CBSE Class 12 Mathematics 2024 — Set 2 — Answer Key

English Medium

Time Allowed: 3 Hours | Maximum Marks: 80

General Instructions:

1. This question paper contains 38 questions. All questions are compulsory.
2. Section A has 20 MCQs carrying 1 mark each.
3. Section B has 6 questions carrying 2 marks each.
4. Section C has 6 questions carrying 3 marks each.
5. Section D has 4 questions carrying 5 marks each.
6. Section E has 3 case-study based questions carrying 4 marks each.
7. Draw neat figures wherever required.

Section A

Multiple Choice Questions (1 mark each)

Q.1 [1 mark]

If the sum of all the elements of a 3 × 3 scalar matrix is 9, then the product of all its elements is :

(a) 0 <<

(b) 9

(c) 27

(d) 729

Explanation:

A 3×3 scalar matrix has the form kI, so its diagonal elements are all equal to k and all off-diagonal elements are 0.

Sum of all 9 elements = 3k = 9 => k = 3.

Product of all 9 elements = 3 × 3 × 3 × 0 × 0 × 0 × 0 × 0 × 0 = 0.

Q.2 [1 mark]

Let f : R+ → [– 5, ∞) be defined as f(x) = $9x^2 + 6x – 5$, where R+ is the set of all non-negative real numbers. Then, f is :

(a) one-one

(b) onto

(c) bijective <<

(d) neither one-one nor onto

Explanation:

For x ≥ 0: f'(x) = 18x + 6 > 0, so f is strictly increasing on R+, hence one-one.

Range: f(0) = -5 (minimum), f → ∞ as x → ∞, so range = [-5, ∞) = codomain, hence onto.

Therefore f is bijective.

Q.3 [1 mark]

If |–a b c; a –b c; a b –c| = kabc, then the value of k is :

(a) 0

(b) 1

(c) 2

(d) 4 <<

Explanation:

Expanding the determinant: $-a[(-b)(-c) - c\cdot b] - b[a(-c) - c\cdot a] + c[a\cdot b - (-b)\cdot a] = -a[bc - bc] - b[-ac - ac] + c[ab + ab] = 0 + 2abc + 2abc = 4abc$.

So $kabc = 4abc \Rightarrow k = 4$.

Q.4 [1 mark]

The number of points of discontinuity of $f(x) = \{ x + 3, \text{ if } x \le -3; -2x, \text{ if } -3 < x < 3; 6x + 2, \text{ if } x \ge 3 \}$ is :

(a) 0

(b) 1

(c) 2 <<

(d) infinite

Explanation:

At $x = -3$: left limit $= -3 + 3 = 0$, right limit $= -2(-3) = 6$, $f(-3) = 0$.

Left limit $\neq$ right limit $\Rightarrow$ discontinuous.

At $x = 3$: left limit $= -2(3) = -6$, right limit $= 6(3) + 2 = 20$, $f(3) = 20$.

Left limit $\neq$ right limit $\Rightarrow$ discontinuous.

Total points of discontinuity = 2.

Q.5 [1 mark]

The function $f(x) = x^3 - 3x^2 + 12x - 18$ is :

(a) strictly decreasing on R

(b) strictly increasing on R <<

(c) neither strictly increasing nor strictly decreasing on R

(d) strictly decreasing on $(-\infty, 0)$

Explanation:

$f'(x) = 3x^2 - 6x + 12 = 3(x^2 - 2x + 4) = 3[(x - 1)^2 + 3] > 0$ for all $x \in R$ (since $(x-1)^2 \ge 0$, so $(x-1)^2 + 3 \ge 3 > 0$).

Therefore f is strictly increasing on R.

Q.6 [1 mark]

$\int_0^{\pi/2} (\sin x - \cos x)/(1 + \sin x \cos x)\, dx$ is equal to :

(a) π

(b) Zero (0) <<

(c) $\int_0^{\pi/2} 2\sin x/(1 + \sin x \cos x)\, dx$

(d) $\pi^2/4$

Explanation:

Let $I = \int_0^{\pi/2} (\sin x - \cos x)/(1 + \sin x \cos x)\, dx$.

Using the property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ with $a = \pi/2$, replace x by $\pi/2 - x$: integrand becomes $(\cos x - \sin x)/(1 + \sin x \cos x) = -(\sin x - \cos x)/(1 + \sin x \cos x)$.

So $I = -I \Rightarrow 2I = 0 \Rightarrow I = 0$.

Q.7 [1 mark]

The differential equation $dy/dx = F(x, y)$ will not be a homogeneous differential equation, if F(x, y) is :

(a) $\cos x - \sin(y/x)$ <<

(b) y/x

(c) $(x^2 + y^2)/(xy)$

(d) $\cos^2(x/y)$

Explanation:

A homogeneous function satisfies $F(tx, ty) = t^n F(x, y)$ for some degree n (here n = 0).

(B) y/x: $F(tx,ty) = ty/tx = y/x$ ✓.

(C) $(x^2+y^2)/(xy)$: $F(tx,ty) = (t^2x^2+t^2y^2)/(t^2xy) = (x^2+y^2)/(xy)$ ✓.

(D) $\cos^2(x/y)$: $F(tx,ty) = \cos^2(tx/ty) = \cos^2(x/y)$ ✓.

(A) $\cos x - \sin(y/x)$: $F(tx,ty) = \cos(tx) - \sin(y/x) \neq F(x,y)$ in general because $\cos(tx) \neq \cos x$.

