CBSE Class 12 Mathematics 2024 — Set 1 — Answer Key

English Medium

Time Allowed: 3 Hours | Maximum Marks: 80

General Instructions:

1. This question paper contains 38 questions. All questions are compulsory.
2. Section A has 20 MCQs carrying 1 mark each.
3. Section B has 6 questions carrying 2 marks each.
4. Section C has 6 questions carrying 3 marks each.
5. Section D has 4 questions carrying 5 marks each.
6. Section E has 3 case-study based questions carrying 4 marks each.
7. Draw neat figures wherever required.

Section A

Multiple Choice Questions (1 mark each)

Q.1 [1 mark]

A function f : R+ → R (where R+ is the set of all non-negative real numbers) defined by f(x) = 4x + 3 is:

(a) one-one but not onto <<

(b) onto but not one-one

(c) both one-one and onto

(d) neither one-one nor onto

Explanation:

f(x) = 4x + 3 on R+ ($x \geq 0$).

One-one: if $4x_1+3 = 4x_2+3$, then $x_1 = x_2$ — yes.

Onto R: range = $[3, \infty) \neq R$ (e.g., $y = 0 \notin$ range), so not onto R.

Answer: one-one but not onto.

Q.2 [1 mark]

If a matrix has 36 elements, the number of possible orders it can have, is:

(a) 13

(b) 3

(c) 5

(d) 9 <<

Explanation:

Factor pairs of 36 ($p \times q$ with $p \cdot q = 36$): (1,36),(2,18),(3,12),(4,9),(6,6),(9,4),(12,3),(18,2),(36,1) = 9 possible orders.

Q.3 [1 mark]

Which of the following statements is true for the function $f(x) = x^2 + 3$ for $x \neq 0$, and $f(0) = 1$?

(a) f(x) is continuous and differentiable $\forall\ x \in \mathbb{R}$

(b) f(x) is continuous $\forall\ x \in \mathbb{R}$

(c) f(x) is continuous and differentiable $\forall\ x \in \mathbb{R} - \{0\}$ <<

(d) f(x) is discontinuous at infinitely many points

Explanation:

At x = 0: lim f(x) = 0 + 3 = 3 ≠ f(0) = 1, so f is discontinuous at x = 0.

For x ≠ 0, f(x) = x² + 3 is a polynomial — continuous and differentiable everywhere on $\mathbb{R} - \{0\}$.

Option (C) is correct.

Q.4 [1 mark]

Let f(x) be a continuous function on [a, b] and differentiable on (a, b). Then, this function f(x) is strictly increasing in (a, b) if:

(a) f′(x) < 0, ∀ x ∈ (a, b)

(b) f′(x) > 0, ∀ x ∈ (a, b) <<

(c) f′(x) = 0, ∀ x ∈ (a, b)

(d) f(x) > 0, ∀ x ∈ (a, b)

Explanation:

By the standard theorem, f is strictly increasing on (a,b) iff f′(x) > 0 for all x ∈ (a, b).

Q.5 [1 mark]

If the matrix equation [[x + y, 2], [5, xy]] = [[6, 2], [5, 8]] holds, then the value of (24/x + 24/y) is:

(a) 7

(b) 6

(c) 8

(d) 18 <<

Explanation:

Equating entries: x + y = 6 and xy = 8.

So 24/x + 24/y = 24(x + y)/(xy) = 24 × 6/8 = 24 × 3/4 = 18.

Q.6 [1 mark]

∫[a to b] f(x) dx is equal to:

(a) $\int_a^b f(a - x)\,dx$

(b) $\int_a^b f(a + b - x)\,dx$ <<

(c) $\int_a^b f(x - (a + b))\,dx$

(d) $\int_a^b f((a - x) + (b - x))\,dx$

Explanation:

By the substitution $x \to a + b - x$, the limits remain a to b and $dx \to -dx$ (with sign flip giving the same integral).

Thus $\int_a^b f(x)\,dx = \int_a^b f(a + b - x)\,dx$.

Option (B).

Q.7 [1 mark]

Let θ be the angle between two unit vectors â and b̂ such that $\sin\theta = 3/5$. Then, â · b̂ is equal to:

(a) ±3/5

(b) ±3/4

(c) ±4/5 <<

(d) ±4/3

Explanation:

$\sin\theta = 3/5 \Rightarrow \cos\theta = \pm 4/5$ (since $\sin^2\theta + \cos^2\theta = 1$: $9/25 + \cos^2\theta = 1 \Rightarrow \cos^2\theta = 16/25$).

For unit vectors, â · b̂ = |â||b̂|cos θ = 1·1·(±4/5) = ±4/5.

Q.8 [1 mark]

The integrating factor of the differential equation $(1 - x^2)(dy/dx) + xy = ax$, $-1 < x < 1$, is:

(a) $1/(x^2 - 1)$

(b) $1/\sqrt{(x^2 - 1)}$

(c) $1/(1 - x^2)$

(d) $1/\sqrt{(1 - x^2)}$ <<

Explanation:

Divide by $(1 - x^2)$: $dy/dx + x/(1 - x^2) \cdot y = ax/(1 - x^2)$.

$P(x) = x/(1 - x^2)$.

IF = e^($\int x/(1-x^2)\,dx$).

