CBSE Class 12 Mathematics 2024 — Set 3 — Answer Key

English Medium

Time Allowed: 3 Hours | Maximum Marks: 80

General Instructions:

1. This question paper contains 38 questions. All questions are compulsory.
2. Section A has 20 MCQs carrying 1 mark each.
3. Section B has 6 questions carrying 2 marks each.
4. Section C has 6 questions carrying 3 marks each.
5. Section D has 4 questions carrying 5 marks each.
6. Section E has 3 case-study based questions carrying 4 marks each.
7. Draw neat figures wherever required.

Section A

Multiple Choice Questions (1 mark each)

Q.1 [1 mark]

If A = [aij] is an identity matrix, then which of the following is true?

(a) aij = 0 if i = j, and aij = 1 if i ≠ j

(b) aij = 1 for all i, j

(c) aij = 0 for all i, j

(d) aij = 0 if i ≠ j, and aij = 1 if i = j <<

Explanation:

By definition, an identity matrix has 1s on the main diagonal (where i = j) and 0s everywhere else (where i ≠ j).

So aij = 1 if i = j and aij = 0 if i ≠ j.

Option (D) states exactly this.

Q.2 [1 mark]

Let R+ denote the set of all non-negative real numbers. Then the function f : R+ → R+ defined as f(x) = x^2 + 1 is:

(a) one-one but not onto <<

(b) onto but not one-one

(c) both one-one and onto

(d) neither one-one nor onto

Explanation:

One-one: f(x) = x^2 + 1.

For x_1, x_2 ∈ R+ (x ≥ 0), if $x_1^2 = x_2^2$ then $x_1 = x_2$ (since both are non-negative).

So f is one-one.

Onto: f(x) = x^2 + 1 ≥ 1 for all x ∈ R+, so the range is [1, ∞), which is a proper subset of R+ = [0, ∞).

The value 0 (for example) has no pre-image.

So f is NOT onto.

Answer: one-one but not onto.

Q.3 [1 mark]

Let A = [[a, b], [c, d]] be a square matrix such that adj A = A. Then (a + b + c + d) is equal to:

(a) 2a <<

(b) 2b

(c) 2c

(d) 0

Explanation:

For a 2×2 matrix A = [[a, b], [c, d]], the adjugate is adj A = [[d, -b], [-c, a]].

Given adj A = A: d = a, -b = b => b = 0, -c = c => c = 0, a = d.

So the matrix is diagonal with b = 0, c = 0, and d = a.

Then a + b + c + d = a + 0 + 0 + a = 2a.

Q.4 [1 mark]

A function f(x) = |1 – x + |x|| is:

(a) discontinuous at x = 1 only

(b) discontinuous at x = 0 only

(c) discontinuous at x = 0, 1

(d) continuous everywhere <<

Explanation:

For x ≥ 0: |x| = x, so f(x) = |1 – x + x| = |1| = 1 (constant, continuous).

For x < 0: |x| = –x, so f(x) = |1 – x + (–x)| = |1 – 2x|.

At x = 0: from the right, f = 1; from the left, |1 – 0| = 1.

Both pieces give f(0) = 1 and are continuous on their respective domains.

Since both pieces are continuous and agree at x = 0, f is continuous everywhere.

Absolute value and constant functions are always continuous, so f is continuous everywhere.

Q.5 [1 mark]

If the sides of a square are decreasing at the rate of 1·5 cm/s, the rate of decrease of its perimeter is:

(a) 1·5 cm/s

(b) 6 cm/s <<

(c) 3 cm/s

(d) 2·25 cm/s

Explanation:

Let side = x.

Perimeter P = 4x.

dP/dt = 4 · dx/dt.

Given dx/dt = –1.5 cm/s (decreasing).

So dP/dt = 4 × (–1.5) = –6 cm/s.

The rate of decrease of perimeter is 6 cm/s.

Q.6 [1 mark]

$\int_{-a}^{a} f(x)\,dx = 0$, if:

(a) f(–x) = f(x)

(b) f(–x) = –f(x) <<

(c) f(a – x) = f(x)

(d) f(a – x) = –f(x)

Explanation:

By the property of definite integrals over symmetric interval [–a, a]: $\int_{-a}^{a} f(x)\,dx = 0$ if and only if f is an odd function, i.e., f(–x) = –f(x).

If f(–x) = f(x) (even function), the integral equals $2\int_{0}^{a} f(x)\,dx \neq 0$ in general.

Q.7 [1 mark]

$x \log x \cdot dy/dx + y = 2 \log x$ is an example of a:

(a) variable separable differential equation

(b) homogeneous differential equation

(c) first order linear differential equation <<

(d) differential equation whose degree is not defined

Explanation:

Rewrite: dy/dx + y/(x log x) = 2/x.

This is of the form dy/dx + P(x)y = Q(x) where P(x) = 1/(x log x) and Q(x) = 2/x.

This is a first order linear differential equation.

Q.8 [1 mark]

If $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} - \hat{k}$, then $\vec{a}$ and $\vec{b}$ are:

(a) collinear vectors which are not parallel

(b) parallel vectors

(c) perpendicular vectors <<

(d) unit vectors

Explanation:

Compute $\vec{a} \cdot \vec{b} = (2)(1) + (-1)(1) + (1)(-1) = 2 - 1 - 1 = 0$.

