.

(iii) For a quadrilateral with sides a, b, c, d (any positive values) to exist:
The necessary and sufficient condition is that each side is strictly less than the sum of the other three sides.
By symmetry, the binding condition is: the largest side must be less than the sum of the other three.
If we label the sides so that $a \le b \le c \le d$, the condition is: $d < a + b + c$.
(The other three conditions $a < b+c+d$, $b < a+c+d$, $c < a+b+d$ are automatically satisfied when $d < a+b+c$ and all values are positive.)

So: a quadrilateral with sides a, b, c, d exists if and only if the largest side is less than the sum of the other three.

Q20 (*) Counting diagonals of a polygon. (i) How should we define a diagonal of an n-gon? How many diagonals does an n-gon have? Make a table for small values of n. (ii) Can you guess a formula for the number of diagonals? How many diagonals get added when we increase the number of sides by 1? Justify why your formula is true for all n.

SOLUTION

(i) Definition: A diagonal of an n-gon is a line segment joining two non-adjacent vertices.

Table of diagonals:

n = 3 (triangle): 0 diagonals.
n = 4 (quadrilateral): 2 diagonals.
n = 5 (pentagon): 5 diagonals.
n = 6 (hexagon): 9 diagonals.
n = 7 (heptagon): 14 diagonals.
n = 8 (octagon): 20 diagonals.

(ii) Formula: The number of diagonals of an n-gon is $n(n-3)/2$.

Derivation: Each vertex can be connected to $n-1$ other vertices. Subtract the 2 adjacent vertices to get $n-3$ non-adjacent vertices. So from each vertex, there are $n-3$ diagonals. Total = $n(n-3)$, but each diagonal is counted twice (once from each endpoint), so the number of diagonals = $n(n-3)/2$.

Verification: n=4: $4(1)/2 = 2$ ✓; n=5: $5(2)/2 = 5$ ✓; n=6: $6(3)/2 = 9$ ✓.

How many diagonals are added going from n to n+1?
Diagonals for n+1: $(n+1)(n-2)/2$.
Diagonals for n: $n(n-3)/2$.
Difference: $[(n+1)(n-2) - n(n-3)] / 2 = [n^2-n-2 - n^2+3n] / 2 = [2n-2] / 2 = n-1$.

So adding one vertex (going from n-gon to (n+1)-gon) adds exactly $n-1$ new diagonals: the new vertex connects to all vertices except its two neighbors, giving $n+1-2 = n-1$ new diagonals.

Proof by induction: Base case n=3: $3(0)/2 = 0$ diagonals ✓. If the formula holds for n, then adding the (n+1)th vertex creates $n-1$ new diagonals, giving $n(n-3)/2 + (n-1) = [n(n-3) + 2(n-1)] / 2 = [n^2-3n+2n-2] / 2 = [n^2-n-2] / 2 = (n+1)(n-2)/2$, which matches the formula for n+1. ✓

Q21 (*) Sum of angles of a polygon. What is the sum of angles of a (planar non-self-intersecting) n-gon? We know the answer is 180° for n = 3 and 360° for n = 4. Find the next few values. Then find a formula in terms of n and prove it.

SOLUTION

Values for small n:
n = 3: 180°
n = 4: 360°
n = 5: 540°
n = 6: 720°
n = 7: 900°
n = 8: 1080°

Formula: The sum of interior angles of a planar non-self-intersecting n-gon = $(n - 2) \times 180°$.

Proof: Triangulate the n-gon by drawing $(n-3)$ non-crossing diagonals from one vertex. This divides the n-gon into $(n-2)$ triangles.

More precisely: Choose any vertex V of the n-gon. Draw diagonals from V to all non-adjacent

vertices. Since there are n−3 non-adjacent vertices (excluding V's two neighbors), this creates n−2 triangles that together cover the interior of the n-gon exactly once.

Each triangle has an angle sum of 180°. The angles of all the triangles together account for exactly the interior angles of the n-gon (since every interior angle of the n-gon appears as an angle in exactly one of the triangles, and there are no extra angles at the interior diagonals since they are full diagonals sharing only with adjacent triangles).

Total sum of interior angles = (n−2) × 180°. ✓

Verification: n=4: (4−2)×180° = 360° ✓; n=5: 3×180° = 540° ✓; n=6: 4×180° = 720° ✓.

Q22 (*) Multiple converses to a theorem. [See the textbook for the full statement with four named conditions: (P MID) P is midpoint of AB, (Q MID) Q is midpoint of AC, (PRLL) PQ ∥ BC, (HALF) PQ = BC/2.] (i) 'If (PRLL) and (HALF) are true, then (P MID) and (Q MID) are true.' Write this out and prove it. (ii) Write the converse of the Midpoint Theorem in the form that gives Theorem 7. Write Theorem 7 in terms of the four named conditions. (iii) Examine the remaining combination: 'If (P MID) and (HALF) are true, can we conclude (Q MID) and/or (PRLL)?'

SOLUTION

Setup: In △ABC, P is on AB and Q is on AC.
Four conditions: (P MID): P is midpoint of AB; (Q MID): Q is midpoint of AC; (PRLL): PQ ∥ BC; (HALF): PQ = BC/2.

(i) Statement: 'If PQ ∥ BC and PQ = BC/2, then P is the midpoint of AB and Q is the midpoint of AC.'