Let $u = 1-x^2$, $du = -2x\,dx$, so $\int x/(1-x^2)\,dx = -(1/2)\ln|1-x^2| = \ln(1/\sqrt{(1-x^2)})$.

Thus IF = $1/\sqrt{(1-x^2)}$.

Q.9 [1 mark]

If the direction cosines of a line are $\sqrt{3}$ k, $\sqrt{3}$ k, $\sqrt{3}$ k, then the value of k is:

(a) ±1

(b) $\pm\sqrt{3}$

(c) ±3

(d) ±1/3 <<

Explanation:

Sum of squares of direction cosines = 1: $(\sqrt{3}\,k)^2 + (\sqrt{3}\,k)^2 + (\sqrt{3}\,k)^2 = 3k^2 + 3k^2 + 3k^2 = 9k^2 = 1 \Rightarrow k^2 = 1/9 \Rightarrow k = \pm 1/3$.

Answer is option (D): ±1/3.

Q.10 [1 mark]

A linear programming problem deals with the optimization of a/an:

(a) logarithmic function

(b) linear function <<

(c) quadratic function

(d) exponential function

Explanation:

By definition, a linear programming problem optimizes a linear objective function subject to linear constraints.

Q.11 [1 mark]

If $P(A|B) = P(A'|B)$, then which of the following statements is true?

(a) $P(A) = P(A')$

(b) $P(A) = 2\,P(B)$

(c) $P(A \cap B) = (1/2)\, P(B)$ <<

(d) $P(A \cap B) = 2\, P(B)$

Explanation:

$P(A|B) = P(A \cap B)/P(B)$ and $P(A'|B) = P(A' \cap B)/P(B)$.

Setting them equal: $P(A \cap B) = P(A' \cap B)$.

Now $P(A \cap B) + P(A' \cap B) = P(B)$, so $2P(A \cap B) = P(B)$, hence $P(A \cap B) = (1/2)P(B)$.

Option (C).

Q.12 [1 mark]

The value of the determinant $|(x+1, x-1), (x^2+x+1, x^2-x+1)|$ is equal to:

(a) $2x^3$

(b) 2 <<

(c) 0

(d) $2x^3 - 2$

Explanation:

$\Delta = (x+1)(x^2-x+1) - (x-1)(x^2+x+1)$.

$(x+1)(x^2-x+1) = x^3+1$ (sum of cubes).

$(x-1)(x^2+x+1) = x^3-1$ (difference of cubes).

So $\Delta = (x^3+1) - (x^3-1) = 2$.

Q.13 [1 mark]

The derivative of $\sin(x^2)$ with respect to x, at $x = \sqrt{\pi}$, is:

(a) 1

(b) –1

(c) $-2\sqrt{\pi}$ <<

(d) $2\sqrt{\pi}$

Explanation:

$d/dx[\sin(x^2)] = \cos(x^2) \cdot 2x$.

At $x = \sqrt{\pi}$: $\cos((\sqrt{\pi})^2) \cdot 2\sqrt{\pi} = \cos(\pi) \cdot 2\sqrt{\pi} = (-1)(2\sqrt{\pi}) = -2\sqrt{\pi}$.

Q.14 [1 mark]

The order and degree of the differential equation $[1 + (dy/dx)^2]^3 = d^2y/dx^2$, respectively, are:

(a) 1, 2

(b) 2, 3

(c) 2, 1 <<

(d) 2, 6

Explanation:

The highest-order derivative is d^2y/dx^2, so the order is 2.

Expanding the left side gives a polynomial in dy/dx, and d^2y/dx^2 appears to the power 1, so the degree is 1.

Order 2, degree 1.

Q.15 [1 mark]

The vector with terminal point A(2, –3, 5) and initial point B(3, –4, 7) is:

(a) $\hat{i} - \hat{j} + 2\hat{k}$

(b) $\hat{i} + \hat{j} + 2\hat{k}$

(c) $-\hat{i} - \hat{j} - 2\hat{k}$

(d) $-\hat{i} + \hat{j} - 2\hat{k}$ <<

Explanation:

Vector from initial point B to terminal point A = A – B = $(2-3)\hat{i} + (-3-(-4))\hat{j} + (5-7)\hat{k} = -\hat{i} + \hat{j} - 2\hat{k}$.

Option (D).

Q.16 [1 mark]

The distance of point P(a, b, c) from the y-axis is:

(a) b

(b) b^2

(c) $\sqrt{(a^2 + c^2)}$ <<

(d) $a^2 + c^2$

Explanation:

Distance from y-axis = $\sqrt{(a^2 + c^2)}$ (the perpendicular distance ignores the y-coordinate component).

Q.17 [1 mark]

The number of corner points of the feasible region determined by constraints $x \geq 0$, $y \geq 0$, $x + y \geq 4$ is:

(a) 0

(b) 1

(c) 2 <<

(d) 3

Explanation:

The constraints $x \geq 0$, $y \geq 0$, $x + y \geq 4$ define an unbounded feasible region.

The corner points are (4, 0) (where $x+y=4$ meets the x-axis) and (0, 4) (where $x+y=4$ meets the y-axis) — exactly 2 corner points.

Q.18 [1 mark]

If A and B are two non-zero square matrices of same order such that $(A + B)^2 = A^2 + B^2$, then:

(a) AB = O

(b) AB = –BA <<

(c) BA = O

(d) AB = BA

Explanation:

$(A+B)^2 = A^2 + AB + BA + B^2$.