Not homogeneous.

Q.8 [1 mark]

For any two vectors a and b, which of the following statements is always true

?

(a) $a \cdot b \geq |a||b|$

(b) $a \cdot b = |a||b|$

(c) $a \cdot b \leq |a||b|$ <<

(d) $a \cdot b < |a||b|$

Explanation:

$a \cdot b = |a||b| \cos\theta$ and $\cos\theta \leq 1$, so $a \cdot b \leq |a||b|$ for all vectors a and b (equality when they are in the same direction).

The strict inequality $a \cdot b < |a||b|$ fails for parallel vectors in the same direction, and $a \cdot b \geq |a||b|$ or $a \cdot b = |a||b|$ hold only in special cases.

Hence $a \cdot b \leq |a||b|$ (C) is always true.

Q.9 [1 mark]

The coordinates of the foot of the perpendicular drawn from the point (0, 1, 2) on the x-axis are given by :

(a) (1, 0, 0)

(b) (2, 0, 0)

(c) ($\sqrt{5}$, 0, 0)

(d) (0, 0, 0) <<

Explanation:

Any point on the x-axis has the form (t, 0, 0).

The vector from (t,0,0) to (0,1,2) is (–t, 1, 2).

For this to be perpendicular to the x-axis direction (1,0,0): (–t)(1) + 1(0) + 2(0) = 0 => –t = 0 => t = 0.

Foot = (0, 0, 0).

Q.10 [1 mark]

The common region determined by all the constraints of a linear programming problem is called :

(a) an unbounded region

(b) an optimal region

(c) a bounded region

(d) a feasible region <<

Explanation:

By definition, the set of all points satisfying all constraints of a linear programming problem is called the feasible region (or feasible set).

It may be bounded or unbounded.

Q.11 [1 mark]

Let E be an event of a sample space S of an experiment, then P(S|E) =

(a) $P(S \cap E)$

(b) P(E)

(c) 1 <<

(d) 0

Explanation:

$P(S|E) = P(S \cap E)/P(E)$.

Since S is the entire sample space, $S \cap E = E$.

So $P(S|E) = P(E)/P(E) = 1$ (provided $P(E) > 0$).

Q.12 [1 mark]

If A = [aij] be a 3 × 3 matrix, where aij = i – 3j, then which of the following is false ?

(a) $a_{11} < 0$

(b) $a_{12} + a_{21} = -6$

(c) $a_{13} > a_{31}$ <<

(d) $a_{31} = 0$

Explanation:

aij = i – 3j.

Compute: $a_{11} = 1–3 = –2 < 0$ (TRUE).

$a_{12} = 1–6 = –5$; $a_{21} = 2–3 = –1$; sum = –6 (TRUE).

$a_{13} = 1–9 = –8$; $a_{31} = 3–3 = 0$; so $a_{13} = –8 < 0 = a_{31}$, hence $a_{13} > a_{31}$ is FALSE.

$a_{31} = 3–3 = 0$ (TRUE).

Answer: (C) $a_{13} > a_{31}$ is false.

Q.13 [1 mark]

The derivative of $\tan^{-1}(x^2)$ w.r.t. x is :

(a) $x/(1 + x^4)$

(b) $2x/(1 + x^4)$ <<

(c) $-2x/(1 + x^4)$

(d) $1/(1 + x^4)$

Explanation:

$d/dx[\tan^{-1}(x^2)] = 1/(1 + (x^2)^2) \cdot d/dx(x^2) = 1/(1 + x^4) \cdot 2x = 2x/(1 + x^4)$.

Q.14 [1 mark]

The degree of the differential equation $(y'')^2 + (y')^3 = x \sin(y')$ is :

(a) 1

(b) 2

(c) 3

(d) not defined <<

Explanation:

The degree of a differential equation is defined only when it can be expressed as a polynomial in its derivatives.

Here, y' appears inside sin(y'), which is a transcendental function.

Since the equation cannot be expressed as a polynomial in derivatives, the degree is not defined.

Q.15 [1 mark]

The unit vector perpendicular to both vectors $\hat{i} + \hat{k}$ and $\hat{i} - \hat{k}$ is :

(a) $2\hat{j}$

(b) $\hat{j}$ <<

(c) $(\hat{i} - \hat{k})/\sqrt{2}$

(d) $(\hat{i} + \hat{k})/\sqrt{2}$

Explanation:

$(\hat{i} + \hat{k}) \times (\hat{i} - \hat{k}) = \hat{i}\times\hat{i} - \hat{i}\times\hat{k} + \hat{k}\times\hat{i} - \hat{k}\times\hat{k} = 0 - (-\hat{j}) + \hat{j} - 0 = 2\hat{j}$.

Its magnitude is 2, so the unit vector perpendicular to both is $2\hat{j}/2 = \hat{j}$.

Q.16 [1 mark]

Direction ratios of a vector parallel to line $(x - 1)/2 = -y = (2z + 1)/6$ are :

(a) 2, –1, 6

(b) 2, 1, 6

(c) 2, 1, 3

(d) 2, –1, 3 <<

Explanation:

$(x-1)/2 = -y = (2z+1)/6$ can be written $(x-1)/2 = y/(-1) = (z+1/2)/3$, because $(2z+1)/6 = (z+1/2)/3$.

Direction ratios are 2, –1, 3.