Since the dot product is 0, $\vec{a}$ and $\vec{b}$ are perpendicular.

$|\vec{a}| = \sqrt{(4+1+1)} = \sqrt{6} \neq 1$ and $|\vec{b}| = \sqrt{(1+1+1)} = \sqrt{3} \neq 1$, so they are not unit vectors.

They are not parallel (not scalar multiples of each other).

Q.9 [1 mark]

If α, β and γ are the angles which a line makes with positive directions of x, y and z axes respectively, then which of the following is NOT true?

(a) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

(b) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

(c) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1$

(d) $\cos\alpha + \cos\beta + \cos\gamma = 1$ <<

Explanation:

We know $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ (true, by definition of direction cosines).

$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = (1 - \cos^2\alpha) + (1 - \cos^2\beta) + (1 - \cos^2\gamma) = 3 - 1 = 2$ (true).

$\cos 2\alpha + \cos 2\beta + \cos 2\gamma = (2\cos^2\alpha - 1) + (2\cos^2\beta - 1) + (2\cos^2\gamma - 1) = 2(1) - 3 = -1$ (true).

$\cos\alpha + \cos\beta + \cos\gamma = 1$ is NOT necessarily true in general (e.g., if $\alpha = \beta = \gamma = \cos^{-1}(1/\sqrt{3})$, then $\cos\alpha + \cos\beta + \cos\gamma = 3/\sqrt{3} = \sqrt{3} \neq 1$).

So (D) is NOT true.

Q.10 [1 mark]

The restrictions imposed on decision variables involved in an objective function of a linear programming problem are called:

(a) feasible solutions

(b) constraints <<

(c) optimal solutions

(d) infeasible solutions

Explanation:

In a linear programming problem, the restrictions (inequalities or equalities) imposed on the decision variables are called constraints.

Feasible solutions satisfy all constraints;

the optimal solution maximises or minimises the objective function.

Q.11 [1 mark]

Let E and F be two events such that P(E) = 0·1, P(F) = 0·3, P(E ∪ F) = 0·4, then P(F|E) is:

(a) 0·6

(b) 0·4

(c) 0·5

(d) 0 <<

Explanation:

P(E ∩ F) = P(E) + P(F) – P(E ∪ F) = 0.1 + 0.3 – 0.4 = 0.

P(F|E) = P(E ∩ F) / P(E) = 0 / 0.1 = 0.

So P(F|E) = 0.

E and F are mutually exclusive events.

Q.12 [1 mark]

If A and B are two skew symmetric matrices, then (AB + BA) is:

(a) a skew symmetric matrix

(b) a symmetric matrix <<

(c) a null matrix

(d) an identity matrix

Explanation:

A and B are skew symmetric, so A^T = –A and B^T = –B.

Consider (AB + BA)^T = (AB)^T + (BA)^T = B^TA^T + A^TB^T = (–B)(–A) + (–A)(–B) = BA + AB = AB + BA.

Since (AB + BA)^T = AB + BA, the matrix AB + BA is symmetric.

Q.13 [1 mark]

If |1 3 1; k 0 1; 0 0 1| = ±6, then the value of k is:

(a) 2

(b) –2

(c) ±2 <<

(d) ∓2

Explanation:

Expanding the 3×3 determinant along row 1: det = 1·(0·1 – 1·0) – 3·(k·1 – 1·0) + 1·(k·0 – 0·0) = 1·0 – 3k + 0 = –3k.

Setting –3k = ±6 gives k = ∓2, i.e., k = –2 or k = 2, so k = ±2.

Q.14 [1 mark]

The derivative of 2^x with respect to 3^x is:

(a) (3/2)^x · (log 2)/(log 3)

(b) (2/3)^x · (log 3)/(log 2)

(c) (2/3)^x · (log 2)/(log 3) <<

(d) (3/2)^x · (log 3)/(log 2)

Explanation:

Let u = 2^x and v = 3^x.

du/dx = 2^x · log 2.

dv/dx = 3^x · log 3.

du/dv = (du/dx)/(dv/dx) = (2^x · log 2)/(3^x · log 3) = (2/3)^x · (log 2)/(log 3).

Q.15 [1 mark]

If $|\vec{a}| = 2$ and $-3 \le k \le 2$, then $|k\vec{a}| \in$:

(a)

(b)

(c)

(d) <<

Explanation:

$|k\vec{a}| = |k| \cdot |\vec{a}| = |k| \cdot 2 = 2|k|$.

Since $-3 \le k \le 2$, $|k|$ ranges from 0 to 3.

So $2|k| \in [0, 6]$.

The magnitude is always non-negative, so the answer is [0, 6].

Q.16 [1 mark]

If a line makes an angle of $\pi/4$ with the positive directions of both x-axis and z-axis, then the angle which it makes with the positive direction of y-axis is:

(a) 0

(b) $\pi/4$

(c) $\pi/2$ <<

(d) π

Explanation:

Direction cosines satisfy $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Here $\alpha = \gamma = \pi/4$.

$\cos^2(\pi/4) + \cos^2\beta + \cos^2(\pi/4) = 1 \Rightarrow 1/2 + \cos^2\beta + 1/2 = 1 \Rightarrow \cos^2\beta = 0 \Rightarrow \cos\beta = 0 \Rightarrow \beta = \pi/2$.