Setting equal to $A^2 + B^2$: $AB + BA = O$, so $AB = –BA$.

Option (B).

Q.19 [1 mark]

Assertion (A): For matrix $A = [[1, \cos\theta, 1], [–\cos\theta, 1, \cos\theta], [–1, –\cos\theta, 1]]$, where $\theta \in [0, 2\pi]$, $|A| \in [2, 4]$. Reason (R): $\cos\theta \in [–1, 1]$, $\forall\ \theta \in [0, 2\pi]$.

(a) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A). <<

(b) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of Assertion (A).

(c) Assertion (A) is true, but Reason (R) is false.

(d) Assertion (A) is false, but Reason (R) is true.

Explanation:

Expand $|A|$ along R1: $|A| = 1(1 - (-\cos^2\theta)) - \cos\theta(-\cos\theta - (-\cos\theta)) + 1(\cos^2\theta - (-1))$ $= 1(1 + \cos^2\theta) - \cos\theta(0) + 1(\cos^2\theta + 1) = 2 + 2\cos^2\theta$.

Since $\cos\theta \in [-1,1]$, $\cos^2\theta \in [0,1]$, so $|A| \in [2, 4]$.

Both A and R are true and R correctly explains A.

Q.20 [1 mark]

Assertion (A): A line in space cannot be drawn perpendicular to x, y and z axes simultaneously. Reason (R): For any line making angles α, β, γ with the positive directions of x, y and z axes respectively, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

(a) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of Assertion (A). <<

(b) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of Assertion (A).

(c) Assertion (A) is true, but Reason (R) is false.

(d) Assertion (A) is false, but Reason (R) is true.

Explanation:

If a line were perpendicular to all 3 axes, then $\alpha = \beta = \gamma = 90°$, giving $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 0 \neq 1$, contradicting R.

So A is true and R is the correct explanation.

Section B

Very Short Answer Questions (2 marks each)

Q.21 [2 marks]

(a) Check whether the function $f(x) = x^2|x|$ is differentiable at $x = 0$ or not.

Answer: $f(x) = x^2|x|$ is differentiable at $x = 0$ (derivative = 0 from both sides)

Explanation:

For $x > 0$: $f(x) = x^3$, $f'(x) = 3x^2$.

LHD at 0: $\lim_{h\to 0^-} [f(0+h)-f(0)]/h = \lim h\cdot|h| = \lim h(-h)/h = \lim(-h) = 0$.

RHD: $\lim_{h\to 0^+} h\cdot h/h = \lim h = 0$.

LHD = RHD = 0, so differentiable at $x = 0$.

OR

Q.21 [2 marks]

(b) If $y = \sqrt{(\tan\sqrt{x})}$, prove that $\sqrt{x}\,(dy/dx) = (1 + y^4)/(4y)$.

Answer: Proved (see explanation)

Explanation:

Here $y = \sqrt{(\tan\sqrt{x})}$, so $y^2 = \tan\sqrt{x}$.

Differentiate w.r.t. x: $2y(dy/dx) = \sec^2(\sqrt{x}) \cdot d(\sqrt{x})/dx = \sec^2(\sqrt{x}) \cdot 1/(2\sqrt{x})$.

Hence $dy/dx = \sec^2(\sqrt{x})/(4y\sqrt{x})$.

Now $\sec^2(\sqrt{x}) = 1 + \tan^2(\sqrt{x}) = 1 + (y^2)^2 = 1 + y^4$.

Therefore $\sqrt{x} \cdot (dy/dx) = \sqrt{x} \cdot (1 + y^4)/(4y\sqrt{x}) = (1 + y^4)/(4y)$.

QED.

Q.22 [2 marks]

Show that the function $f(x) = 4x^3 - 18x^2 + 27x - 7$ has neither maxima nor minima.

Answer: No maxima or minima (see explanation)

Explanation:

$f'(x) = 12x^2 - 36x + 27 = 3(4x^2 - 12x + 9) = 3(2x - 3)^2$.

Setting $f'(x) = 0$: $(2x - 3)^2 = 0 \Rightarrow x = 3/2$ is the only critical point.

$f''(x) = 24x - 36$; $f''(3/2) = 36 - 36 = 0$.

Since $f'(x) = 3(2x-3)^2 \geq 0$ for all x (never negative), f' does not change sign at $x = 3/2$.

Hence it is neither a maximum nor a minimum (it is a point of inflection).

Q.23 [2 marks]

(a) Find: $\int x\sqrt{(1 + 2x)}\, dx$

Answer: (1/10)(1+2x)^(5/2) – (1/6)(1+2x)^(3/2) + C

Explanation:

Let u = 1+2x, x = (u–1)/2, dx = du/2.

Then $\int x\sqrt{(1+2x)}\, dx = \int (u–1)/2 \cdot \sqrt{u} \cdot du/2 = (1/4)\int(u^{(3/2)} – u^{(1/2)})\, du = (1/4)[(2/5)u^{(5/2)} – (2/3)u^{(3/2)}] + C = (1/10)(1+2x)^{(5/2)} – (1/6)(1+2x)^{(3/2)} + C.$

OR

Q.23 [2 marks]

(b) Evaluate: $\int_{0}^{\pi^2/4} (\sin\sqrt{x})/\sqrt{x}\, dx$

Answer: 2

Explanation:

Let u = √x, x = u², dx = 2u du.