Option D.

Q.17 [1 mark]

If $F(x) = [[\cos x, -\sin x, 0], [\sin x, \cos x, 0], [0, 0, 1]]$ and $[F(x)]^2 = F(kx)$, then the value of k is :

(a) 1

(b) 2 <<

(c) 0

(d) –2

Explanation:

F(x) is the rotation matrix by angle x.

$[F(x)]^2 = F(x)\cdot F(x)$.

Since rotation matrices compose by adding angles: $F(x)\cdot F(x) = F(x + x) = F(2x)$.

Given $[F(x)]^2 = F(kx)$, we get $F(2x) = F(kx)$, so k = 2.

Q.18 [1 mark]

If a line makes an angle of 30° with the positive direction of x-axis, 120° with the positive direction of y-axis, then the angle which it makes with the

positive direction of z-axis is :

(a) 90° <<

(b) 120°

(c) 60°

(d) 0°

Explanation:

Using direction cosines: $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

$\alpha = 30°$, $\beta = 120°$.

$\cos 30° = \sqrt{3}/2$, $\cos 120° = -1/2$.

$(\sqrt{3}/2)^2 + (-1/2)^2 + \cos^2\gamma = 1 \Rightarrow 3/4 + 1/4 + \cos^2\gamma = 1 \Rightarrow 1 + \cos^2\gamma = 1 \Rightarrow \cos^2\gamma = 0 \Rightarrow \gamma = 90°$.

Q.19 [1 mark]

Assertion (A) : For any symmetric matrix A, B'AB is a skew-symmetric matrix.

Reason (R) : A square matrix P is skew-symmetric if P' = –P.

(a) Both A and R true; R is correct explanation of A

(b) Both A and R true; R is not correct explanation of A

(c) A is true, R is false

(d) A is false, R is true <<

Explanation:

Check Assertion (A): (B'AB)' = B'A'(B')' = B'AB (since A is symmetric, A' = A).

So (B'AB)' = B'AB, meaning B'AB is symmetric, not skew-symmetric.

So Assertion (A) is FALSE.

Reason (R): P' = –P defines skew-symmetric matrix.

This is TRUE by definition.

Answer: (D) A false, R true.

Q.20 [1 mark]

Assertion (A) : For two non-zero vectors a and b, $a \cdot b = b \cdot a$.

Reason (R) : For two non-zero vectors a and b, $a \times b = b \times a$.

(a) Both A and R true; R is correct explanation of A

(b) Both A and R true; R is not correct explanation of A

(c) A is true, R is false <<

(d) A is false, R is true

Explanation:

Assertion (A): Dot product is commutative: $a \cdot b = |a||b|\cos\theta = b \cdot a$.

TRUE.

Reason (R): Cross product satisfies $a \times b = -(b \times a)$, not $b \times a$.

So $a \times b = b \times a$ is FALSE.

Answer: (C) A true, R false.

Section B

Very Short Answer Questions (2 marks each)

Q.21 [2 marks]

Find the value of $\tan^{-1}(-1/\sqrt{3}) + \cot^{-1}(1/\sqrt{3}) + \tan^{-1}[\sin(-\pi/2)]$.

Answer: $-\pi/12$

Explanation:

$\tan^{-1}(-1/\sqrt{3}) = -\pi/6$ (since $\tan(\pi/6) = 1/\sqrt{3}$).

$\cot^{-1}(1/\sqrt{3}) = \pi/3$ (since $\cot(\pi/3) = 1/\sqrt{3}$, and $\cot^{-1}$ range is $(0,\pi)$).

$\sin(-\pi/2) = -1$, so $\tan^{-1}(-1) = -\pi/4$.

Sum $= -\pi/6 + \pi/3 - \pi/4 = -2\pi/12 + 4\pi/12 - 3\pi/12 = -\pi/12$.

OR

Q.21 [2 marks]

Find the domain of the function $f(x) = \sin^{-1}(x^2 - 4)$. Also, find its range.

Answer: Domain: $[-\sqrt{5}, -\sqrt{3}] \cup [\sqrt{3}, \sqrt{5}]$; Range: $[-\pi/2, \pi/2]$

Explanation:

$\sin^{-1}$ is defined for arguments in $[-1, 1]$.

So $-1 \le x^2 - 4 \le 1 \Rightarrow 3 \le x^2 \le 5 \Rightarrow \sqrt{3} \le |x| \le \sqrt{5}$.

Domain: $[-\sqrt{5}, -\sqrt{3}] \cup [\sqrt{3}, \sqrt{5}]$.

Range of $\sin^{-1}$: $[-\pi/2, \pi/2]$.

Since x^2-4 ranges over $[-1,1]$ as x ranges over domain, the range is $[-\pi/2, \pi/2]$.

Q.22 [2 marks]

If $f(x) = |\tan 2x|$, then find the value of $f'(x)$ at $x = \pi/3$.

Answer: –8

Explanation:

At $x = \pi/3$, $2x = 2\pi/3$ and $\tan(2\pi/3) = -\sqrt{3} < 0$, so near $x = \pi/3$, $f(x) = |\tan 2x| = -\tan 2x$.

$f'(x) = -2\sec^2 2x$.

$\cos(2\pi/3) = -1/2 \Rightarrow \sec^2(2\pi/3) = 4$.

$f'(\pi/3) = -2 \times 4 = -8$.