The line makes angle $\pi/2$ with the y-axis.

Q.17 [1 mark]

Which group of constraints represents the feasible region shown in the figure?

(a) $x + 2y \le 76$, $2x + y \ge 104$, $x, y \ge 0$

(b) $x + 2y \le 76$, $2x + y \le 104$, $x, y \ge 0$

(c) $x + 2y \ge 76$, $2x + y \le 104$, $x, y \ge 0$ <<

(d) $x + 2y \geq 76, 2x + y \geq 104, x, y \geq 0$

Explanation:

Figure: shaded triangle-like region touches the y-axis between y=38 (line x+2y=76) and y=104 (line 2x+y=104), bounded to the right by their intersection (44,16).

Test point (0,60): x+2y=120>=76 and 2x+y=60<=104.

So region is above x+2y=76 (>=) and below 2x+y=104 (<=): x+2y>=76, 2x+y<=104, x,y>=0 = option (C).

Q.18 [1 mark]

If A = diag(2, 3, 5) (i.e., A = [[2,0,0],[0,3,0],[0,0,5]]), then A^{-1} is:

(a) diag(1/2, 1/3, 1/5) <<

(b) 30 · diag(1/2, 1/3, 1/5)

(c) (1/30) · [[2,0,0],[0,3,0],[0,0,5]]

(d) (1/30) · diag(1/2, 1/3, 1/5)

Explanation:

For a diagonal matrix A = diag(a, b, c), A^{-1} = diag(1/a, 1/b, 1/c) provided a, b, c ≠ 0.

Here A = diag(2, 3, 5), so A^{-1} = diag(1/2, 1/3, 1/5).

This corresponds to option (A).

Q.19 [1 mark]

Assertion (A): Every scalar matrix is a diagonal matrix.

Reason (R): In a diagonal matrix, all the diagonal elements are 0.

(a) Both A and R are true and R is the correct explanation of A.

(b) Both A and R are true, but R is NOT the correct explanation of A.

(c) A is true, but R is false. <<

(d) A is false, but R is true.

Explanation:

Assertion (A): A scalar matrix is of the form kI, which has k on the diagonal and 0 off-diagonal — so it IS a diagonal matrix.

Assertion A is TRUE.

Reason (R): In a diagonal matrix, all the diagonal elements are 0.

This is FALSE — a diagonal matrix has non-zero entries allowed on the diagonal (only off-diagonal elements are 0).

So A is true, R is false.

Answer: (C).

Q.20 [1 mark]

Assertion (A): Projection of $\vec{a}$ on $\vec{b}$ is same as projection of $\vec{b}$ on $\vec{a}$.

Reason (R): Angle between $\vec{a}$ and $\vec{b}$ is same as angle between $\vec{b}$ and $\vec{a}$ numerically.

(a) Both A and R are true and R is the correct explanation of A.

(b) Both A and R are true, but R is NOT the correct explanation of A.

(c) A is true, but R is false.

(d) A is false, but R is true. <<

Explanation:

Assertion (A): Projection of $\vec{a}$ on $\vec{b}$ = $|\vec{a}|\cos\theta$.

Projection of $\vec{b}$ on $\vec{a}$ = $|\vec{b}|\cos\theta$.

These are equal only if $|\vec{a}| = |\vec{b}|$, which is not always true.

So Assertion A is FALSE.

Reason (R): The angle between $\vec{a}$ and $\vec{b}$ equals the angle between $\vec{b}$ and $\vec{a}$ numerically.

This is TRUE (angle is commutative).

So A is false but R is true.

Answer: (D).

Section B

Very Short Answer Questions (2 marks each)

Q.21 [2 marks]

Evaluate: $\sec^2(\tan^{-1}(1/2)) + \operatorname{cosec}^2(\cot^{-1}(1/3))$

Answer: 85/36

Explanation:

Let $\alpha = \tan^{-1}(1/2)$, so $\tan \alpha = 1/2$.

Then $\sec^2\alpha = 1 + \tan^2\alpha = 1 + 1/4 = 5/4$.

Let $\beta = \cot^{-1}(1/3)$, so $\cot \beta = 1/3$.

Then $\text{cosec}^2\beta = 1 + \cot^2\beta = 1 + (1/3)^2 = 1 + 1/9 = 10/9$.

Sum = 5/4 + 10/9.

LCM(4, 9) = 36 ⇒ 45/36 + 40/36 = 85/36.

Q.22 [2 marks]

(a) If x = e^(x/y), prove that dy/dx = (log x – 1)/(log x)².

Answer: Proof

Explanation:

Given x = e^(x/y).

Taking log both sides: log x = x/y, so y = x/log x.

Differentiate with respect to x using quotient rule: dy/dx = [log x · 1 – x · (1/x)] / (log x)² = (log x – 1)/(log x)².

Proved.

OR

Q.22 [2 marks]

(b) Check the differentiability of f(x) = {x² + 1, 0 ≤ x < 1; 3 – x, 1 ≤ x ≤ 2} at x = 1.

Answer: Not differentiable at x = 1

Explanation:

LHD at x = 1: $\lim_{h\to0^-}$ [f(1+h) – f(1)]/h = $\lim_{h\to0^-}$ [(1+h)² + 1 – (3–1)]/h = lim [(1 + 2h + h²) + 1 – 2]/h = lim [2h + h²]/h = lim (2 + h) = 2.