When x=0, u=0; when x=π²/4, u=π/2.

$\int (\sin u)/u \cdot 2u\, du = 2\int_{0}^{\pi/2} \sin u\, du = 2[–\cos u]_0^{(\pi/2)} = 2(–\cos(\pi/2) + \cos 0) = 2(0 + 1) = 2.$

Q.24 [2 marks]

If $\vec{a}$ and $\vec{b}$ are two non-zero vectors such that $(\vec{a} + \vec{b}) \perp \vec{a}$ and $(2\vec{a} + \vec{b}) \perp \vec{b}$, then prove that $|\vec{b}| = \sqrt{2}\,|\vec{a}|$.

Answer: Proved: $|\vec{b}| = \sqrt{2}\,|\vec{a}|$

Explanation:

$(\vec{a} + \vec{b}) \perp \vec{a} \Rightarrow (\vec{a} + \vec{b})\cdot\vec{a} = 0 \Rightarrow |\vec{a}|^2 + \vec{a}\cdot\vec{b} = 0 \Rightarrow \vec{a}\cdot\vec{b} = –|\vec{a}|^2.$

$(2\vec{a} + \vec{b}) \perp \vec{b} \Rightarrow (2\vec{a} + \vec{b})\cdot\vec{b} = 0 \Rightarrow 2\vec{a}\cdot\vec{b} + |\vec{b}|^2 = 0 \Rightarrow 2(–|\vec{a}|^2) + |\vec{b}|^2 = 0 \Rightarrow |\vec{b}|^2 = 2|\vec{a}|^2 \Rightarrow |\vec{b}| = \sqrt{2}|\vec{a}|.$

Q.25 [2 marks]

In the given figure, ABCD is a parallelogram. If $\vec{AB} = 2\hat{i} – 4\hat{j} + 5\hat{k}$ and $\vec{DB} = 3\hat{i} – 6\hat{j} + 2\hat{k}$, then find $\vec{AD}$ and hence find the area of parallelogram ABCD.

Answer: $\overrightarrow{AD} = -\hat{i} + 2\hat{j} + 3\hat{k}$; Area = $11\sqrt{5}$ square units

Explanation:

In parallelogram ABCD, $\overrightarrow{DB} = B - D = \overrightarrow{AB} - \overrightarrow{AD}$ (since $\overrightarrow{AB} = \overrightarrow{DC}$).

Hence $\overrightarrow{AD} = \overrightarrow{AB} - \overrightarrow{DB} = (2-3)\hat{i} + (-4+6)\hat{j} + (5-2)\hat{k} = -\hat{i} + 2\hat{j} + 3\hat{k}$.

Area = $|\overrightarrow{AB} \times \overrightarrow{AD}|$, where $\overrightarrow{AB} \times \overrightarrow{AD} = |\hat{i}\ \hat{j}\ \hat{k}$;

2 –4 5;

–1 2 3| = $\hat{i}(-4\cdot3 - 5\cdot2) - \hat{j}(2\cdot3 - 5\cdot(-1)) + \hat{k}(2\cdot2 - (-4)(-1)) = \hat{i}(-12-10) - \hat{j}(6+5) + \hat{k}(4-4) = -22\hat{i} - 11\hat{j} + 0\hat{k}$.

$|\overrightarrow{AB} \times \overrightarrow{AD}| = \sqrt{(22^2 + 11^2)} = \sqrt{(484+121)} = \sqrt{605} = 11\sqrt{5}$.

Area of parallelogram ABCD = $11\sqrt{5}$ square units.

Q.26 [2 marks]

(a) A relation R on set A = {1, 2, 3, 4, 5} is defined as R = {(x, y) : $|x^2 - y^2| < 8$}. Check whether the relation R is reflexive, symmetric and transitive.

Answer: Reflexive and symmetric, but NOT transitive

Explanation:

Reflexive: $|x^2-x^2| = 0 < 8$ for all $x \in A$ — yes.

Symmetric: if $|x^2-y^2| < 8$, then $|y^2-x^2| < 8$ — yes.

Transitive: Take x=1,y=2,z=3: |1–4|=3<8 ✓, |4–9|=5<8 ✓, but |1–9|=8 which is NOT < 8.

So $(1,3) \notin R$ even though $(1,2) \in R$ and $(2,3) \in R$.

Not transitive.

OR

Q.26 [2 marks]

(b) A function f is defined from R → R as f(x) = ax + b, such that f(1) = 1 and f(2) = 3. Find function f(x). Hence, check whether function f(x) is one-one and onto or not.

Answer: f(x) = 2x – 1; one-one and onto

Explanation:

f(1) = a + b = 1, f(2) = 2a + b = 3.

Subtracting: a = 2, b = –1.

So f(x) = 2x – 1.

One-one: $f(x_1) = f(x_2) \Rightarrow 2x_1 - 1 = 2x_2 - 1 \Rightarrow x_1 = x_2$.

Yes.

Onto: for any $y \in R$, $x = (y+1)/2 \in R$, $f(x) = y$.

Yes.

f is bijective.

Section C

Short Answer Questions (3 marks each)

Q.27 [3 marks]

(a) If $\sqrt{(1 - x^2)} + \sqrt{(1 - y^2)} = a(x - y)$, prove that $dy/dx = \sqrt{(1 - y^2)}/\sqrt{(1 - x^2)}$.