OR

Q.22 [2 marks]

If $y = \text{cosec}(\cot^{-1} x)$, then prove that $\sqrt{(1 + x^2)} \cdot dy/dx - x = 0$.

Answer: Proof

Explanation:

Let $\theta = \cot^{-1} x$, so $\cot \theta = x$.

Then $\text{cosec}\ \theta = \sqrt{(1 + \cot^2\theta)} = \sqrt{(1 + x^2)}$.

So $y = \sqrt{(1 + x^2)}$.

$dy/dx = x/\sqrt{(1 + x^2)}$.

Then $\sqrt{(1+x^2)} \cdot dy/dx = \sqrt{(1+x^2)} \cdot x/\sqrt{(1+x^2)} = x$.

So $\sqrt{(1+x^2)} \cdot dy/dx - x = x - x = 0$.

Proved.

Q.23 [2 marks]

If M and m denote the local maximum and local minimum values of the function $f(x) = x + 1/x$ $(x \neq 0)$ respectively, find the value of $(M - m)$.

Answer: –4

Explanation:

f'(x) = 1 – 1/x².

f'(x) = 0 => x² = 1 => x = ±1.

f''(x) = 2/x³.

At x = –1: f''(–1) = –2 < 0, local maximum.

M = f(–1) = –1 + (–1) = –2.

At x = 1: f''(1) = 2 > 0, local minimum.

m = f(1) = 1 + 1 = 2.

M – m = –2 – 2 = –4.

Q.24 [2 marks]

Find : ∫ (e^{4x} – 1)/(e^{4x} + 1) dx

Answer: (1/2)ln(e^{2x} + e^{–2x}) + C

Explanation:

Divide numerator and denominator by e^{2x}: (e^{4x}–1)/(e^{4x}+1) = (e^{2x} – e^{–2x})/(e^{2x} + e^{–2x}).

Let t = e^{2x} + e^{–2x}, so dt = 2(e^{2x} – e^{–2x}) dx, i.e.

(e^{2x} – e^{–2x}) dx = dt/2.

Then the integral = ∫(1/t)(dt/2) = (1/2)ln|t| + C = (1/2)ln(e^{2x} + e^{–2x}) + C.

(Check: d/dx of (1/2)ln(e^{2x}+e^{–2x}) = tanh(2x) = the integrand.)

Q.25 [2 marks]

Show that f(x) = e^x – e⁻^x + x – tan⁻¹ x is strictly increasing in its domain.

Answer: Proof

Explanation:

f'(x) = e^x + e⁻^x + 1 – 1/(1+x²).

We need f'(x) > 0.

Note e^x + e⁻^x ≥ 2 (AM-GM).

Also 1 – 1/(1+x²) = x²/(1+x²) ≥ 0.

So f'(x) = e^x + e⁻^x + x²/(1+x²) ≥ 2 + 0 > 0 for all x.

Hence f is strictly increasing on R.

Q.26 [2 marks]

If $x = e^{\cos 3t}$ and $y = e^{\sin 3t}$, prove that $dy/dx = -(y \log x)/(x \log y)$.

Answer: Proof

Explanation:

$\log x = \cos 3t$, $\log y = \sin 3t$.

$dx/dt = e^{\cos 3t} \cdot (-3 \sin 3t) = -3x \sin 3t$.

$dy/dt = e^{\sin 3t} \cdot (3 \cos 3t) = 3y \cos 3t$.

$dy/dx = (dy/dt)/(dx/dt) = (3y \cos 3t)/(-3x \sin 3t) = -y \cos 3t/(x \sin 3t) = -(y \cdot \log x)/(x \cdot \log y)$, since $\cos 3t = \log x$ and $\sin 3t = \log y$.

OR

Q.26 [2 marks]

Show that : $d/dx(|x|) = x/|x|$, $x \neq 0$.

Answer: Proof

Explanation:

For $x > 0$: $|x| = x$, so $d/dx(|x|) = 1 = x/x = x/|x|$.

For $x < 0$: $|x| = -x$, so $d/dx(|x|) = -1 = x/(-x) = x/|x|$ (since $|x| = -x$ when $x < 0$).

In both cases $d/dx(|x|) = x/|x|$.

Section C

Short Answer Questions (3 marks each)

Q.27 [3 marks]

Evaluate : $\int_{-2}^{2} \sqrt{(2-x)/(2+x)}\, dx$

Answer: 2π

Explanation:

Let $I = \int_{-2}^{2} \sqrt{(2-x)/(2+x)}\, dx$.

Rationalise by multiplying numerator and denominator inside the root by $(2-x)$: $\sqrt{(2-x)/(2+x)} = (2-x)/\sqrt{(2+x)(2-x)} = (2-x)/\sqrt{(4-x^2)}$ (valid on $(-2,2)$ where $2-x > 0$).

So $I = \int_{-2}^{2} (2-x)/\sqrt{(4-x^2)}\, dx = 2\int_{-2}^{2} 1/\sqrt{(4-x^2)}\, dx - \int_{-2}^{2} x/\sqrt{(4-x^2)}\, dx$.

The second integrand $x/\sqrt{(4-x^2)}$ is odd, so its integral over the symmetric interval $[-2,2]$

is 0.

The first: $2\int_{-2}^{2} 1/\sqrt{(4-x^2)}\,dx = 2[\sin^{-1}(x/2)]_{-2}^{2} = 2(\sin^{-1}1 - \sin^{-1}(-1)) = 2(\pi/2 + \pi/2) = 2\pi$.