RHD at x = 1: $\lim_{h\to0^+}$ [f(1+h) – f(1)]/h = $\lim_{h\to0^+}$ [3 – (1+h) – 2]/h = lim [–h]/h = –1.

LHD = 2 ≠ –1 = RHD, so f is NOT differentiable at x = 1.

Q.23 [2 marks]

(a) Evaluate: $\int_0^{\pi/2}$ sin 2x cos 3x dx

Answer: –2/5

Explanation:

Use product-to-sum: sin 2x cos 3x = (1/2)[sin(2x+3x) + sin(2x–3x)] = (1/2)[sin 5x – sin x].

So $\int_0^{\pi/2} \sin 2x \cos 3x\, dx = (1/2)\int_0^{\pi/2} [\sin 5x - \sin x]\, dx = (1/2)[-\cos 5x/5 + \cos x]_0^{\pi/2} = (1/2)\{[-\cos(5\pi/2)/5 + \cos(\pi/2)] - [-\cos 0/5 + \cos 0]\} = (1/2)\{[0 + 0] - [-1/5 + 1]\} = (1/2)\{0 - 4/5\} = -2/5$.

OR

Q.23 [2 marks]

(b) Given $d/dx\, F(x) = 1/\sqrt{(2x - x^2)}$ and $F(1) = 0$, find $F(x)$.

Answer: $F(x) = \sin^{-1}(x - 1)$

Explanation:

$F'(x) = 1/\sqrt{(2x - x^2)} = 1/\sqrt{(1 - (x-1)^2)}$.

This is the derivative of $\sin^{-1}(x - 1)$.

So $F(x) = \sin^{-1}(x - 1) + C$.

Using $F(1) = 0$: $\sin^{-1}(1 - 1) + C = \sin^{-1}(0) + C = 0 + C = 0$, so $C = 0$.

Therefore $F(x) = \sin^{-1}(x - 1)$.

Q.24 [2 marks]

Find the position vector of point C which divides the line segment joining points A and B having position vectors $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively in the ratio 4:1 externally. Further, find $|\overrightarrow{AB}| : |\overrightarrow{BC}|$.

Answer: Position vector of C = $(-5\hat{i} + 2\hat{j} + 5\hat{k})/3$; $|\overrightarrow{AB}| : |\overrightarrow{BC}| = 3 : 1$

Explanation:

$A = \hat{i} + 2\hat{j} - \hat{k}$, $B = -\hat{i} + \hat{j} + \hat{k}$.

External division in ratio 4:1: $C = (4 \cdot B - 1 \cdot A)/(4-1) = (4(-\hat{i}+\hat{j}+\hat{k}) - (\hat{i}+2\hat{j}-\hat{k}))/3 = (-5\hat{i} + 2\hat{j} + 5\hat{k})/3$.

$\overrightarrow{AB} = B - A = -2\hat{i} - \hat{j} + 2\hat{k}$, so $|\overrightarrow{AB}| = \sqrt{(4+1+4)} = 3$.

$\overrightarrow{BC} = C - B = (-2/3)\hat{i} + (-1/3)\hat{j} + (2/3)\hat{k}$, so $|\overrightarrow{BC}| = \sqrt{(4/9 + 1/9 + 4/9)} = \sqrt{1} = 1$.

Therefore $|\overrightarrow{AB}| : |\overrightarrow{BC}| = 3 : 1$.

(Consistent with external division 4:1: |AC|/|BC| = 4, and |AB| = |AC| – |BC| = 3|BC|.)

Q.25 [2 marks]

Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors. Prove that $|\vec{a} \times \vec{b}| \leq |\vec{a}||\vec{b}|$. State the condition under which equality holds, i.e., $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|$.

Answer: Equality holds when $\vec{a}$ and $\vec{b}$ are perpendicular ($\theta = \pi/2$)

Explanation:

We know $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$.

Since $\sin\theta \leq 1$ for all θ, we have $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta \leq |\vec{a}||\vec{b}| \cdot 1 = |\vec{a}||\vec{b}|$.

Equality holds when $\sin\theta = 1$, i.e., $\theta = \pi/2$, meaning $\vec{a}$ and $\vec{b}$ are perpendicular to each other.

Q.26 [2 marks]

(a) If $x\cos(p + y) + \cos p \cdot \sin(p + y) = 0$, prove that $\cos p \cdot dy/dx = -\cos^2(p + y)$, where p is a constant.

Answer: Proof

Explanation:

Given: $x\cos(p + y) + \cos p \cdot \sin(p + y) = 0$.

Rearranging: $x = -\cos p \cdot \sin(p + y)/\cos(p + y) = -\cos p \cdot \tan(p + y)$.

Differentiating with respect to x: $1 = -\cos p \cdot \sec^2(p + y) \cdot dy/dx$.

So $dy/dx = -1/[\cos p \cdot \sec^2(p + y)] = -\cos^2(p + y)/\cos p$.

Thus $\cos p \cdot dy/dx = -\cos^2(p + y)$.

Proved.

OR

Q.26 [2 marks]

(b) Find the value of a and b so that the function f defined as:

$f(x) = \{(x-2)/|x-2| + a, \text{ if } x < 2;\ a + b, \text{ if } x = 2;\ (x-2)/|x-2| + b, \text{ if } x > 2\}$

is a continuous function.