Answer: Proved (see explanation)

Explanation:

Substitute $x = \sin \alpha$, $y = \sin \beta$.

Then $\sqrt{(1-\sin^2\alpha)} + \sqrt{(1-\sin^2\beta)} = a(\sin \alpha - \sin \beta) \Rightarrow \cos \alpha + \cos \beta = a(\sin \alpha - \sin \beta)$.

Using sum-to-product: $2\cos((\alpha+\beta)/2)\cos((\alpha-\beta)/2) = 2a \cdot \cos((\alpha+\beta)/2)\sin((\alpha-\beta)/2)$.

Dividing: $\cot((\alpha-\beta)/2) = a \Rightarrow \alpha - \beta = 2\cot^{-1}(a) = \text{constant}$.

Differentiating: $d\alpha/dx - d\beta/dy \cdot dy/dx = 0$ (where $d\alpha/dx = 1/\sqrt{(1-x^2)}$, $d\beta/dy = 1/\sqrt{(1-y^2)}$).

So $dy/dx = (d\alpha/dx)/(d\beta/dy) = \sqrt{(1-y^2)}/\sqrt{(1-x^2)}$.

QED.

OR

Q.27 [3 marks]

(b) If y = (tan x)^x, then find dy/dx.

Answer: dy/dx = (tan x)^x [ln(tan x) + x · sec x · csc x]

Explanation:

Take logarithm: ln y = x ln(tan x).

Differentiate: $(1/y)(dy/dx) = \ln(\tan x) + x \cdot (1/\tan x) \cdot \sec^2 x = \ln(\tan x) + x \cdot \sec x \cdot \csc x$.

Therefore dy/dx = (tan x)^x [ln(tan x) + x·(sec²x/tan x)] = (tan x)^x [ln(tan x) + x·sec x·csc x].

Q.28 [3 marks]

(a) Find: $\int x^2/[(x^2 + 4)(x^2 + 9)]\, dx$

Answer: $-(2/5)\tan^{-1}(x/2) + (3/5)\tan^{-1}(x/3) + C$

Explanation:

Write $x^2/[(x^2+4)(x^2+9)] = A/(x^2+4) + B/(x^2+9)$, so $x^2 = A(x^2+9) + B(x^2+4)$.

Putting $x^2 = -4$ gives $A = -4/5$; putting $x^2 = -9$ gives $B = 9/5$.

Integral = $-(4/5)\cdot(1/2)\tan^{-1}(x/2) + (9/5)\cdot(1/3)\tan^{-1}(x/3) + C = -(2/5)\tan^{-1}(x/2) + (3/5)\tan^{-1}(x/3) + C$.

OR

Q.28 [3 marks]

(b) Evaluate: $\int_1^3 (|x - 1| + |x - 2| + |x - 3|)\, dx$

Answer: 5

Explanation:

Split at x=1,2,3.

On [1,2]: |x–1|=x–1, |x–2|=2–x, |x–3|=3–x.

Sum = (x–1)+(2–x)+(3–x) = 4–x.

On [2,3]: |x–1|=x–1, |x–2|=x–2, |x–3|=3–x.

Sum = (x–1)+(x–2)+(3–x) = x.

Integral = $\int_1^2 (4-x)dx + \int_2^3 x\, dx = [4x - x^2/2]_1^2 + [x^2/2]_2^3 = ((8-2)-(4-1/2)) + (9/2-4/2) = (6-3.5) + 2.5 = 2.5 + 2.5 = 5$.

Q.29 [3 marks]

Find the particular solution of the differential equation given by $x^2(dy/dx) - xy = x^2 \cos^2(y/(2x))$, given that when x = 1, y = π/2.

Answer: $2\tan(y/(2x)) = \ln|x| + 2$ (equivalently $\tan(y/(2x)) = (\ln|x|)/2 + 1$)

Explanation:

Homogeneous DE.

Put $v = y/x$, so $y = vx$ and $dy/dx = v + x(dv/dx)$.

Substituting: $x^2(v + x\,dv/dx) - x(vx) = x^2\cos^2(v/2) \Rightarrow x^3(dv/dx) + vx^2 - vx^2 = x^2\cos^2(v/2) \Rightarrow x(dv/dx) = \cos^2(v/2)$.

Separate: $\sec^2(v/2)\,dv = dx/x$.

Integrate: $2\tan(v/2) = \ln|x| + C$.

Apply $x = 1$, $y = \pi/2$ (so $v = \pi/2$): $2\tan(\pi/4) = \ln 1 + C \Rightarrow 2 = C$.

Hence $2\tan(y/(2x)) = \ln|x| + 2$.

Q.30 [3 marks]

Solve the following linear programming problem graphically: Maximise $z = 500x + 300y$, subject to constraints $x + 2y \le 12$, $2x + y \le 12$, $4x + 5y \ge 20$, $x \ge 0$, $y \ge 0$.

Answer: Maximum $z = 3200$ at $(4, 4)$

Explanation:

Corner points of the feasible region (satisfying $x+2y\le12$, $2x+y\le12$, $4x+5y\ge20$, $x,y\ge0$): $x+2y=12$ and $2x+y=12$ intersect where $x=y=4$, giving $(4,4)$;

the region also has vertices $(0,6)$, $(0,4)$, $(5,0)$ and $(6,0)$.