Therefore $I = 2\pi$.

OR

Q.27 [3 marks]

Find : $\int 1/(x[(\log x)^2 - 3\log x - 4])\,dx$

Answer: $(1/5)\ln|\log x - 4| - (1/5)\ln|\log x + 1| + C$

Explanation:

Let $t = \log x$, $dt = dx/x$.

Integral = $\int 1/(t^2 - 3t - 4)\,dt = \int 1/((t-4)(t+1))\,dt$.

Partial fractions: $1/((t-4)(t+1)) = (1/5)[1/(t-4) - 1/(t+1)]$.

Integral = $(1/5)[\ln|t-4| - \ln|t+1|] + C = (1/5)\ln|\log x - 4| - (1/5)\ln|\log x + 1| + C = (1/5)\ln|(\log x - 4)/(\log x + 1)| + C$.

Q.28 [3 marks]

Find the particular solution of the differential equation given by $2xy + y^2 - 2x^2\,dy/dx = 0$; $y = 2$, when $x = 1$.

Answer: $y = 2x/(1 - \ln x)$

Explanation:

Rewrite: $2x^2\,dy/dx = 2xy + y^2$.

This is homogeneous.

Let $v = y/x$, so $y = vx$ and $dy/dx = v + x\,dv/dx$.

Then $2x^2(v + x\,dv/dx) = 2x(vx) + (vx)^2 = 2vx^2 + v^2x^2$.

Divide by x^2: $2(v + x\,dv/dx) = 2v + v^2$, so $2x\,dv/dx = v^2$.

Separate: $2/v^2\,dv = dx/x$.

Integrate: $-2/v = \ln|x| + C$, i.e.

$-2x/y = \ln|x| + C$.

At $x = 1$, $y = 2$: $-2(1)/2 = \ln 1 + C \Rightarrow C = -1$.

So $-2x/y = \ln x - 1 \Rightarrow 2x/y = 1 - \ln x \Rightarrow y = 2x/(1 - \ln x)$.

OR

Q.28 [3 marks]

Find the general solution of the differential equation : $y\,dx = (x + 2y^2)\,dy$

Answer: $x = 2y^2 + Cy$

Explanation:

Rewrite: $y\,dx - x\,dy = 2y^2\,dy$.

Divide by y^2: $(y\,dx - x\,dy)/y^2 = 2\,dy$.

Note $d(x/y) = (y\,dx - x\,dy)/y^2$.

So $d(x/y) = 2\,dy$.

Integrate: $x/y = 2y + C \Rightarrow x = 2y^2 + Cy$.

Q.29 [3 marks]

The position vectors of vertices of $\triangle ABC$ are $A(2\hat{i} - \hat{j} + \hat{k})$, $B(\hat{i} - 3\hat{j} - 5\hat{k})$ and $C(3\hat{i} - 4\hat{j} - 4\hat{k})$. Find all the angles of $\triangle ABC$.

Answer: $\angle C = 90°$, $\angle A = \cos^{-1}(\sqrt{35}/\sqrt{41}) \approx 22.5°$, $\angle B = \cos^{-1}(\sqrt{6}/\sqrt{41}) \approx 67.5°$

Explanation:

$A = (2, -1, 1)$, $B = (1, -3, -5)$, $C = (3, -4, -4)$.

$AB = (-1, -2, -6)$, $|AB| = \sqrt{41}$; $BC = (2, -1, 1)$, $|BC| = \sqrt{6}$; $CA = (-1, 3, 5)$, $|CA| = \sqrt{35}$.

$CB \cdot CA = (-2, 1, -1)\cdot(-1, 3, 5) = 2 + 3 - 5 = 0$, so $\angle C = 90°$ (also $6 + 35 = 41$).

$\cos A = AB \cdot AC/(|AB||AC|) = (-1, -2, -6)\cdot(1, -3, -5)/(\sqrt{41}\cdot\sqrt{35}) = 35/(\sqrt{41}\cdot\sqrt{35}) = \sqrt{35}/\sqrt{41} \Rightarrow \angle A = \cos^{-1}(\sqrt{35}/\sqrt{41}) \approx 22.5°$.

$\cos B = BA \cdot BC/(|BA||BC|) = (1, 2, 6)\cdot(2, -1, 1)/(\sqrt{41}\cdot\sqrt{6}) = 6/(\sqrt{41}\cdot\sqrt{6}) = \sqrt{6}/\sqrt{41} \Rightarrow \angle B = \cos^{-1}(\sqrt{6}/\sqrt{41}) \approx 67.5°$.

Q.30 [3 marks]

A pair of dice is thrown simultaneously. If X denotes the absolute difference of the numbers appearing on top of the dice, then find the probability distribution of X.

Answer: $P(X=0)=6/36=1/6$; $P(X=1)=10/36=5/18$; $P(X=2)=8/36=2/9$; $P(X=3)=6/36=1/6$; $P(X=4)=4/36=1/9$; $P(X=5)=2/36=1/18$

Explanation:

Total outcomes = 36.

X = $|d_1 - d_2|$.

X=0: (1,1)(2,2)(3,3)(4,4)(5,5)(6,6) = 6.

X=1: (1,2)(2,1)(2,3)(3,2)(3,4)(4,3)(4,5)(5,4)(5,6)(6,5) = 10.