Answer: a = 1, b = –1

Explanation:

For $x < 2$: $(x–2)/|x–2| = (x–2)/(–(x–2)) = –1$, so LHL = $\lim_{x\to 2^-} f(x) = –1 + a$.

For $x > 2$: $(x–2)/|x–2| = 1$, so RHL = $\lim_{x\to 2^+} f(x) = 1 + b$.

$f(2) = a + b$.

Continuity at $x = 2$ requires LHL = f(2) = RHL.

From LHL = f(2): $–1 + a = a + b \Rightarrow b = –1$.

From RHL = f(2): $1 + b = a + b \Rightarrow a = 1$.

Check: LHL = $–1 + 1 = 0$, RHL = $1 + (–1) = 0$, $f(2) = 1 + (–1) = 0$.

All equal ✓.

So $a = 1$ and $b = –1$.

Section C

Short Answer Questions (3 marks each)

Q.27 [3 marks]

(a) Find the intervals in which the function $f(x) = (\log x)/x$ is strictly increasing or strictly decreasing.

Answer: Increasing on $(0, e)$; Decreasing on (e, ∞)

Explanation:

$f(x) = (\log x)/x$ for $x > 0$.

$f'(x) = [x \cdot (1/x) – \log x \cdot 1]/x^2 = (1 – \log x)/x^2$.

$f'(x) > 0$ when $1 – \log x > 0$, i.e., $\log x < 1$, i.e., $x < e$.

$f'(x) < 0$ when $x > e$.

$f'(e) = 0$.

So f is strictly increasing on $(0, e)$ and strictly decreasing on (e, ∞).

OR

Q.27 [3 marks]

(b) Find the absolute maximum and absolute minimum values of the function f given by $f(x) = x/2 + 2/x$, on the interval $[1, 2]$.

Answer: Absolute maximum = 5/2; Absolute minimum = 2

Explanation:

f(x) = x/2 + 2/x.

f'(x) = 1/2 – 2/x².

Setting f'(x) = 0: 1/2 = 2/x² => x² = 4 => x = 2 (taking positive value, since x ∈ [1, 2]).

Evaluate at critical point and endpoints: f(1) = 1/2 + 2/1 = 1/2 + 2 = 5/2.

f(2) = 2/2 + 2/2 = 1 + 1 = 2.

f(2) is at the boundary (x = 2 is both critical point and endpoint).

So on [1, 2]: max = f(1) = 5/2, min = f(2) = 2.

Q.28 [3 marks]

Find: $\int (x^2 + 1)/[(x^2 + 2)(x^2 + 4)]\, dx$

Answer: $-(1/(2\sqrt{2}))\tan^{-1}(x/\sqrt{2}) + (3/4)\tan^{-1}(x/2) + C$

Explanation:

Partial fractions (substitute t = x²): $(x^2+1)/[(x^2+2)(x^2+4)] = A/(x^2+2) + B/(x^2+4)$, so $x^2+1 = A(x^2+4) + B(x^2+2)$.

Set x² = –2: –1 = 2A ⇒ A = –1/2.

Set x² = –4: –3 = –2B ⇒ B = 3/2.

Now $\int 1/(x^2+2)\, dx = (1/\sqrt{2})\tan^{-1}(x/\sqrt{2})$ and $\int 1/(x^2+4)\, dx = (1/2)\tan^{-1}(x/2)$.

So the integral = $-(1/2)\cdot(1/\sqrt{2})\tan^{-1}(x/\sqrt{2}) + (3/2)\cdot(1/2)\tan^{-1}(x/2) + C = -(1/(2\sqrt{2}))\tan^{-1}(x/\sqrt{2}) + (3/4)\tan^{-1}(x/2) + C$.

Q.29 [3 marks]

(a) Find: $\int [(2 + \sin 2x)/(1 + \cos 2x)] \cdot e^x\, dx$

Answer: $e^x \tan x + C$

Explanation:

Note: 1 + cos 2x = 2cos²x and sin 2x = 2 sin x cos x.

So $(2 + \sin 2x)/(1 + \cos 2x) = (2 + 2 \sin x \cos x)/(2\cos^2 x) = 1/\cos^2 x + \sin x/\cos x = \sec^2 x + \tan x$.

So the integral = $\int e^x(\tan x + \sec^2 x)\, dx$.

This is of the form $\int e^x[f(x) + f'(x)]\, dx = e^x f(x) + C$, with f(x) = tan x and f'(x) = sec²x.

Therefore the integral = $e^x \tan x + C$.

OR

Q.29 [3 marks]

(b) Evaluate: $\int_0^{\pi/4} 1/(\sin x + \cos x)\, dx$

Answer: $(1/\sqrt{2}) \ln|\tan(3\pi/8)|$

Explanation:

$\sin x + \cos x = \sqrt{2} \sin(x + \pi/4) = \sqrt{2} \cos(x - \pi/4)$.

So $1/(\sin x + \cos x) = 1/(\sqrt{2} \cos(x - \pi/4))$.

$\int_0^{\pi/4} 1/(\sin x + \cos x)\, dx = (1/\sqrt{2}) \int_0^{\pi/4} \sec(x - \pi/4)\, dx$.

Let $u = x - \pi/4$, $du = dx$.