Evaluate $z = 500x + 300y$: $z(0,4)=1200$, $z(0,6)=1800$, $z(5,0)=2500$, $z(6,0)=3000$, $z(4,4)=2000+1200=3200$.

Maximum $z = 3200$ at $(4, 4)$.

Q.31 [3 marks]

E and F are two independent events such that $P(\bar{E}) = 0{\cdot}6$ and $P(E \cup F) = 0{\cdot}6$. Find $P(F)$ and $P(\bar{E} \cup F)$.

Answer: $P(F) = 1/3$ and $P(\bar{E} \cup F) = 13/15$

Explanation:

$P(E) = 1 - 0{\cdot}6 = 0{\cdot}4$.

For independent events $P(E \cup F) = P(E) + P(F) - P(E)P(F)$, so $0{\cdot}6 = 0{\cdot}4 + 0{\cdot}6\cdot P(F)$, giving $P(F) = 1/3$.

Then $P(E \cap F) = 0{\cdot}4 \times 1/3 = 2/15$, and $P(\bar{E} \cup F) = 1 - P(E \cap F) = 1 - 2/15 = 13/15$.

Q.32 [3 marks]

(a) If A = [[1, –2, 0], [2, –1, –1], [0, –2, 1]], find A^{-1} and use it to solve the system of equations: x – 2y = 10, 2x – y – z = 8, –2y + z = 7.

Answer: x = 0, y = –5, z = –3

Explanation:

|A| = 1[(–1)(1) – (–1)(–2)] + 2[(2)(1) – 0] + 0 = –3 + 4 = 1 ≠ 0, so A^{-1} exists.

Cofactors: C_{11} = –3, C_{12} = –2, C_{13} = –4;

C_{21} = 2, C_{22} = 1, C_{23} = 2;

C_{31} = 2, C_{32} = 1, C_{33} = 3.

adj A = [[–3, 2, 2], [–2, 1, 1], [–4, 2, 3]], so A^{-1} = adj A / |A| = [[–3, 2, 2], [–2, 1, 1], [–4, 2, 3]].

The system is AX = B with B = [10, 8, 7]^T, so X = A^{-1}B:

x = –30 + 16 + 14 = 0; y = –20 + 8 + 7 = –5; z = –40 + 16 + 21 = –3.

Check: x – 2y = 10 ✓, 2x – y – z = 5 + 3 = 8 ✓, –2y + z = 10 – 3 = 7 ✓.

OR

Q.32 [3 marks]

(b) If A = [[–1, a, 2], [1, 2, x], [3, 1, 1]] and A^{-1} = [[1, –1, 1], [–8, 7, –5], [b, y, 3]], find the value of (a + x) – (b + y).

Answer: 3

Explanation:

Since A·A^{-1} = I, we have |A|·|A^{-1}| = 1.

Also A·A^{-1} row-by-row gives equations.

Row 1 of A times Col 1 of A^{-1} = 1: –1(1)+a(–8)+2b=1 => –1–8a+2b=1.

Row 1 of A times Col 3 of A^{-1} = 0: –1(1)+a(–5)+2(3)=0 => –1–5a+6=0 => 5–5a=0 => a=1.

Row 3 of A times Col 3 of A^{-1} = 1: 3(1)+1(–5)+1(3)=3–5+3=1✓.

Row 2 of A times Col 1 of A^{-1} = 0: 1(1)+2(–8)+x(b)=0.

Row 2 of A times Col 3 of A^{-1} = 0: 1(1)+2(–5)+x(3)=0 => 1–10+3x=0 => 3x=9 => x=3.

Back to a=1: Row 1 times Col 1: –1–8+2b=1 => 2b=10 => b=5.

Row 3 times Col 2: 3(–1)+1(7)+1(y)=0 => –3+7+y=0 => y=–4.

(a+x)–(b+y)=(1+3)–(5+(–4))=4–1=3.

Verify Row2·Col1: 1–16+3b=0 => 3b=15 => b=5✓.

So (a+x)–(b+y) = (1+3)–(5–4) = 4–1 = 3.

Section D

Long Answer Questions (5 marks each)

Q.33 [5 marks]

(a) Evaluate: $\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16 \sin 2x}\, dx$

Answer: $\frac{1}{20} \ln 3$

Explanation:

Let $t = \sin x - \cos x$, so $dt = (\sin x + \cos x)\, dx$ and $\sin 2x = 1 - t^2$.

Limits: $x = 0$ gives $t = -1$; $x = \pi/4$ gives $t = 0$.

Integral $= \int_{-1}^{0} \frac{dt}{9 + 16(1 - t^2)} = \int_{-1}^{0} \frac{dt}{25 - 16t^2}$.

Using $\int \frac{dt}{a^2 - b^2t^2} = \frac{1}{2ab} \ln\left|\frac{a + bt}{a - bt}\right|$ with $a = 5$, $b = 4$: $= \frac{1}{40}\left[\ln\left|\frac{5+4t}{5-4t}\right|\right]$ from -1 to $0 = \frac{1}{40}(0 - \ln(1/9)) = \frac{1}{40} \ln 9 = \frac{1}{20} \ln 3$.