X=2: (1,3)(3,1)(2,4)(4,2)(3,5)(5,3)(4,6)(6,4) = 8.

X=3: (1,4)(4,1)(2,5)(5,2)(3,6)(6,3) = 6.

X=4: (1,5)(5,1)(2,6)(6,2) = 4.

X=5: (1,6)(6,1) = 2.

Check: 6+10+8+6+4+2=36 ✓.

P(X=0)=1/6, P(X=1)=5/18, P(X=2)=2/9, P(X=3)=1/6, P(X=4)=1/9, P(X=5)=1/18.

Q.31 [3 marks]

Find : $\int x^2 \cdot \sin^{-1}(x^{3/2})\, dx$

Answer: $(x^3/3)\sin^{-1}(x^{3/2}) - (1/6)\sin^{-1}(x^{3/2}) + (1/6)x^{3/2}\sqrt{(1-x^3)} + C$

Explanation:

Let $u = x^3$, $du = 3x^2\, dx$.

Then $\int x^2 \sin^{-1}(x^{3/2})\, dx = (1/3)\int \sin^{-1}(\sqrt{u})\, du$.

Put $v = \sqrt{u}$ ($u = v^2$, $du = 2v\, dv$): $= (1/3)\int \sin^{-1}(v) \cdot 2v\, dv = (2/3)\int v \sin^{-1}(v)\, dv$.

Integrate by parts ($\int v \sin^{-1}v\, dv = (v^2/2)\sin^{-1}v - (1/2)\int v^2/\sqrt{(1-v^2)}\, dv$): with $\int v^2/\sqrt{(1-v^2)}\, dv = (1/2)\sin^{-1}v - (1/2)v\sqrt{(1-v^2)}$, we get $\int v \sin^{-1}v\, dv = (v^2/2)\sin^{-1}v - (1/4)\sin^{-1}v + (1/4)v\sqrt{(1-v^2)}$.

Multiply by 2/3 and restore $v = x^{3/2}$ ($v^2 = x^3$): $\int x^2 \sin^{-1}(x^{3/2})\, dx = (x^3/3)\sin^{-1}(x^{3/2}) - (1/6)\sin^{-1}(x^{3/2}) + (1/6)x^{3/2}\sqrt{(1-x^3)} + C$.

(Verified: differentiating gives $x^2 \sin^{-1}(x^{3/2})$.)

Q.32 [3 marks]

Show that a function f : R → R defined by $f(x) = 2x/(1 + x^2)$ is neither one-one nor onto. Further, find set A so that the given function f : R → A becomes an onto function.

Answer: A = [–1, 1]

Explanation:

Not one-one: f(2) = 4/5 and f(1/2) = (2·1/2)/(1+1/4) = 1/(5/4) = 4/5.

So f(2) = f(1/2) but 2 ≠ 1/2.

Not onto (with codomain R): Range of f.

f'(x) = (2(1+x²) – 2x·2x)/(1+x²)² = 2(1–x²)/(1+x²)².

Critical points: x = ±1.

f(1) = 2/2 = 1, f(–1) = –2/2 = –1.

Range = [–1,1] ≠ R.

So not onto with codomain R.

Set A = [–1, 1] makes f onto.

OR

Q.32 [3 marks]

A relation R is defined on N × N (where N is the set of natural numbers) as : (a, b) R (c, d) <=> a – c = b – d. Show that R is an equivalence relation.

Answer: Proof

Explanation:

Reflexive: (a,b) R (a,b) since a–a = b–b = 0.

✓.

Symmetric: If (a,b) R (c,d) then a–c = b–d => c–a = d–b => (c,d) R (a,b).

✓.

Transitive: If (a,b) R (c,d) and (c,d) R (e,f) then a–c = b–d and c–e = d–f.

Adding: (a–c)+(c–e) = (b–d)+(d–f) => a–e = b–f => (a,b) R (e,f).

✓.

So R is an equivalence relation.

Section D

Long Answer Questions (5 marks each)

Q.33 [5 marks]

Find the equation of the line which bisects the line segment joining points A(2, 3, 4) and B(4, 5, 8) and is perpendicular to the lines (x–8)/3 = (y+19)/(–16) = (z–10)/7 and (x–15)/3 = (y–29)/8 = (z–5)/(–5).

Answer: $\vec{r} = (3\hat{i} + 4\hat{j} + 6\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$, i.e. $(x–3)/2 = (y–4)/3 = (z–6)/6$

Explanation:

Midpoint M of AB = ((2+4)/2, (3+5)/2, (4+8)/2) = (3, 4, 6);

the required line passes through M.

Its direction (a,b,c) is perpendicular to both given lines, so it is parallel to $d_1 \times d_2$ where $d_1 = (3, –16, 7)$, $d_2 = (3, 8, –5)$.

$d_1 \times d_2 = i((–16)(–5) – 7·8) – j(3(–5) – 7·3) + k(3·8 – (–16)·3) = i(80–56) – j(–15–21) + k(24+48) = 24\hat{i} + 36\hat{j} + 72\hat{k} = 12(2\hat{i} + 3\hat{j} + 6\hat{k})$.

Take direction (2, 3, 6).

Check: (2,3,6)·(3,–16,7) = 6–48+42 = 0 ✓ and (2,3,6)·(3,8,–5) = 6+24–30 = 0 ✓.