When $x = 0$, $u = -\pi/4$; $x = \pi/4$, $u = 0$.

$= (1/\sqrt{2}) \int_{-\pi/4}^{0} \sec u\, du = (1/\sqrt{2}) [\ln|\sec u + \tan u|]_{-\pi/4}^{0} = (1/\sqrt{2}) \{\ln|1 + 0| - \ln|\sqrt{2} - 1|\} = (1/\sqrt{2}) \{0 - \ln(\sqrt{2} - 1)\} = (1/\sqrt{2}) \ln(1/(\sqrt{2}-1)) = (1/\sqrt{2}) \ln(\sqrt{2}+1)$.

So $\int_0^{\pi/4} 1/(\sin x + \cos x)\, dx = (1/\sqrt{2}) \ln(\sqrt{2} + 1)$.

Q.30 [3 marks]

Solve the following linear programming problem graphically:

Maximise $z = 4x + 3y$, subject to the constraints:

$x + y \le 800$

$2x + y \le 1000$

$x \le 400$

$x, y \ge 0$

Answer: Maximum $z = 2600$ at $(200, 600)$

Explanation:

Corner points of the feasible region: $(0, 0)$, $(400, 0)$, $(400, 200)$, $(200, 600)$, $(0, 800)$.

Check: at $x = 400$, $2(400) + y = 1000 \Rightarrow y = 200$ ✓;

solving $x + y = 800$ and $2x + y = 1000$ gives $x = 200$, $y = 600$ ✓.

Evaluate $z = 4x + 3y$: $z(0,0) = 0$;

$z(400,0) = 1600$;

$z(400,200) = 1600 + 600 = 2200$;

$z(200,600) = 800 + 1800 = 2600$;

$z(0,800) = 2400$.

Maximum z = 2600 at (200, 600).

Q.31 [3 marks]

The chances of P, Q and R getting selected as CEO of a company are in the ratio 4:1:2 respectively. The probabilities for the company to increase its profits from the previous year under the new CEO, P, Q or R are 0·3, 0·8 and 0·5 respectively. If the company increased the profits from the previous year, find the probability that it is due to the appointment of R as CEO.

Answer: 1/3

Explanation:

P(P) = 4/7, P(Q) = 1/7, P(R) = 2/7.

P(profit|P) = 0.3, P(profit|Q) = 0.8, P(profit|R) = 0.5.

Total probability of profit: P(profit) = (4/7)(0.3) + (1/7)(0.8) + (2/7)(0.5) = (1.2 + 0.8 + 1.0)/7 = 3.0/7.

By Bayes' theorem: P(R|profit) = P(R)·P(profit|R)/P(profit) = (2/7 × 0.5)/(3/7) = (1/7)/(3/7) = 1/3.

Q.32 [3 marks]

A relation R on set A = {–4, –3, –2, –1, 0, 1, 2, 3, 4} be defined as R = {(x, y) : x + y is an integer divisible by 2}. Show that R is an equivalence relation. Also, write the equivalence class [2].

Answer: [2] = {–4, –2, 0, 2, 4}

Explanation:

Reflexive: For any $x \in A$, $x + x = 2x$ is divisible by 2.

So $(x, x) \in R$ for all x.

R is reflexive.

Symmetric: If $(x, y) \in R$, then x + y is divisible by 2.

Since x + y = y + x, $(y, x) \in R$.

R is symmetric.

Transitive: If $(x, y) \in R$ and $(y, z) \in R$, then x + y = 2p and y + z = 2q for integers p, q.

Adding: x + 2y + z = 2(p + q).

So x + z = 2(p + q) – 2y = 2(p + q – y), which is divisible by 2.

So (x, z) ∈ R.

R is transitive.

Hence R is an equivalence relation.

Equivalence class [2]: {x ∈ A : x + 2 is divisible by 2} = {x ∈ A : x is even} = {–4, –2, 0, 2, 4}.

Section D

Long Answer Questions (5 marks each)

Q.33 [5 marks]

(a) It is given that function $f(x) = x^4 – 62x^2 + ax + 9$ attains local maximum value at x = 1. Find the value of 'a', hence obtain all other points where the given function f(x) attains local maximum or local minimum values.

Answer: a = 120; local maximum at x = 1; local minimum at x = –6 and x = 5

Explanation:

$f(x) = x^4 – 62x^2 + ax + 9$.

$f'(x) = 4x^3 – 124x + a$.

Since f has local max at x = 1, f'(1) = 0: 4 – 124 + a = 0 => a = 120.

So $f'(x) = 4x^3 – 124x + 120 = 4(x^3 – 31x + 30)$.

Factor: x = 1 is a root.

$x^3 – 31x + 30 = (x – 1)(x^2 + x – 30) = (x – 1)(x + 6)(x – 5)$.

So $f'(x) = 4(x – 1)(x + 6)(x – 5)$.

Critical points: x = 1, x = –6, x = 5.

$f''(x) = 12x^2 – 124$.

$f''(1) = 12 – 124 = –112 < 0$ → local max at x = 1 ✓.

$f''(–6) = 12(36) – 124 = 432 – 124 = 308 > 0$ → local min at x = –6.

$f''(5) = 12(25) – 124 = 300 – 124 = 176 > 0$ → local min at x = 5.