OR

Q.33 [5 marks]

(b) Evaluate: $\int_0^{\pi/2} \sin 2x \cdot \tan^{-1}(\sin x)\, dx$

Answer: $\frac{\pi}{2} - 1$

Explanation:

Let $I = \int_0^{\pi/2} \sin 2x \cdot \tan^{-1}(\sin x)\, dx = 2\int_0^{\pi/2} \sin x \cos x \cdot \tan^{-1}(\sin x)\, dx$.

Let $t = \sin x$, $dt = \cos x\, dx$.

When $x=0$: $t=0$; $x=\pi/2$: $t=1$.

$I = 2\int_0^1 t \cdot \tan^{-1}(t)\, dt$.

Use IBP: $u=\tan^{-1}t$, $dv=2t\, dt$ => $du=dt/(1+t^2)$, $v=t^2$.

$I = [t^2 \tan^{-1}t]_0^1 - \int_0^1 t^2/(1+t^2)\, dt = (\pi/4) - \int_0^1 (1 - 1/(1+t^2))dt = \pi/4 - [t - \tan^{-1}t]_0^1$

$= \pi/4 - (1 - \pi/4) = \pi/4 - 1 + \pi/4 = \pi/2 - 1$.

Q.34 [5 marks]

Using integration, find the area of the ellipse $x^2/16 + y^2/4 = 1$, included between the lines $x = -2$ and $x = 2$.

Answer: $4\sqrt{3} + 8\pi/3$ square units

Explanation:

For the ellipse $x^2/16 + y^2/4 = 1$: $y = 2\sqrt{(1 - x^2/16)} = (1/2)\sqrt{(16 - x^2)}$.

The area between $x = -2$ and $x = 2$ (above and below the x-axis) is Area $= 2\int_{-2}^{2} (1/2)\sqrt{(16 - x^2)}\, dx = \int_{-2}^{2} \sqrt{(16 - x^2)}\, dx$.

Using $\int\sqrt{(a^2-x^2)}\, dx = (x/2)\sqrt{(a^2-x^2)} + (a^2/2)\sin^{-1}(x/a)$ with $a = 4$: $[(x/2)\sqrt{(16-x^2)} + 8\sin^{-1}(x/4)]$ from -2 to 2.

At $x = 2$: $(1)\sqrt{12} + 8\sin^{-1}(1/2) = 2\sqrt{3} + 8(\pi/6) = 2\sqrt{3} + 4\pi/3$.

At $x = -2$: $-2\sqrt{3} - 4\pi/3$.

Difference $= 4\sqrt{3} + 8\pi/3$.

Area $= 4\sqrt{3} + 8\pi/3$ square units.

Q.35 [5 marks]

The image of point P(x, y, z) with respect to line $x/1 = (y-1)/2 = (z-2)/3$ is $P'(1, 0, 7)$. Find the coordinates of point P.

Answer: P = (1, 6, 3)

Explanation:

The midpoint M of PP' lies on the line, and PP' is perpendicular to the line's direction (1, 2, 3).

A general point of the line is M = (t, 1+2t, 2+3t).

Since M is the midpoint of P(x,y,z) and P′(1,0,7): x = 2t–1, y = 2+4t, z = 6t–3.

Then $PP' = P' - P = (2-2t, -2-4t, 10-6t)$.

Perpendicularity: $(2-2t)(1) + (-2-4t)(2) + (10-6t)(3) = 0 \Rightarrow 2-2t-4-8t+30-18t = 0 \Rightarrow 28 - 28t = 0 \Rightarrow t = 1$.

Hence $P = (2\cdot1-1, 2+4\cdot1, 6\cdot1-3) = (1, 6, 3)$.

Check: midpoint (1, 3, 5) lies on the line $(1/1 = (3-1)/2 = (5-2)/3 = 1)$ and $PP' = (0, -6, 4)$ is perpendicular to (1, 2, 3) since $0-12+12 = 0$.

Q.36(a) [5 marks]

Case Study 1: A camera is installed on a pole at height 5 m. It detects a car travelling away from the pole at 20 m/s. At any point x m away from the base of the pole, the angle of elevation of the speed camera from the car C is θ.

(i) Express θ in terms of the height of the camera installed on the pole and x.

Answer: $\theta = \tan^{-1}(5/x)$

Explanation:

The camera is at height 5 m.

At horizontal distance x from the base, the angle of elevation from the car to the camera: $\tan\theta = 5/x$, so $\theta = \tan^{-1}(5/x)$.

Q.36(b) [5 marks]

Case Study 1 (continued): Find $d\theta/dx$.

Answer: $d\theta/dx = -5/(x^2 + 25)$

Explanation:

$\theta = \tan^{-1}(5/x)$.

$d\theta/dx = 1/(1+(5/x)^2) \cdot d(5/x)/dx = 1/(1+25/x^2) \cdot (-5/x^2) = (x^2/(x^2+25)) \cdot (-5/x^2) = -5/(x^2+25)$.

Q.36(c) [5 marks]

Case Study 1 (iii)(a): Find the rate of change of angle of elevation with respect to time at an instant when the car is 50 m away from the pole.

Answer: $-4/101$ rad/s (magnitude $4/101$ rad/s)

Explanation:

$d\theta/dt = (d\theta/dx)\cdot(dx/dt)$.

At $x = 50$: $d\theta/dx = -5/(50^2 + 25) = -5/2525 = -1/505$.