Equation: $(x–3)/2 = (y–4)/3 = (z–6)/6$.

Q.34 [5 marks]

Solve the following system of equations, using matrices : $2/x + 3/y + 10/z = 4$, $4/x – 6/y + 5/z = 1$, $6/x + 9/y – 20/z = 2$, where $x, y, z \neq 0$.

Answer: $x = 2, y = 3, z = 5$

Explanation:

Let $a = 1/x$, $b = 1/y$, $c = 1/z$.

System: $2a+3b+10c=4$, $4a–6b+5c=1$, $6a+9b–20c=2$.

Matrix A = [[2,3,10],[4,-6,5],[6,9,-20]].

$\det(A) = 2[(-6)(-20)-(5)(9)] – 3[(4)(-20)-(5)(6)] + 10[(4)(9)-(-6)(6)] = 2[120-45] – 3[-80-30] + 10[36+36] = 2(75) – 3(-110) + 10(72) = 150 + 330 + 720 = 1200$.

Cofactor $C_{11} = 120-45=75$, $C_{12} = –(–80–30)=110$, $C_{13}=36+36=72$.

$C_{21}=–(–60–90)=150$, $C_{22}=(–40–60)=–100$, $C_{23}=–(18–18)=0$.

$C_{31}=(15+60)=75$, $C_{32}=–(10–40)=30$, $C_{33}=(–12–12)=–24$.

adj(A)=[[75,150,75],[110,-100,30],[72,0,-24]].

$X = A^{-1}B = (1/1200)\text{adj}(A)\cdot B$.

$a=(1/1200)(75·4+150·1+75·2)=(1/1200)(300+150+150)=600/1200=1/2$.

$b=(1/1200)(110·4+(-100)·1+30·2)=(1/1200)(440-100+60)=400/1200=1/3$.

$c=(1/1200)(72·4+0·1+(-24)·2)=(1/1200)(288-48)=240/1200=1/5$.

So $1/x=1/2, 1/y=1/3, 1/z=1/5 \Rightarrow x=2, y=3, z=5$.

OR

Q.34 [5 marks]

If A = [[1, cot x], [–cot x, 1]], show that $A'A^{-1}$ = [[–cos 2x, –sin 2x], [sin 2x, –cos 2x]].

Answer: Proof

Explanation:

$\det(A) = 1 + \cot^2 x = \operatorname{cosec}^2 x$.

$A^{-1} = (1/\operatorname{cosec}^2 x)[[1, -\cot x],[\cot x, 1]] = \sin^2 x \cdot [[1,-\cot x],[\cot x,1]]$.

$A' = [[1,-\cot x],[\cot x,1]]$.

$A'A^{-1} = [[1,-\cot x],[\cot x,1]] \cdot (\sin^2 x \cdot [[1,-\cot x],[\cot x,1]])$.

$= \sin^2 x \cdot [[1-\cot^2 x, -\cot x-\cot x],[\cot x+\cot x, -\cot^2 x+1]]$.

$= \sin^2 x \cdot [[1-\cot^2 x, -2\cot x],[2\cot x, 1-\cot^2 x]]$.

Now $\sin^2 x(1-\cot^2 x) = \sin^2 x - \cos^2 x = -\cos 2x$.

$\sin^2 x \cdot 2\cot x = 2\sin^2 x \cdot (\cos x/\sin x) = 2\sin x \cos x = \sin 2x$.

So $A'A^{-1}$ = [[-cos 2x, -sin 2x],[sin 2x, -cos 2x]].

Proved.

Q.35 [5 marks]

If A_1 denotes the area of region bounded by $y^2 = 4x$, x = 1 and x-axis in the first quadrant and A_2 denotes the area of region bounded by $y^2 = 4x$, x = 4, find $A_1 : A_2$.

Answer: 1 : 16

Explanation:

A_1 (first quadrant only) = $\int_0^1 2\sqrt{x}\,dx = (4/3)[x^{(3/2)}]_0^1 = 4/3$.

A_2 is the whole region enclosed by $y^2 = 4x$ and x = 4 (both above and below the x-axis): $A_2 = 2\int_0^4 2\sqrt{x}\,dx = (8/3)[x^{(3/2)}]_0^4 = (8/3)(8) = 64/3$.

$A_1 : A_2 = (4/3) : (64/3) = 1 : 16$.

Q.36(a) [5 marks]

Case Study – 1 (Fuel Consumption): The relation between fuel consumption F (l/100 km) and speed V (km/h) is $F = V^2/500 – V/4 + 14$.

(i) Find F, when V = 40 km/h. [1 mark]

Answer: 7.2

Explanation:

F = $(40)^2/500 – 40/4 + 14 = 1600/500 – 10 + 14 = 3.2 – 10 + 14 = 7.2$ l/100 km.

Q.36(b) [5 marks]

Case Study – 1 (Fuel Consumption): $F = V^2/500 – V/4 + 14$.

(ii) Find dF/dV. [1 mark]

Answer: $V/250 – 1/4$

Explanation:

$dF/dV = d/dV(V^2/500 – V/4 + 14) = 2V/500 – 1/4 = V/250 – 1/4$.

Q.36(c) [5 marks]

Case Study – 1 (Fuel Consumption): $F = V^2/500 – V/4 + 14$.

(iii)(a) Find the speed V for which fuel consumption F is minimum. [2 marks]

Answer: V = 62.5 km/h

Explanation:

$dF/dV = V/250 – 1/4 = 0 \Rightarrow V = 250/4 = 62.5$ km/h.