So f attains local maximum at x = 1 (a = 120), local minimum at x = –6 and x = 5.

OR

Q.33 [5 marks]

(b) The perimeter of a rectangular metallic sheet is 300 cm. It is rolled along

one of its sides to form a cylinder. Find the dimensions of the rectangular sheet so that volume of the cylinder so formed is maximum.

Answer: Length = 100 cm, width = 50 cm (the 100 cm side is rolled to form the circumference; cylinder height = 50 cm)

Explanation:

Let the side rolled to form the circumference be l and the other side b.

2(l + b) = 300, so b = 150 – l.

Radius r = l/(2π), height h = b = 150 – l.

Volume V = πr²h = l²(150 – l)/(4π).

dV/dl = l(300 – 3l)/(4π).

Setting dV/dl = 0: l = 0 (rejected) or l = 100.

d²V/dl² = (300 – 6l)/(4π) = –75/π < 0 at l = 100, so V is maximum.

Hence l = 100 cm, b = 50 cm; radius = 50/π cm, height = 50 cm.

Q.34 [5 marks]

Using integration, find the area of the region enclosed between the circle $x^2 + y^2 = 16$ and the lines $x = -2$ and $x = 2$.

Answer: $16\pi/3 + 8\sqrt{3}$ sq units

Explanation:

Circle $x^2 + y^2 = 16$ has radius 4;

for each x the vertical strip runs from $y = -\sqrt{(16-x^2)}$ to $y = +\sqrt{(16-x^2)}$, height $2\sqrt{(16-x^2)}$.

So area = $\int_{-2}^{2} 2\sqrt{(16 - x^2)}\, dx = 4\int_{0}^{2} \sqrt{(16 - x^2)}\, dx$ (even integrand).

Using $\int\sqrt{(a^2-x^2)}\, dx = (x/2)\sqrt{(a^2-x^2)} + (a^2/2)\sin^{-1}(x/a)$ with a = 4: $4[(x/2)\sqrt{(16-x^2)} + 8\sin^{-1}(x/4)]_0^2$.

At x = 2: $(1)\sqrt{12} + 8\sin^{-1}(1/2) = 2\sqrt{3} + 8(\pi/6) = 2\sqrt{3} + 4\pi/3$.

At x = 0: 0.

Area = $4[2\sqrt{3} + 4\pi/3] = 8\sqrt{3} + 16\pi/3$ sq units.

Q.35 [5 marks]

(a) Find the equation of the line passing through the point of intersection of the lines $x/1 = (y-1)/2 = (z-2)/3$ and $(x-1)/0 = y/(-3) = (z-7)/2$ and

perpendicular to these given lines.

Answer: (x–1)/13 = (y–3)/(–2) = (z–5)/(–3)

Explanation:

Line 1: x/1 = (y–1)/2 = (z–2)/3 = t ⇒ x = t, y = 1+2t, z = 2+3t; direction d_1 = (1, 2, 3).

Line 2: (x–1)/0 = y/(–3) = (z–7)/2 = s ⇒ x = 1, y = –3s, z = 7+2s;

direction d_2 = (0, –3, 2).

Intersection: from x, t = 1 ⇒ (1, 3, 5); on line 2, s = –1 gives (1, 3, 5) ✓.

Point of intersection P = (1, 3, 5).

Direction of required line = $d_1 \times d_2$ = |î ĵ k̂;

1 2 3;

0 –3 2| = î(2·2 – 3·(–3)) – ĵ(1·2 – 3·0) + k̂(1·(–3) – 2·0) = 13î – 2ĵ – 3k̂.

Required line through (1, 3, 5) with direction (13, –2, –3): (x–1)/13 = (y–3)/(–2) = (z–5)/(–3).

OR

Q.35 [5 marks]

(b) Two vertices of the parallelogram ABCD are given as A(–1, 2, 1) and B(1, –2, 5). If the equation of the line passing through C and D is (x–4)/1 = (y+7)/(–2) = (z–8)/2, then find the distance between sides AB and CD. Hence, find the area of parallelogram ABCD.

Answer: Distance = √26/3; Area = 2√26 sq units

Explanation:

AB direction: B – A = (2, –4, 4) = 2(1, –2, 2).

CD direction from the line: (1, –2, 2).

AB ∥ CD (scalar multiples), confirming ABCD is a parallelogram.

Take A(–1, 2, 1) on AB and C(4, –7, 8) (the point on line CD at parameter 0).

AC = (5, –9, 7).

Distance between the parallel lines = |AC × d|/|d| with d = (1, –2, 2).

AC × d = |î ĵ k̂;

5 –9 7;

1 –2 2| = î((–9)(2) – 7(–2)) – ĵ((5)(2) – 7(1)) + k̂((5)(–2) – (–9)(1)) = î(–4) – ĵ(3) + k̂(–1) = (–4, –3, –1).

|AC × d| = √(16+9+1) = √26; |d| = √(1+4+4) = 3.

Distance = √26/3.

|AB| = √(4+16+16) = 6.

Area = base × height = |AB| × distance = 6 × (√26/3) = 2√26 sq units.

Q.36(a) [5 marks]

Case Study 1: A student may spend 1 to 6 hours per day in upskilling. The probability distribution is:

$P(X = x) = kx^2$, for x = 1, 2, 3

$P(X = x) = 2kx$, for x = 4, 5, 6

$P(X = x) = 0$, otherwise

(i) Express the probability distribution in the form of a probability distribution table.