With $dx/dt = 20$ m/s: $d\theta/dt = (-1/505)(20) = -20/505 = -4/101$ rad/s.

The angle of elevation decreases at $4/101$ rad/s.

OR

Q.36(c) [5 marks]

Case Study 1 (iii)(b): If the rate of change of angle of elevation with respect to time of another car at a distance of 50 m from the base of the pole is $3/101$ rad/s, then find the speed of the car.

Answer: 15 m/s

Explanation:

$d\theta/dt = (d\theta/dx)\cdot(dx/dt)$.

At $x=50$: $d\theta/dx = -5/(2500+25) = -5/2525 = -1/505$.

Given $|d\theta/dt| = 3/101$.

Speed $= |d\theta/dt|/|d\theta/dx| = (3/101)\cdot 505 = (3\times 505)/101 = 3\times 5 = 15$ m/s.

Q.37(a) [4 marks]

Case Study 2: An airplane observes severe, moderate or light turbulence with equal probabilities (1/3 each). The probability of reaching the destination late due to severe, moderate and light turbulence is 55%, 37% and 17% respectively. (i) Find the probability that an airplane reached its destination late.

Answer: 109/300

Explanation:

$P(\text{late}) = P(S)\cdot P(L|S) + P(M)\cdot P(L|M) + P(L)\cdot P(L|Lt) = (1/3)(0.55) + (1/3)(0.37) + (1/3)(0.17) = (1/3)(0.55+0.37+0.17) = (1/3)(1.09) = 1.09/3 = 109/300$.

Q.37(b) [4 marks]

Case Study 2 (continued): If the airplane reached its destination late, find the probability that it was due to moderate turbulence.

Answer: 37/109

Explanation:

By Bayes' theorem: $P(\text{moderate} \mid \text{late}) = P(\text{moderate})\cdot P(\text{late}|\text{moderate})/P(\text{late}) = (1/3)(0.37)/(109/300) = (0.37/3)\cdot(300/109) = (37/300)\cdot(300/109) = 37/109$.

Q.38(a) [4 marks]

Case Study 3: The domain of sine function is R and sin: R → R is neither one-one nor onto. Let sine function be defined from set A to [–1, 1] such that the inverse $\sin^{-1} x$ is defined from [–1, 1] to A. (i) If A is the interval other than

the principal value branch, give an example of one such interval.

Answer: $[\pi/2, 3\pi/2]$

Explanation:

The principal value branch of $\sin^{-1}$ is $[-\pi/2, \pi/2]$.

Any other interval where sine is bijective to $[-1,1]$ is a valid answer, e.g. $[\pi/2, 3\pi/2]$.

Q.38(b) [4 marks]

Case Study 3 (continued): If $\sin^{-1}(x)$ is defined from $[-1, 1]$ to its principal value branch, find the value of $\sin^{-1}(-1/2) - \sin^{-1}(1)$.

Answer: $-\pi/6 - \pi/2 = -2\pi/3$

Explanation:

$\sin^{-1}(-1/2) = -\pi/6$ (principal value).

$\sin^{-1}(1) = \pi/2$.

So $\sin^{-1}(-1/2) - \sin^{-1}(1) = -\pi/6 - \pi/2 = -\pi/6 - 3\pi/6 = -4\pi/6 = -2\pi/3$.

Q.38(c) [4 marks]

Case Study 3 (iii)(a): Draw the graph of $\sin^{-1} x$ from $[-1, 1]$ to its principal value branch $[-\pi/2, \pi/2]$.

Answer: Graph: $\sin^{-1} x$ is a strictly increasing curve from $(-1, -\pi/2)$ to $(1, \pi/2)$, symmetric about the origin, passing through $(0, 0)$, $(1/2, \pi/6)$, $(-1/2, -\pi/6)$, $(\sqrt{2}/2, \pi/4)$, $(1, \pi/2)$, $(-1, -\pi/2)$.

Explanation:

Key points on graph of $y = \sin^{-1} x$: $(-1, -\pi/2)$, $(-\sqrt{3}/2, -\pi/3)$, $(-1/2, -\pi/6)$, $(0, 0)$, $(1/2, \pi/6)$, $(\sqrt{3}/2, \pi/3)$, $(1, \pi/2)$.

The curve is strictly increasing, concave for $x < 0$ and convex for $x > 0$, with vertical tangents at $x = \pm 1$.

OR

Q.38(c) [4 marks]

Case Study 3 (iii)(b): Find the domain and range of $f(x) = 2\sin^{-1}(1 - x)$.

Answer: Domain: [0, 2]; Range: [–π, π]

Explanation:

For $\sin^{-1}(1–x)$ to be defined: $–1 \le 1–x \le 1 \Rightarrow 0 \le x \le 2$.

Domain = [0, 2].

Range: when x=0, $f=2\sin^{-1}(1)=2(\pi/2)=\pi$;

when x=2, $f=2\sin^{-1}(–1)=2(–\pi/2)=–\pi$;

when x=1, $f=2\sin^{-1}(0)=0$.

Since $\sin^{-1}$ is increasing, as x goes from 0 to 2, (1–x) goes from 1 to –1, so $\sin^{-1}(1–x)$ goes from π/2 to –π/2.

Range of $f = 2\cdot[–\pi/2, \pi/2] = [–\pi, \pi]$.