$d^2F/dV^2 = 1/250 > 0$, confirming minimum.

So F is minimum at V = 62.5 km/h.

OR

Q.36(c) [5 marks]

Case Study – 1 (Fuel Consumption): $F = V^2/500 – V/4 + 14$.

(iii)(b) Find the quantity of fuel required to travel 600 km at the speed V at which $dF/dV = –0.01$. [2 marks]

Answer: 37.2 litres (V = 60 km/h, F = 6.2 l/100 km)

Explanation:

dF/dV = V/250 – 1/4 = –0.01 ⇒ V/250 = 0.25 – 0.01 = 0.24 ⇒ V = 60 km/h.

F at V = 60: F = 3600/500 – 60/4 + 14 = 7.2 – 15 + 14 = 6.2 l/100 km.

Fuel for 600 km = (6.2/100) × 600 = 6.2 × 6 = 37.2 litres.

Q.37(a) [4 marks]

Case Study – 2 (Diet Problem): A dietician wishes to minimize cost Z = 16x + 20y for two foods X (x kg at Rs. 16/kg) and Y (y kg at Rs. 20/kg). The feasible region is shown in Figure-2.

(i) Identify and write all the constraints which determine the given feasible region in Figure-2. [2 marks]

Answer: Constraints: 3x + y ≥ 8, x + y ≥ 6, x + 2y ≥ 10, x ≥ 0, y ≥ 0

Explanation:

From Figure-2 the boundary lines pass through the corner points D(0,8), C(1,5), B(2,4), A(10,0).

Line DC: 3x + y = 8; line CB: x + y = 6; line BA: x + 2y = 10.

The feasible (unbounded) region lies above all three, giving 3x + y ≥ 8, x + y ≥ 6, x + 2y ≥ 10, with x ≥ 0, y ≥ 0.

Q.37(b) [4 marks]

Case Study – 2 (Diet Problem): Minimize cost Z = 16x + 20y subject to the constraints shown in Figure-2.

(ii) Find the values of x and y at which cost is minimum, and find the minimum cost. [2 marks]

Answer: Minimum cost Z = Rs. 112 at (x, y) = (2, 4)

Explanation:

Evaluate Z = 16x + 20y at the corner points of the feasible region: D(0,8) →

160; C(1,5) →

16 + 100 = 116; B(2,4) →

32 + 80 = 112; A(10,0) →

160.

The minimum is Z = 112 at (2, 4).

Since the region is unbounded, check 16x + 20y < 112 has no point in the feasible region — it does not, so Z = 112 is the true minimum.

Q.38(a) [4 marks]

Case Study – 3 (Airplane Safety): The probability of an airplane crash is 0.00001%. There are 95% chances of survivors after a plane crash. In case of no crash, all travellers survive. Let E_1 = event of plane crash, E_2 = event of no crash, A = event that passengers survive.

(i) Find the probability that the airplane will not crash. [1 mark]

Answer: $P(E_2) = 1 - 0.0000001 = 0.9999999$

Explanation:

$P(E_1) = 0.00001\% = 0.00001/100 = 0.0000001$.

$P(E_2) = 1 - P(E_1) = 1 - 0.0000001 = 0.9999999$.

Q.38(b) [4 marks]

Case Study – 3 (Airplane Safety): P(crash) = 0.00001%, P(survive | crash) = 95%, P(survive | no crash) = 1.

(ii) Find $P(A|E_1) + P(A|E_2)$. [1 mark]

Answer: 1.95

Explanation:

$P(A|E_1) = 95\% = 0.95$ (probability of survival given crash).

$P(A|E_2) = 1$ (all survive if no crash).

$P(A|E_1) + P(A|E_2) = 0.95 + 1 = 1.95$.

Q.38(c) [4 marks]

Case Study – 3 (Airplane Safety): $P(E_1) = 0.0000001$, $P(E_2) = 0.9999999$, $P(A|E_1) = 0.95$, $P(A|E_2) = 1$.

(iii)(a) Find P(A). [2 marks]

Answer: $P(A) = 0.999999995$

Explanation:

By the total probability theorem: $P(A) = P(E_1)\cdot P(A|E_1) + P(E_2)\cdot P(A|E_2) = 0.0000001 \times 0.95 + 0.9999999 \times 1 = 0.000000095 + 0.9999999 = 0.999999995$.

OR

Q.38(c) [4 marks]

Case Study – 3 (Airplane Safety): $P(E_1) = 0.0000001$, $P(E_2) = 0.9999999$, $P(A|E_1) = 0.95$, $P(A|E_2) = 1$, $P(A) = 0.999999995$.

(iii)(b) Find $P(E_2|A)$. [2 marks]

Answer: $P(E_2|A) = 199999980/199999999 \approx 0.99999990$

Explanation:

By Bayes' theorem: $P(E_2|A) = P(E_2)\cdot P(A|E_2)/P(A) = (0.9999999 \times 1)/0.999999995 = 0.9999999/0.999999995$.

Write as fractions: numerator $0.9999999 = 9999999/10000000$;

denominator $0.999999995 = 999999995/1000000000 = 199999999/200000000$.

So $P(E_2|A) = (9999999/10000000) \times (200000000/199999999) = (9999999 \times 20)/199999999 = 199999980/199999999 \approx 0.99999990$.