Answer: Table with x: 1,2,3,4,5,6 and P(X=x): k, 4k, 9k, 8k, 10k, 12k

Explanation:

$P(X=1) = k(1)^2 = k$.

P(X=2) = k(4) = 4k.

P(X=3) = k(9) = 9k.

P(X=4) = 2k(4) = 8k.

P(X=5) = 2k(5) = 10k.

P(X=6) = 2k(6) = 12k.

The probability distribution table:

| X | 1 | 2 | 3 | 4 | 5 | 6 |

| P(X=x) | k | 4k | 9k | 8k | 10k | 12k |

Q.36(b) [5 marks]

Case Study 1 (continued): (ii) Find the value of k.

Answer: k = 1/44

Explanation:

Sum of all probabilities = 1: k + 4k + 9k + 8k + 10k + 12k = 1 => 44k = 1 => k = 1/44.

Q.36(c) [5 marks]

Case Study 1 (continued): (iii)(a) Find the mean number of hours spent by the student.

Answer: Mean = 190/44 = 95/22

Explanation:

E(X) = Σ x·P(X=x) = 1·k + 2·4k + 3·9k + 4·8k + 5·10k + 6·12k = k(1 + 8 + 27 + 32 + 50 + 72) = 190k.

With k = 1/44, E(X) = 190/44 = 95/22.

OR

Q.36(c) [5 marks]

Case Study 1 (continued): (iii)(b) Find P(1 < X < 6).

Answer: P(1 < X < 6) = 31/44

Explanation:

P(1 < X < 6) = P(X=2) + P(X=3) + P(X=4) + P(X=5) = 4k + 9k + 8k + 10k = 31k = 31/44.

Q.37(a) [4 marks]

Case Study 2: A bacteria sample is observed to grow exponentially. The differential equation is dP/dt = kP, where P is the population at time t.

(i) Obtain the general solution of the given differential equation and express it as an exponential function of t.

Answer: $P = P_0 e^{(kt)}$

Explanation:

dP/dt = kP.

Separating variables: dP/P = k dt.

Integrating both sides: ln P = kt + C.

So $P = e^{(kt+C)} = e^C \cdot e^{(kt)} = A \cdot e^{(kt)}$, where $A = e^C$ is an arbitrary constant.

This is the general solution expressed as an exponential function of t: $P = A \cdot e^{(kt)}$.

Q.37(b) [4 marks]

Case Study 2 (continued): (ii) If population of bacteria is 1000 at t = 0, and 2000 at t = 1, find the value of k.

Answer: k = ln 2

Explanation:

From general solution P = A·e^(kt).

At t = 0, P = 1000: 1000 = A·e⁰ = A.

So A = 1000.

At t = 1, P = 2000: 2000 = 1000·e^k => e^k = 2 => k = ln 2.

Q.38(a) [4 marks]

Case Study 3: A school offered monthly scholarship of Rs. 3,000 each to some girl students and Rs. 4,000 each to meritorious achievers. In all, 50 students were given scholarships and total monthly expenditure was Rs. 1,80,000.

(i) Express the given information algebraically using matrices.

Answer: [[1,1],[3000,4000]] · [[x],[y]] = [[50],[180000]]

Explanation:

Let x = number of girl students (Rs.

3,000 each) and y = number of meritorious achievers (Rs.

4,000 each).

Total students: x + y = 50.

Total expenditure: 3000x + 4000y = 180000.

In matrix form: [[1, 1],[3000, 4000]]·[[x],[y]] = [[50],[180000]].

Q.38(b) [4 marks]

Case Study 3 (continued): (ii) Check whether the system of matrix equations so obtained is consistent or not.

Answer: Consistent (determinant ≠ 0)

Explanation:

The coefficient matrix A = [[1,1],[3000,4000]].

det(A) = 1×4000 – 1×3000 = 4000 – 3000 = 1000 ≠ 0.

Since the determinant is non-zero, the system is consistent and has a unique solution.

Q.38(c) [4 marks]

Case Study 3 (continued): (iii)(a) Find the number of scholarships of each kind given by the school, using matrices.

Answer: Girl students = 20, Meritorious achievers = 30

Explanation:

A = [[1,1],[3000,4000]], det(A) = 1000.

A^{-1} = (1/1000)[[4000,–1],[–3000,1]].

[[x],[y]] = A^{-1}·[[50],[180000]] = (1/1000)[[4000×50 – 1×180000],[–3000×50 + 1×180000]] = (1/1000)[[200000–180000],[–150000+180000]] = (1/1000)[[20000],[30000]] = [[20],[30]].

So x = 20 girl students and y = 30 meritorious achievers.

OR

Q.38(c) [4 marks]

Case Study 3 (continued): (iii)(b) Had the amount of scholarship given to each girl child and meritorious student been interchanged, what would be the monthly expenditure incurred by the school?

Answer: Rs. 1,70,000

Explanation:

From part (iii)(a): x = 20 girl students, y = 30 meritorious achievers.

If amounts are interchanged: girl students get Rs.

4,000 each and meritorious achievers get Rs.

3,000 each.

New monthly expenditure = 4000×20 + 3000×30 = 80000 + 90000 = Rs.

1,70,000.