Proof: Suppose PQ ∥ BC and PQ = BC/2 (with P on AB, Q on AC).
Let M be the midpoint of AB and N be the midpoint of AC. By the Midpoint Theorem, MN ∥ BC and MN = BC/2.
So PQ ∥ BC, MN ∥ BC → PQ ∥ MN.
Also PQ = MN (both = BC/2).
P is on AB and M is the midpoint of AB; Q is on AC and N is the midpoint of AC.
Since PQ = MN and PQ ∥ MN, quadrilateral PQNM is a parallelogram (opposite sides equal and parallel).
So PM ∥ QN and PM = QN.
But PM and QN are both along lines AB and AC respectively. PM lies on AB and QN lies on AC.
For PQNM to be a parallelogram with PM ∥ QN: since AB and AC are not parallel (they meet at A), the only way PM ∥ QN is if PM = 0 and QN = 0, i.e., P = M and Q = N.
Therefore P is the midpoint of AB and Q is the midpoint of AC. ✓

(ii) Theorem 7 as a converse: The Midpoint Theorem states: 'If (P MID) and Q is any point on AC, then if (Q MID) then (PRLL) [and (HALF)].'
Converse (keeping 'P is midpoint of AB' fixed): 'If (P MID) and (PRLL), then (Q MID) [and (HALF)].'

This is precisely Theorem 7: the line through the midpoint P of AB, drawn parallel to BC, bisects AC at Q (so Q is the midpoint of AC).

In terms of the four conditions: Theorem 7 is: '(P MID) + (PRLL) => (Q MID) [+ (HALF)].'

(iii) Remaining combination: 'If (P MID) and (HALF) are true, can we conclude (Q MID) and/or (PRLL)?'
Statement: P is the midpoint of AB, and PQ = BC/2. Must Q be the midpoint of AC? Must PQ ∥ BC?

The answer is NO.
Fixing only P (midpoint of AB) and the length PQ = BC/2 leaves Q free to be any point at distance BC/2 from P. The set of such points is a whole circle of radius BC/2 centred at P.

Counter-example: Take ∆ABC with A=(0,0), B=(2,0), C=(1,2). Then P = (1,0) and |BC| = √5, so PQ must equal √5/2. Choose Q straight below P, say Q = (1, −√5/2). This Q satisfies PQ = √5/2, but PQ is vertical while BC has direction (−1,2), so PQ is not parallel to BC, and Q does not even lie on AC — so it is certainly not the midpoint of AC.

Hence (P MID) together with (HALF) alone forces neither (Q MID) nor (PRLL): the direction of PQ is completely unconstrained.

Q23 (*) Validity of tiling methods. Show using reasoning that each of the three tiling methods we saw produces a tiling of the plane using the given 4-gon. Prove that when we place new copies using any of the three procedures, the entire plane is covered with no gaps and no overlaps.

SOLUTION

We need to verify three methods produce valid tilings — no gaps, no overlaps.

Method 1 (180° rotation about edge midpoints):
Start with one copy of SOME. For each edge, rotate a new copy of SOME by 180° about the midpoint of that edge.
No overlap: A 180° rotation about the midpoint of edge OM maps O to M and M to O. The rotated copy shares exactly the edge OM with the original, lies on the opposite side, and cannot overlap the original (since the rotation takes the interior of one copy to the exterior of the other).
No gap: After placing one copy on each side of each edge, the edges of adjacent copies match perfectly (by the definition of the 180° rotation). At each vertex where copies meet, the angles of all meeting copies sum to 360° (since all four angles of SOME sum to 360°, and the 180° rotation procedure places all four angles around each vertex exactly once). So no gap at any vertex.
Infinite coverage: By repeating at every edge, the tiling extends to cover all of $\mathbb{R}^2$.

Method 2 (Varignon parallelogram grid):
Tile the plane with the Varignon parallelogram PQRS of SOME. For each parallelogram cell, reconstruct a copy of SOME (using the reconstruction from Exercise Set 12.3 Q5). Since the reconstruction is unique and congruent to SOME, the copies fit exactly into each parallelogram cell. The parallelogram tiling has no gaps or overlaps, so the SOME tiling also has no gaps or

overlaps.

Method 3 (Translation along diagonals — Exercise 18):
From Exercise 18: two copies of SOME sharing a diagonal endpoint (O = E′) are exact translates of each other (translation by the diagonal vector EO). This translation is a rigid motion, so the translated copy has the same shape and size, shares exactly the vertex O = E′, and lies in a new region. At each 'gap' vertex, exactly four copies of SOME meet with angles summing to 360°. Translating along both diagonals in both directions tiles the plane as a 3×3, 5×5, … grid, covering all of $\mathbb{R}^2$.

Q24 (*) What fraction of the square is shaded? [Fig. 12.43 shows a square with an inner rotated square (shaded) whose vertices lie at the midpoints of the outer square's sides. Equal tick marks confirm the midpoint division.]

SOLUTION

From Fig. 12.43: The outer square has its four sides each divided into two equal halves (shown by equal tick marks). The inner shaded square has its four vertices at the midpoints of the outer square's sides.

Let the outer square have side length 2. Then each half-side has length 1.

The inner square's vertices are at the midpoints: (1, 0), (2, 1), (1, 2), (0, 1) (placing the outer square with vertices at (0,0), (2,0), (2,2), (0,2)).

Side length of inner square: distance from (1,0) to (2,1) = $\sqrt{((2-1)^2 + (1-0)^2)} = \sqrt{(1+1)} = \sqrt{2}$.

Area of inner square = $(\sqrt{2})^2 = 2$.
Area of outer square = $2^2 = 4$.

Fraction shaded = 2/4 = 1/2.

The shaded inner square is exactly half the area of the outer square.